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COMPATIBILITY OF THE α-PRODUCT OF SIGNED GRAPHS

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Abstract

A signed graph is a graph whose edges are assigned to a sign, positive or negative, by a function $\sigma : E \to \{+1, -1\}$. E M El-Kholy et.al. defined the α-product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as the graph $G_1 \square G_2$ with vertex set $V_1 \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \square G_2$ if and only if u_1 is adjacent to v_1 in G_1 and u_2 is not adjacent to v_2 in G_2, or u_1 is not adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2. This paper discusses structural properties and connectivity of the α-product of graphs. The α-product of two signed graphs is the signed graph with the vertex set and edge set as that of the unsigned product and the sign of an edge in the α -product is determined by the sign of the adjacency in the component signed graphs. This work also examines the compatibility of α-products for graph classes such as complete graphs and cycles.

Mathematics Subject Classification (2020): 05C12, 05C22, 05C75, 05C76

Key Words: Signed Graphs, $\alpha-$product of two signed graphs, Distance compatibility.

1. Introduction

A graph $G = (V, E)$ consists of a finite non-empty set V of vertices, and another set E of unordered pairs of distinct vertices called edges. The number of vertices in a graph is denoted as $n(G)$, and the number of edges as $m(G)$. Adjacency between u and v is indicated by $u \sim v$ and the corresponding edge is denoted by $e = uv$. The number of vertices adjacent to u in G is termed the degree of u and is denoted by $d_G(u)$. The distance $d(u, v)$ between two vertices u and v is the length of a shortest path between them, if any; otherwise $d(u, v) = \infty$. A shortest $u - v$ path is called a $u - v$ geodesic. A graph G is geodetic if each pair of vertices in G is joined by a unique shortest path. All graphs considered in this paper are finite and simple.

A **signed graph** is a pair $\Sigma = (G, \sigma)$ where $G = (V, E)$ is a graph and $\sigma : E \to \{+1, -1\}$ is a sign function. Here, each edge in G is assigned a sign, either positive or negative, by the function σ. The (unsigned) graph G is said to be the underlying graph of Σ, and the function σ is called the signature of Σ. A signed graph is *homogeneous* if it is all-positive or all-negative. Otherwise, it is *heterogeneous*. An all-positive signed graph is denoted by $+G$ and an all-

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negative graph by $-G$. The sign of a path P in the signed graph $\Sigma = (G, \sigma)$ is defined as $\sigma(P) = \prod_{e \in E(P)} \sigma(e)$.

In [4], S Hameed et.al. defines auxiliary signs of $u - v$ paths as:

$(s1)$ $\sigma_{\max}(u, v) = -1$ if all the shortest u-v paths are negative, and $+1$ otherwise

$(s2)$ $\sigma_{\min}(u, v) = +1$ if all the shortest u-v paths are positive, and -1 otherwise,

and the signed distances between u and v is defined as,

$(d1)$ $d_{max}(u, v) = \sigma_{\max}(u, v)\, d(u, v)$

$(d2)$ $d_{min}(u, v) = \sigma_{\min}(u, v)\, d(u, v)$

where $d(u, v)$ is the usual distance between u and v

Definition 1.1. *[4] Two vertices u and v in a signed graph Σ are said to be distance compatible if $d_{min}(u, v) = d_{max}(u, v)$, and Σ is said to be compatible if every pair of vertices are compatible.*

Hence, a signed graph achieves compatibility when the shortest paths between any two vertices consistently bears the same sign. Since there is only one path between any two vertices of a geodetic graph, a geodetic signed graph is compatible.

2. α-product of graphs

α-product of two graphs G_1 and G_2 is defined [1] as the graph $G_1 \square G_2$ with vertex set $V_1 \times V_2$ where V_1 and V_2 are the vertex sets of G_1 and G_2 respectively, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \square G_2$ if and only if

(i) $u_1 \sim v_1$ in G_1 and $u_2 \nsim v_2$ in G_2

(ii) $u_1 \nsim v_1$ in G_1 and $u_2 \sim v_2$ in G_2

where $\sim$, $\nsim$ denotes the adjacency and non-adjacency respectively.

The definition implies that, $G_1 \square G_2 \cong G_2 \square G_1$ and $G_1 \square G_2$ has $n(G_1)n(G_2)$ vertices and $n(G_1)^2 m(G_2) + n(G_2)^2 m(G_1) - 4m(G_1)m(G_2)$ edges. The degree of a vertex (u_1, u_2) in $G_1 \square G_2$ is $d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1) \times number\ of\ vertices\ non-adjacent\ to - u_2\ in\ G_2 + d_{G_2}(u_2) \times number\ of\ vertices\ non-adjacent\ to\ u_1\ in\ G_1$.

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Theorem 2.1. *Let G_1 and G_2 be two non-trivial connected graphs, then every pair of vertices in $G_1 \square G_2$ has a distance less than or equal to* 2.

Proof. Consider two distinct vertices (u_i, v_j), (u_k, v_l) in $G_1 \square G_2$. To prove the theorem, it is enough to prove that these vertices are connected by a path of length less than or equal to two.

If $u_i \sim u_k$ and $v_j \nsim v_l$ or $u_i \nsim u_k$ and $v_j \sim v_l$, then by definition, $(u_i, v_j) \sim (u_k, v_l)$. If $u_i \sim u_k$ and $v_j \sim v_l$, then $(u_i, v_j)(u_i, v_l)(u_k, v_l)$ is a path of length 2.

If $u_i \nsim u_k$ and $v_j \nsim v_l$, then, since G_1 is connected and non-trivial, a vertex $u_m \in G_1$ such that $u_m \sim u_i$. If $u_m \sim u_k$, then $(u_i, v_j)(u_m, v_l)(u_k, v_l)$ is a path of length two. If $u_m \nsim u_k$, consider a vertex $v_n \sim v_l$. If $v_n \sim v_j$, then $(u_i, v_j)(u_k, v_n)(u_k, v_l)$ is a required path. If $v_n \nsim v_j$, then $(u_i, v_j)(u_m, v_n)(u_k, v_l)$ connects (u_i, v_j) and (u_k, v_l).

It is immediate from the definition that, the number of isolated vertices in $G_1 \square G_2$ is the product of the number of isolated vertices in G_1 and the number of isolated vertices in G_2. Furthermore, the α-product of an empty graph on n vertices with any graph consisting of k connected components result in a graph with exactly kn connected components.

Theorem 2.2. *Let G_1 and G_2 be two non-empty graphs. If at least one of these graphs contains no isolated vertices, then the α-product $G_1 \square G_2$ is connected.*

Proof. Assume that G_1 contains no isolated vertices. The following cases are consid- ered.

Case 1: Both G_1 and G_2 are connected

By Theorem 2.1, every pair of vertices in $G_1 \square G_2$ are connected by a path of length at most 2. Hence the graph is connected.

Case 2: G_1 is disconnected and G_2 is connected.

If $u_1, u_2 \in V(G_1)$ lie in the same connected component, their corresponding vertices in the product are connected as in case 1.

Now suppose u_1 and u_2 belong to different components of G_1, and let $v_1, v_2 \in V(G_2)$ be arbitrary. Since G_2 is non-empty connected graph there exists $v' \in V(G_2)$ such that $v_1 \sim v'$. Then the path $(u_1, v_1)(u_2, v')(u_2, v_1)$ connects (u_1, v_1) and (u_2, v_1). As G_2 is connected (u_2, v_1) is connected to (u_2, v_2). Hence all vertices of the form (u_1, v_1) and (u_2, v_2) are connected in $G_1 \square G_2$, and the product is connected.

Case 3: Both G_1 and G_2 are disconnected.

Let $(u_1, v_1), (u_2, v_2) \in V(G_1 \square G_2)$. Now consider the relative positions of $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$ in their respective components. If $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$ are in the same respective components, the result follows as in case1. If u_1, u_2 are in different components of G_1 and v_1, v_2 are in the same component of G_2, the result follows similarly to case 2.

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Now consider the case where u_1, u_2 are in different components of G_1 and v_1, v_2 are in different components of G_2; a path in $G_1 \square G_2$ connecting (u_1, v_1), and (u_2, v_2) will be constructed. Since G_1 has no isolated vertices let $u'_1 \sim u_1$ and $u'_2 \sim u_2$. Because G_2 is non-empty pick adjacent vertices $v'_1 \sim v'_2$. If $v_1 v'_1 v'_2$ (or $v_2 v'_1 v'_2$) do not form a triangle in G_2, then $v'_1 \nsim v_1$ and $v'_2 \nsim v_2$ (v'_1 may be equal to v_2), and the path $(u_1, v_1)(u'_1, v_1)(u_1, v'_1)(u_2, v'_2)(u'_2, v_2)(u_2, v_2)$ is a path in $G_1 \square G_2$ joining (u_1, v_1), and (u_2, v_2). If $v_1 v'_1 v'_2$ forms a triangle, then the path $(u_1, v_1)(u'_1, v_1)(u'_1, v'_1)(u'_2, v_1)(u_2, v_2)$ provides the required connection.

3. α-product of signed graphs

To introduce the concept of the α-product of signed graphs, it is necessary to define the sign of each edge in the product. The following discussion focuses on the definition and properties of the α-product of signed graphs.

Definition 3.1. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs. The α–product $\Sigma_1 \square \Sigma_2$ of the signed graphs Σ_1 and Σ_2 is the signed graph with vertex set and edge set are as that of the underlying graph and the sign of the edge $(u_1, u_2)(v_1, v_2)$ in $\Sigma_1 \square \Sigma_2$ is defined by*

$$\sigma[(u_1, u_2)(v_1, v_2)] = \begin{cases} \sigma_1(u_1, v_1), if\ u_1 \sim v_1. \\ \sigma_2(u_2, v_2),\ \ if\ u_2 \sim v_2 \end{cases}$$

The total number of positive and negative edges in the α-product $(\Sigma_1 \square \Sigma_2, \sigma)$ are as follows:

1. The number of positive edges is

$$m^+(\Sigma_1 \square \Sigma_2) = n(\Sigma_1)^2 \cdot m^+(\Sigma_2) + n(\Sigma_2)^2 . m^+(\Sigma_1) - 2[m^+(\Sigma_1) . m(\Sigma_2) + m^+(\Sigma_2) \cdot m(\Sigma_1)]$$

2. The number of negative edges is

$$m^-(\Sigma_1 \square \Sigma_2) = n(\Sigma_1)^2 \cdot m^-(\Sigma_2) + n(\Sigma_2)^2 \cdot m^-(\Sigma_1) - 2[m^-(\Sigma_1) . m(\Sigma_2) + m^-(\Sigma_2) \cdot m(\Sigma_1)]$$

Here $m^+(\Sigma_i)$ and $m^-(\Sigma_i)$ denote the number of positive and negative edges respectively in the signed graph Σ_i for $i = 1,2$.

Example 3.1. *In Figure 1, Σ_1 is K_3 with a negative edge, Σ_2 is $-K_2$, and $\Sigma_1 \square \Sigma_2$ is their product. Here the dotted lines represent the negative edges.*

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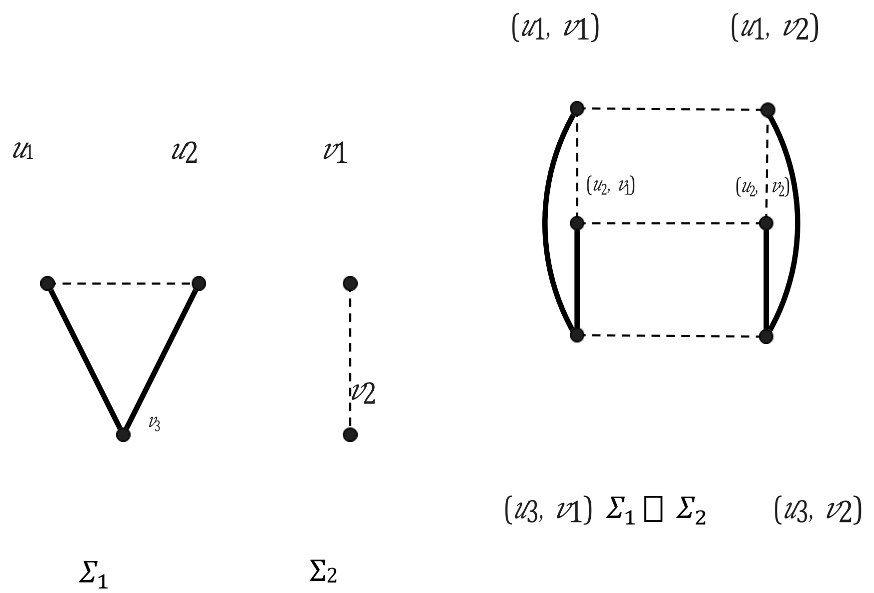

Figure1

4. Compatibility of the α-product of signed graphs

A signed graph is compatible if, for every pair of vertices, all shortest paths between them have the same sign. Furthermore, if the α-product $\Sigma_1 \square \Sigma_2$ is compatible, then the subgraphs Σ_1 and Σ_2 are also compatible, since incompatibility in either would lead to incompatibility in the product.

Example 4.1. Figure 2 shows that α-product of two compatible signed graphs need not be compatible. Since trees are compatible, Γ_1 and Γ_2 are compatible. However, in the product, (u_1, v_2) and (u_2, v_4) are connected by two shortest paths of length 2, namely

$$P_1 = (u_1, v_2)(u_1, v_3)(u_2, v_4)$$

$$and\ P_2 = (u_1, v_2)(u_1, v_4)(u_2, v_4)$$

Where $\sigma(P_1) = \sigma[(u_1, v_2)(u_1, v_3)]\sigma[(u_1, v_3)(u_2, v_4)] = -1. -1 = 1$ and

$\sigma(P_2) = \sigma[(u_1, v_2)(u_1, v_4)]\sigma[(u_1, v_4)(u_2, v_4)] = +1. -1 = -1$. So the vertices (u_1, v_2) and (u_2, v_4) are not compatible. Hence $\Gamma_1 \square \Gamma_2$ is not compatible.

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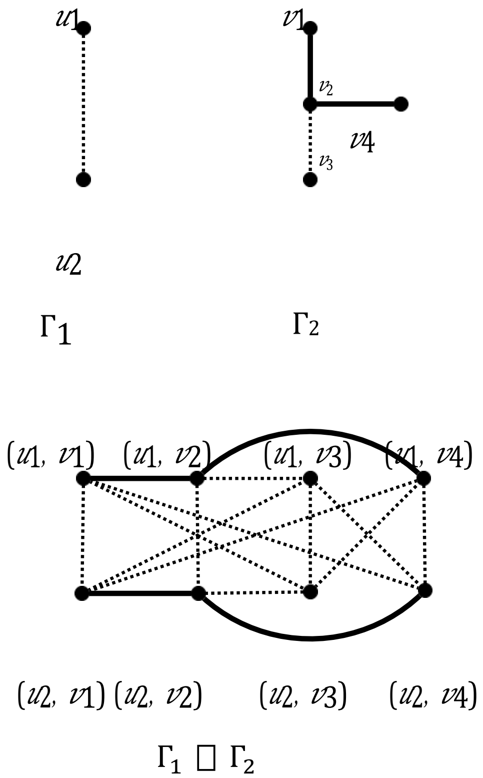

Figure 2

Since all pairs of vertices are adjacent in a complete graph, signed complete graphs are compatible. The compatibility of the α−product of signed complete graphs is ad- dressed in the following theorem.

Theorem 4.2. *Let* $\Sigma_1 = (G_1, \sigma_1)$ *and* $\Sigma_2 = (G_2, \sigma_2)$ *be two signed graphs where* G_1 *and* G_2 *are complete. Then* $\Sigma_1 \square \Sigma_2$*is compatible.*

Proof. Let Σ_1 and Σ_2 be complete signed graphs with underlying graphs $G_1 = K_m$ and $G_2 = K_n$, respectively. The α−product $\Sigma_1 \square \Sigma_2$has signature σ. Let $u_i, i = 1, 2, 3, \dots, m$ and $v_j, j = 1, 2, 3, \dots, n$ be the vertices of G_1 and G_2 respectively. Consider two vertices (u_i, v_j) and (u_k, v_l)in $\Sigma_1 \square \Sigma_2$. If (u_i, v_j) and (u_k, v_l) are adjacent, then the vertices are compatible.

If the (u_i, v_j) and (u_k, v_l) are not adjacent, that is if $u_i \sim u_k$ and $v_j \sim v_l$, then there exist exactly two distinct paths of length 2 from $(u_i, v_j)$to (u_k, v_l) viz. $(u_i, v_j)(u_i, v_l)(u_k, v_l)$ and $(u_i, v_j)(u_k, v_j)(u_k, v_l)$. Since $\sigma[(u_i, v_j)(u_i, v_l)] = \sigma[(u_k, v_j)(u_k, v_l)]$ and $\sigma((u_i, v_l)(u_k, v_l) = \sigma((u_i, v_j)(u_k, v_j))$, both paths have the same sign. It follows that every pair of vertices in $\Sigma_1 \square \Sigma_2$ is compatible; therefore, $\Sigma_1 \square \Sigma_2$is compatible. □

Theorem 4.3. *Let* Σ_1 *and* Σ_2 *be non-empty signed graphs. If there exists in one of the graphs, a path of length two between two non-adjacent vertices such that the two edges in the path have opposite signs, then* $\Sigma_1 \square \Sigma_2$ *is incompatible.*

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Proof. Suppose there exist two non-adjacent vertices u_1 and u_2 of one of the graphs say Σ_1, which are connected by the path $u_1u'u_2$ where the edges u_1u' and $u'u_2$ are of opposite sign. Let u_1u' is a positive and $u'u_2$ is a negative edge. Since Σ_2 is non-empty, there isan edge v_1v_2 in Σ_2. Suppose v_1v_2 is of positive (or negative) sign, then the two shortest paths, $(u_1,v_1)(u',v_1)(u_2,v_1)$and $(u_1,v_1)(u_1,v_2)(u_2,v_1)$ in $\Sigma_1 \square \Sigma_2$ connecting $(u_1,v_1)(u_2,v_1)$ are of opposite sign. Hence $\Sigma_1 \square \Sigma_2$is not compatible.

Theorem 4.3. *Let Σ_1 and Σ_2 be non-empty signed graphs. If there exists in one of the graphs, a path of length two between two non-adjacent vertices such that the two edges in the path have opposite signs, then $\Sigma_1 \square \Sigma_2$ is incompatible.* □

Proof. Suppose there exist two non-adjacent vertices u_1 and u_2 of one of the graphs say Σ_1, which are connected by the path $u_1u'u_2$ where the edges u_1u' and $u'u_2$ are of opposite sign. Let u_1u' is a positive and $u'u_2$ is a negative edge. Since Σ_2 is non-empty, there is an edge v_1v_2 in Σ_2. Suppose v_1v_2 is of positive (or negative) sign, then the two shortest paths, $(u_1,v_1)(u',v_1)(u_2,v_1)$ and $(u_1,v_1)(u_1,v_2)(u_2,v_1)$ in $\Sigma_1 \square \Sigma_2$ connecting (u_1,v_1) and (u_2,v_1) are of opposite sign. Hence $\Sigma_1 \square \Sigma_2$ is not compatible.

Theorem 4.4. *α-product of two cycles C_m and C_n $(m,n \geq 3)$, is compatible if $C_m \square C_n$ have one of the following forms.* □

(1) $+C_m \square + C_n, m,n \geq 3$

(2) $-C_m \square - C_n, m,n \geq 3$

(3) $+C_m \square - C_n, m,n \geq 4$

(4) $C_3 \square + C_4$, *where C_3 has any signature.*

(5) $C_3 \square - C_4$, *where C_3 has any signature.*

(6) $C_3 \square + C_n$, *where C_3 has exactly two negative edges and $n \geq 5$.*

(7) $C_3 \square - C_n$, *where C_3 has exactly one negative edge and $n \geq 5$.*

(8) $C_3 \square C_3$, *where both factors C_3 have arbitrary signatures.*

Proof. In $C_m \square C_n$, the vertices of the cycles C_m and C_n are denoted by $u_1, u_2, \ldots u_m$ and $v_1, v_2, \ldots, v_n$. Also, the signature function of $C_m \square C_n$, C_m and C_n are denoted respectively by σ, σ_1 and, σ_2.

Cases *(1) & (2)*:

Since $+C_m \square + C_n$ and $-C_m \square - C_n$ are homogeneous, the products in *(1)* and *(2)* are compatible.

To prove that the cases in *(3)* to *(7)* yield a compatible product, consider two non- adjacent

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vertices $(u_i, v_j), (u_k, v_l)$ *of* $C_m \square C_n$, where C_m and C_n have the corresponding signatures. Since C_m and C_n are connected, the vertices of $C_m \square C_n$ are connected by a path of length 2. Let $P = (u_i, v_j)(u, v)(u_k, v_l)$ be any 2-path joining the vertices where $u = u_r$ and $v = v_s$ for some $r = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, n$. In all cases there are two possibilities for the choice of u_i and u_k: $u_i \sim u_k$ and $u_i \nsim u_k$. The proof of each case is completed by showing that the vertices (u_i, v_j) and (u_k, v_l) are compatible pairs whenever $u_i \sim u_k$ and $u_i \nsim u_k$.

Case *(3)*: Consider the signed graph $+C_m \square - C_n$ with $m, n \geq 4$.

If $u_i \sim u_k$ (i.e., $v_j \sim v_l$), there are two possible configurations for constructing a path P between (u_i, v_j) and (u_k, v_l): (a) $u_i \sim u, v_j \nsim v, u \nsim u_k, v \sim v_l$ yielding $\sigma(P) = \sigma_1(u_i, u)\sigma_2(v, v_l) = -1$; (b) $u_i \nsim u, v_j \sim v, u \sim u_k, v \nsim v_l$, with $\sigma(P) = -1$. In both cases, all such paths have the same sign, hence the vertices are compatible.

(i) If $u_i \nsim u_k$ (i.e., $v_j \nsim v_l$), again two configurations arise: (a) $u_i \sim u, v_j \nsim v$, $u \sim u_k, v \nsim v_l$, yielding $\sigma(P) = \sigma_1(u_i, u)\sigma_1(u, u_k) = +1$; (b) $u_i \nsim u, v_j \sim v, u \nsim u_k, v \sim v_l$, with $\sigma(P) = +1$. Thus, the vertices remain compatible in this case as well.

Case *(4)*: Consider the signed graph $C_3 \square C_4$, where C_3 has an arbitrary signature.

(i) If $u_i \sim u_k$ (i.e., $v_j \sim v_l$), two possible paths connect (u_i, v_j) and (u_k, v_l):

(a) $u_i \sim u, v_j \nsim v$; $u \nsim u_k$ (or $u = u_k$), $v \sim v_l$, yielding $\sigma(P) = \sigma_1(u_i, u) \cdot \sigma_2(v, v_l) = \sigma_1(u_i, u_k)$;

(b) $u_i \nsim u$ (or $u = u_i$), $v_j \sim v$; $u \sim u_k$, $v \nsim v_l$, yielding $\sigma(P) = \sigma_2(v_j, v) \cdot \sigma_1(u, u_k) = \sigma_1(u_i, u_k)$.

Thus, all such paths have the same sign and hence vertices are compatible.

(i) If $u_i \nsim u_k$ (i.e., $v_j \nsim v_l$), the following cases arise: (a) $u_i \nsim u$ (or $u = u_i$), $v_j \sim v$; $u \nsim u_k$ (or $u = u_k$), $v \sim v_l$; (b) $u_i \sim u$, $v_j \nsim v$; $u \sim u_k$ (possibly $u = u_i$), $v \nsim v_l$. In both cases, $\sigma(P) = +1$, hence the vertex pair is compatible.

Case *(5)*: The compatibility of $C_3 \square - C_4$ can be established using an argument analogous to that in Case (4).

Case *(6)*: Consider the product $C_3 \square + C_n$, where C_3 has exactly two negative edges and $n \geq 5$. Let the negative edges be u_1u_2 and u_2u_3. Note that in C_3, the only non- adjacent vertex to u_i is u_i itself for each $i = 1, 2, 3$.

(i) Suppose $u_i \sim u_k$ ($u_i \neq u_k$) and $v_j \sim v_l$. Since $n \geq 5$, there are three possibilities for forming a path: (a) $u_i \sim u$, $v_j \nsim v$; $u \sim u_k$, $v \nsim v_l$, giving

$\sigma(P) = \sigma_1(u_i, u)\sigma_1(u, u_k) = \sigma_1(u_i, u_k)$, where u is the third vertex in C_3;

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(b) $u_i \nsim u$ (or $u = u_i$), $v_j \sim v$; $u \sim u_k$, $v \nsim v_l$, giving $\sigma(P) = \sigma_2(v_j, v)\sigma_1(u, u_k) = \sigma_1(u_i, u_k)$;

(c) $u_i \sim u, v_j \nsim v$; $u \nsim u_k$ (or $u = u_k$), $v \sim v_l$, giving $\sigma(P) = \sigma_1(u_i, u)\sigma_2(v, v_l) = \sigma_1(u_i, u_k)$;

Thus, all such paths have the same si gn, and the vertices are compatible.

(ii) If $u_i \nsim u_k$ (i.e., $u_i = u_k$) and $v_j \nsim v_l$, then, as in Case (3), any path between (u_i, v_j) and (u_k, v_l) has sign $+1$, so the vertices are compatible.

Case *(7)*: In a similar manner as demonstrated in Case *(6)*, the compatibility of the product in Case *(7)* can be established by utilizing the property

$$\sigma_1(u_i, u) \cdot \sigma_1(u, u_k) = -\sigma_1(u_i, u_k),$$

Which holds for any three distinct vertices u_i, u, u_k of the cycle C_3 when C_3 contains exactly one negative edge.

Case *(8)*: Since C_3 is a complete graph, by Theorem 4.2 $C_3 \square C_3$ is compatible for any signature of C_3.

□

Note that the products of cycles other than the above cases are incompatible. If C_m or C_n is non-homogeneous for $m \geq 4$ or $n \geq 4$, then there are two non-adjacent vertices connected by a path having one positive and one negative edge. Therefore, by Theorem 4.3, $C_m \square C_n$ is incompatible. Incompatible pairs of vertices can be found in the following cases also.

1. $C_3 \square + C_n$ for $n \geq 5$ and C_3 has only one negative edge.

Let u_1u_2 be the negative edge in C_3. Then the vertices (u_1, v_1) and (u_3, v_2) are in-compatible in the product, since the paths $(u_1, v_1)(u_3, v_1)(u_3, v_2)$ and $(u_1, v_1)(u_2, v_4)(u_3, v_2)$are of opposite signs, connecting the non-adjacent vertices (u_1, v_1) and (u_3, v_2).

2. $C_3 \square - C_n$ for $n \geq 5$ and C_3 has exactly two negative edges.

If the negative edges of C_3 are u_1u_2 and u_2u_3, then the paths $(u_1, v_1)(u_3, v_1)(u_3, v_2)$ and $(u_1, v_1)(u_2, v_4)(u_3, v_2)$ are of opposite signs and hence (u_1, v_1) and (u_3, v_2) are incompatible.

Theorem 4.5. *Let Σ_1 and Σ_2 be two signed trees that are homogeneous and if one of them is a star graph, then $\Sigma_1 \square \Sigma_2$ is compatible.*

Proof. If Σ_1 and Σ_2 are of the same sign (if both positive or both negative), then $\Sigma_1 \square \Sigma_2$ will be homogeneous, and hence compatible.

Let Σ_1 and Σ_2 be signed graphs with opposite signs. Since $\Sigma_1 \square \Sigma_2 \cong \Sigma_2 \square \Sigma_1$, without loss of generality, assume that Σ_1 is all positive and Σ_2 is all negative. Further, assume that Σ_2 is a star graph; that is, the distance between any two vertices in Σ_2 is at most 2.

Since Σ_1 and Σ_2 are connected, distance between any two vertices in $\Sigma_1 \square \Sigma_2$ is less than or

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equal two 2. If the vertices (u_i, v_j) and (u_k, v_l) in $\Sigma_1 \square \Sigma_2$ are adjacent, then they form a compatible pair. Let (u_i, v_j) is not adjacent to (u_k, v_l), then the vertices can be connected by a path of length 2. If possible, assume there exist two $(u_i, v_j) - (u_k, v_l)$ paths of length 2 in $\Sigma_1 \square \Sigma_2$ with opposite signs.

Let the positive path be $(u_i, v_j)(u_r, v_s)(u_k, v_l)$. There are two possible sign configurations for the edges, namely:

$$\sigma[(u_i, v_j)(u_r, v_s) = 1 \;,\; \sigma[(u_r, v_s)(u_k, v_l) = 1 \qquad (2)$$

$$\text{Or } \sigma[(u_i, v_j)(u_r, v_s) = 1 \;,\; \sigma[(u_r, v_s)(u_k, v_l) = 1 \qquad (3)$$

Assume the negative path be $(u_i, v_j)(u_x, v_y)(u_k, v_l)$. Then, the edges along this path can have one of two possible sign configurations:

$$\sigma[(u_i, v_j)(u_x, v_y) = -1 \;,\; \sigma[(u_x, v_y)(u_k, v_l) = 1 \qquad (4)$$

$$\text{Or } \sigma[(u_i, v_j)(u_x, v_y) = -1 \;,\; \sigma[(u_x, v_y)(u_k, v_l) = 1 \qquad (5)$$

Claim. No two equations from the pairs (2) and (4), (2) and (5), (3) and (4), or (3) and (5) can simultaneously be satisfied.

Suppose (2) and (4) hold together. Since Σ_2 is all negative, the vertices of Σ_2 satisfy the following conditions,

$$v_j \nsim v_s, v_s \nsim v_l, v_j \sim v_y, v_y \nsim v_l$$

Since Σ_2 is a tree and $v_y \nsim v_l$ there should be at least one vertex in between v_y and v_l. even if there is only one vertex v_p between them, the path $v_j v_y v_p v_l$ is of length 3 in Σ_2, leads to a contradiction.

A similar contradiction can be obtained if (2) and (5) hold together.

Suppose that (3) and (4) hold together, then the vertices of Σ_2 satisfy the conditions,

$$v_j \sim v_s, v_s \sim v_l, v_j \sim v_y, v_y \nsim v_l$$

Then the path $v_y v_j v_s v_l$ of length 3, which contradicts our choice of Σ_2. Similarly (3) and (5) also cannot hold together.

Thus, all $(u_i, v_j) - (u_k, v_l)$ paths of length 2 in $\Sigma_1 \square \Sigma_2$ have the same sign. Hence $\Sigma_1 \square \Sigma_2$ is compatible.

□

Example 4.6. Let P_4 and $-P_4$ be homogeneous trees on the vertex sets u_1, u_2, u_3, u_4 and v_1, v_2, v_3, v_4 respectively. Note that the diameter of P_4 is 3 (≥ 2). Consider the vertices (u_1, v_1) and (u_3, v_4), which can be joined by the paths $(u_1, v_1)(u_2, v_4)(u_3, v_4)$ and $(u_1, v_1)(u_4, v_2)(u_3, v_4)$. Here, the first path is positive and the second negative. Hence $P_4 \square - P_4$ is not compatible.

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Conclusion

This work successfully extends the definition of the α-product to signed graphs, thereby broadening its applicability beyond unsigned graphs. We have thoroughly analyzed the distance compatibility criterion for the α-product of signed complete graphs and signed cycles and established new results concerning compatibility for other families of signed graphs. These findings contribute to a deeper understanding of the structural properties of signed graph products and open new directions for future research in this area.

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