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A NEW MATRIX PARTIAL ORDER AND ITS CHARACTERIZATIONS

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Abstract

A new matrix partial order on the class of matrices having minkowski inverse is obtained in this article. The newly defined Minkowski partial order is based on the involution operator Minkowski adjoint. Several characteriza- tions and properties of Minkowski partial order analogous to secondary partial order are obtained.

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1 Introduction

The generalized inverses play a fundamental role in the study of matrix partial orders. The minus partial order, sharp partial order, star partial order and the core

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EP order are some of the promi- nent partial orders defined using different classes of generalized inverses. Wang et al.,[25] has defined and established a characterization based on the properties of rank of a matrix. For further developments and additional characterizations related to partial or- ders the readers are referred to [4], [12], [3], [15], [2], [24], [17], [7], [18], [1]. Recently, a novel partial order was introduced by [11], with the help of involution operation 'secondary transpose' originally defined by Lee.[13]. Motivated by this work, we propose two new matrix partial orders namely the Minkowski order and the $\sim$ order.

Meenakshi[14] studied the existence of Minkowski inverse using the minkowski inner product proposed by Renardy [22] as a part of his study on singular value decomposition of Mueller matrices. Even- though Minkowski inverse in Minkowski space can be considered as an extension of Moore-Penrose inverse, there are certain differences in the conditions for its existence [6]. The Minkowski metric tensor N is defined as a diagonal matrix $N = diag(1, -I_{(n-1)})$ where $I_{(n-1)}$ is the identity matrix of order $(n-1) \times (n-1)$. Note that$N^{*} = N$ and $N^{2} = I_n$. The adjoint of a matrix X under Minkowski inner product is $X^{\sim} = NX^{*}N$ where X^{*} is the classic conjugate trans- pose. Throughout this article, the symbol $\mathbb{C}^{m\times n}$ is used to represent the set of all complex matrices of order $m \times n$.

Definition 1. [14] For a matrix $X \in \mathbb{C}^{m\times n}$, if there exists a matrix $G \in \mathbb{C}^{n\times m}$ satisfying the following matrix equations:

(1) $XGX = X$ (2) $GXG = G$

(3) $(XG)^{\sim} = XG$ (4) $(GX)^{\sim} = GX$

then is G called Minkowski inverse and denoted by $G = X^{m}$.

Whenever X^{m} exists, it is unique. We denote X^{*}, $Trace(X)$, $r(X)$, $\mathcal{C}(X)$ and $\mathcal{R}(X)$ to represent transpose, trace, rank, column space and row space of X, respectively. Matrices X and Y are said to be space equivalent that is, $X \underset{SP}{\simeq} Y$ if $\mathcal{R}(X) = \mathcal{R}(Y)$ and $\mathcal{C}(X) = \mathcal{C}(Y)$. A matrix G of size $m \times n$ is called a generalized inverse of X of order $n \times m$ if $XGX = X$. X^{-} an arbitrary generalized inverse of X. An $n \times m$ matrix G which satisfy the matrix relations, called as Penrose conditions,

(1) $XGX = X$ (2) $GXG = G$

(3) $(XG)^{*} = XG$ (4) $(GX)^{*} = GX$

is said to be the Moore Penrose inverse of X and represented by

$X^\dagger$. The X that satisfies the conditions (1), (3) and (4) of Penrose equations is called the $\{1, 3, 4\}-$ *inverse* and the conditions (2), (3) and (4) is called the $\{2, 3, 4\}-$ *inverse*, and if X satisfy condition

(2) then it is called $\{2\}-$ *inverse* (Outer inverse).

A square matrix X is said to be $\sim -$*symmetric* if $X^\sim = X$. A matrix X of order $m \times n$ is called $\sim -$*cancellable* if $X^\sim XA = X^\sim XB \Rightarrow XA = XB$ and $AXX^\sim = BXX^\sim \Rightarrow AX = BX$.

A matrix J is called a projector(idempotent) if $J^2 = J$. And J is said to be a $\sim -$*symmetric* projector if $J^\sim = J$. The set of all $\sim -$*symmetric* projectors where $C(J) = W$ is represented by

$\mathbf{J}$ (W).

As we noted that the Minkowski inverse is unique whenever it exists, following theorem gives condition for the same.

Theorem 2. *[14] Let $X \in \mathbb{C}^{m\times n}$. Then, X^{m} exists iff $r(XX^\sim) =$*

$r(X^\sim X) = r(X)$.

Theorem 3. *Consider a matrix X of order $m \times n$. Then the statements below are equivalent.*

(i) X is $\sim$-cancellable. (i.e., If $X^\sim XA = 0$ then $XA = 0$ and if

$BXX^\sim = 0$ then $BX = 0$).

(ii) $r(XX^\sim) = r(X^\sim X) = r(X)$.

(iii) X^{m} exists and

$$X^{\mathrm{m}} = X^\sim(XX^\sim)^- X(X^\sim X)^- X^\sim$$

where $(XX^\sim)^-$ and $(X^\sim X)^-$ are arbitrary generalized in- verses of $XX^\sim$ and $X^\sim X$, respectively.

The concept of minus partial order was originally proposed by Hartwig and Nambooripad[7, 19] on the set of regular elements in a semigroup. Among the various partial orders defined on rectan- gular matrices, the minus partial order is the most dominant. In the context of real or complex matrices the minus partial order is defined as follows:

Definition 4. [18] Let $X, Y \in \mathbb{C}^{m\times n}$. Then Y less than X under minus partial order is denoted by $Y \leq^- X$ and is defined as

$$Y \leq^- X \iff Y^- X = Y^- Y \quad \text{and } XY^- = YY^-$$

The following lemma provides a few characterizations of minus partial order.

Theorem 5. *[15] For $P, Q \in \mathbb{C}^{m\times n}$, the following conditions are equivalent.*

(i) *Q and $P - Q$ are disjoint matrices, i.e., $C(Q) \cap C(P{-}Q) = (0)$ and $R(Q) \cap R(P - Q) = (0)$.*

(ii) *$r(Q) + r(P - Q) = r(P)$.*

(iii) *$Q \leq^{-} P$ with reference to minus order.*

(iv) *$\{P^{-}\} \subseteq \{Q^{-}\}$.*

(v) *$C(Q) \oplus C(P - Q) = C(P)$.*

(vi) *$R(Q) \oplus R(P - Q) = R(P)$.*

Inspired by the study of orders like minus and star defined us- ing generalized inverse and conjugate transpose of a matrix, we introduce a new relation called $\sim$ $-$*order* and m $-$ *order* based on minkowski inner product and minkowski inverse on class of complex rectangular matrices. It can be seen that $\sim$ $-$*order* fails to be a partial order on entire class of complex rectangular matrices, and hence we obtained the necessary and sufficient conditions for the $\sim$ $-$*order* to be a partial order. Also, some properties related to $\sim$ $-$*order* and m $-$ *order* has been discussed.

2 Main Results

We start with discussing some important properties of $\sim$ $-$*symmetric* projectors like its uniqueness. While each subspace P of $\mathbb{C}^n$ pos- sesses an orthogonal projector onto itself, the same does not apply to $\sim$ $-$*symmetric* projector. The justification is provided in the Lemma below.

Lemma 6. *Let P be a subspace of $\mathbb{C}^n$ and let V be any matrix satisfying $C(V) = W$, the following statements hold:*

(i) *If $J, K \in \mathbf{J}(W)$, then $J = K$ (For a given space, $\sim$-symmetric projector is unique, if it exists).*

(ii) *If $J \in \mathbf{J}(W)$, it follows that $R(J) = R(V^{\sim})$.*

(iii) *There exists $J \in \mathbf{J}(W)$ iff $r(V^{\sim}V) = r(V)$, in which case*

$$J = V(V^{\sim}V)^{-}V^{\sim} \qquad \text{for any given } (V^{\sim}V)^{-}.$$

Proof. Assume $J, K \in \mathbf{J}(W)$, it is trivial that $JK = K$ and $KJ = J$ as we have $C(J) = C(K) = W$. Now, J, K are $\sim$- symmetric which gives that

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$$J = KJ = K^{\sim}J^{\sim} = (JK)^{\sim} = K^{\sim} = K,$$

This is proof of statement (i).

Using the definition of $\sim$-symmetric matrix, it follows that $J = J^{\sim} = NJ^{*}N$ and hence $\mathrm{R}(J) = \mathrm{R}(J^{*}N) = \mathrm{C}(NJ)$. Because $\mathrm{C}(V) = \mathrm{C}(J)$, it follows that $\mathrm{R}(J) = \mathrm{C}(NV) = \mathrm{R}(V^{*}N) = \mathrm{R}(NV^{*}N) = \mathrm{R}(V^{\sim})$. Proving statement (ii) of the lemma.

Furthermore, as stated in the Lemma 4.3 of Prasad [20] that any matrices A, X and Y, we can find an $\{2\}-$ *inverse* G satisfying $\mathrm{C}(G) = \mathrm{C}(X)$ and $\mathrm{R}(G) = \mathrm{R}(Y)$ iff $\mathrm{r}(X) = \mathrm{r}(YAX) = \mathrm{r}(Y)$,

then $G = X(YAX)^{-}Y$ for every choice of $(YAX)^{-}$. Hence, statement (iii) of the lemma is based on the fact that $J \in \mathbf{J}(\mathrm{W})$ is $\{2\}-$ *inverse* of the identity matrix I (equivalently, an idempotent matrix) with $C(J) = C(V)$ and $\mathrm{R}(J) = \mathrm{R}(V^{\sim})$, as shown in statement (ii).

□

The next theorem provides characterization of the matrices that have Minkowski inverse. It presents what property of column and row spaces must be satisfied for the existance of the minkowski inverse.

Theorem 7. *Let X an $m \times n$ complex matrix , X^{m} exists iff both $\mathrm{C}(X)$ and $\mathrm{R}(X)$ possess $\sim$-symmetric projectors.*

Proof. Assume X^{m} exists, it follows from its definition that, XX^{m} and $(X^{m}X)^{*}$ are the $\sim$-symmetric projectors corresponding to $\mathrm{C}(X)$ and $\mathrm{R}(X)$, respectively.

On the other hand, when $\mathrm{C}(X)$ possesses an $\sim$-symmetric projector onto itself, then part (iii) of Lemma 6, implies that

$$\mathrm{r}(X^{\sim}X) = \mathrm{r}(X).$$

Following the same argument, if $\mathrm{R}(X)(= \mathrm{C}(X^{*}))$ has an $\sim$-symmetric projector, it follows that

$$\mathrm{r}(X) = \mathrm{r}(X^{*}) = \mathrm{r}(X^{*\sim}X^{*}) = \mathrm{r}(XX^{\sim}).$$

Thus, it follows from [14], given in Theorem 2 that X^{m} exists. □

2.1 The Minkowski order

In this section, we first define $\sim -$*order* on the class of complex rectangular matrices.

Definition 8. Consider the set of all rectangular matrices X, Y over the field $\mathrm{C}^{m\times n}$. The relation $\leq^{\sim}$ defined by $Y \leq^{\sim} X$ if $Y^{\sim}Y = Y^{\sim}X$ and $YY^{\sim} = XY^{\sim}$ is called $\sim$-order.

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It can be seen that $Y \leq^{\sim} X$ if and only if $Y^{m}Y = Y^{m}X$ and $YY^{m} = XY^{m}$ whenever Y^{m} exists.

The proof is as follows:

$Y^{\sim}Y = Y^{\sim}X \implies Y^{m}(Y^{m})^{\sim}Y^{\sim}Y = Y^{m}(Y^{m})^{\sim}Y^{\sim}X$

$\implies (Y^{m}YY^{m})Y = (Y^{m}YY^{m})X$

$\implies Y^{m}Y = Y^{m}X$

$\implies (Y^{\sim}Y)Y^{m}Y = (Y^{\sim}Y)Y^{m}X$

$\implies Y^{\sim}Y = Y^{\sim}X$ $\quad\because Y^{m} \simeq Y^{\sim}$

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So we have $Y^{\sim}Y = Y^{\sim}X$ if and only if $Y^{m}Y = Y^{m}X$. Similarly, it can be proved that $YY^{\sim} = XY^{\sim}$ if and only if $YY^{m} = XY^{m}$, whenever Y^{m} exists.

Therefore,

$$Y \leq^{\sim} X \iff Y^{m}Y = Y^{m}X \text{ and } YY^{m} = XY^{m} \qquad (1)$$ for all matrices Y for which Y^{m} exists.

The Equation (1) leads us to define relation $\leq^{m}$ called Minkowski order.

Definition 9. Consider set of all rectangular matrices X, Y over the field $C^{m\times n}$, the relation $\leq^{m}$ defined by $Y \leq^{m} X$ if Y^{m} exists and $Y^{m}Y = Y^{m}X$ and $YY^{m} = XY^{m}$.

The above relation shows that the relation $\leq^{m}$ is dominated by $\leq^{\sim}$ relation.

Let C^{m} represents the collection of all matrices having Minkowski inverse and $C^{\sim}$ represents class of all matrices having $\sim$-symmetric. From Theorem 2, we have $C^{m} = C^{\sim}$ and equation (1) implies that $\leq^{\sim}$ *and* $\leq^{m}$ coinside in this class of $C^{\sim}(= C^{m})$ i.e,

$$Y \leq^{\sim} X \iff Y \leq^{m} X, \text{ for all } Y \in C^{\sim} = C^{m}. \qquad (2)$$

Then the following lemma gives a characterization for Minkowski order.

Lemma 10. *For $X = Y + Z$ and $Y \in C^{m}$, the statements below are all equivalent to each other:*

(i) $Y^{\sim}(X - Y) = 0$ *and* $(X - Y)Y^{\sim} = 0$

(ii) $Y^{\sim}Y = Y^{\sim}X$ *and* $YY^{\sim} = XY^{\sim}$

(iii) $Y \leq^m X$

(iv) $Y^m Y = Y^m X$ *and* $Y Y^m = X Y^m$. *Proof.* The proof is direct.

The example below illustrates that $\leq^{\sim}$ fails to be a partial order on the class of rectangular matrices.

Example 1. *Consider* $X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Here, $X \leq^{\sim} Y$ *as well as* $Y \leq^{\sim} X$ *since*

$$X^{\sim}X = X^{\sim}Y = Y^{\sim}Y = Y^{\sim}X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$XX^{\sim} = YX^{\sim} = YY^{\sim} = XY^{\sim} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

But $X \neq Y$, *implies that the relation* $\leq^{\sim}$ *fails to be antisymmetric on the class of rectangular matrices.*

Note that the relation $\leq^{\sim}$ is not even a preorder in $C^{2\times 3}$. For example, consider the matrices,

Example 2. $X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *and* $Z = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

Theorem 11. *The relation ‘ $\leq^{\sim}$’ ($\leq^{m}$) defines a partial order on the class of all rectangular matrices that possess Minkowski inverse.*

Proof. Consider a matrices from the class of rectangular matrices having Minkowski inverse. From the definition it is clear that $X \leq^{\sim} X$, and therefore $\leq^{\sim}$ is reflexive. Now, consider X, Y from the class of rectangular matrices having Minkowski inverse satisfying the conditions $X \leq^{\sim} Y$ and $Y \leq^{\sim} X$.

$Y \leq^{\sim} X \Longrightarrow YY^{m} = XY^{m} \Longrightarrow Y = YY^{m}Y = XY^{m}Y$, which implies that $\mathcal{C}(Y) \subset \mathcal{C}(X)$. From $X \leq^{m} Y$, we get $X^{\sim}Y = X^{\sim}X$. Also, $XX^{m} = (XX^{m})^{\sim} = (X^{m})^{\sim}X^{\sim}$. Hence we get

$$X = (X^{m})^{\sim}X^{\sim}X = (X^{m})^{\sim}X^{\sim}Y = XX^{\sim}Y = Y$$

which proves the antisymmetry of $\leq^{\sim}$.

The transitivity property can be proved by considering three matrices X, Y, Z having Minkowski inverse such that $X \leq^{\sim} Y$ and $Y \leq^{\sim} Z$. Now, by definition of $\leq^{\sim}$, we have $XX^{m} = YX^{m}$ and

therefore, $YX^{m}X = XX^{m}X = X$. This proves the $\mathcal{C}(X) \subset \mathcal{C}(Y)$ or equivalently $\mathcal{R}(X^{\sim}) \subset \mathcal{R}(Y^{\sim})$. Since Y^{m} is space equivalent to $Y^{\sim}$, $\mathcal{R}(X^{\sim}) \subset \mathcal{R}(Y^{\sim}) \Longrightarrow \mathcal{R}(X^{m}) \subset \mathcal{R}(Y^{m})$, and therefore

$$X^{m}YY^{m} = X^{m}, \tag{3}$$

Similarly, by taking $X^{m}X = X^{m}Y$, we obtain $\mathcal{C}(X^{m}) \subseteq \mathcal{C}(Y^{m})$ and

$$Y^{m}YX^{m} = X^{m} \tag{4}$$

Hence, we get

$$\begin{aligned}
X^{m}X &= X^{m}Y && \because X \leq^{\sim} Y \\
&= X^{m}\ Y^{m})^{\sim}Y^{\sim}Y && \because Y = YY^{m}Y = (YY^{m})^{\sim}Y \\
&= X^{m}(Y^{m})^{\sim}Y^{\sim}Z = (X^{m}YY^{m})Z && (Y \leq^{\sim} Z\ \ (Y^{m})^{\sim}Y^{\sim} = YY^{m}) \\
&= X^{m}Z,
\end{aligned}$$

Similarly, using equation (4) and $XX^{m} = YX^{m}$ we get $XX^{m} = YX^{m}$. Thus $X \leq^{\sim} Z$ proving that $\leq^{\sim}$ is transitive. Hence, $\leq^{\sim}$ is a partial order on the class of all rectangular matrices that have Minkowski inverse.

The following theorem discusses some properties of $\sim$- order and Minkowski order. It also shows that the Minkowski order is dominated by the minus partial order .

Theorem 12. *Let* $X, Y \in \mathrm{C}^m$. *The assumptions given below are true.*

(i) $Y \leq^{\sim} X$ *implies* $Y \leq^{-} X$.

(ii) $Y \leq^{\sim} X$ *iff if* $Y^{\sim} \leq^{\sim} X^{\sim}$

(iii) For $X \in \mathrm{C}^m$, $Y \leq^{\sim} X$ *iff* $Y^m \leq^{\sim} X^m$

(iv) For $X \in \mathrm{C}^m$, $Y \leq^{\sim} X$ *if and only if* X^m *is a* $\{1, 3, 4\}$*-inverse of* Y*, and in such cases,* $Y^m = X^m Y X^m$.

(v) $Y \leq^{\sim} X$ *iff* Y^m *is a* $\{2, 3, 4\}$*-inverse of* X*, in which case* $Y =$

$XY^m A$

(vi) For X^m *exists,* $Y \leq^{\sim} X$ *iff* Z^m *exists and* $Z = X - Y \leq^{\sim} X$*, in which case* $Z^m = X^m Z X^m$ *and* $X^m = Y^m + Z^m$.

Proof. Assumption (i) is a direct consequence of the definition of minus partial order and Equation (1), since Y^m is a specific gen- eralized inverse of Y . Assumption (ii) follows directly by referring to definition of $\leq^{\sim}$ as provided in definition 8.

Assumption (iii) can be deduced from the property that $Y = (Y^m)^m$ and the equivalent definition of $\leq^{\sim}$ as given in equation (1). For $Y \leq^{\sim} X$, from (i) it follows that $Y \leq^{-} X$. Now, using (iii) $\Rightarrow$ (iv) of Theorem 5, it follows that X^m is a generalized inverse of Y . Also by (iii), we have $Y \leq^{\sim} X \Leftrightarrow Y^m \leq^{\sim} X^m$ and hence $X^m Y = Y^m Y$ and $Y X^m = Y Y^m$, which are $\sim$-symmetric. Thus X^m is a $\{1, 3, 4\}$-inverse of Y . On the other hand, if X^m is a $\{1, 3, 4\}$- inverse of Y , then $Y X^m$ and $(X^m Y)^*$ are $\sim$-symmetric projectors onto $\mathrm{C}(Y)$ and $\mathrm{R}(Y)$, correspondingly. Due to the uniqueness of the $\sim$-symmetric projector onto a given space, it follows that $Y X^m = Y Y^m$ and $X^m Y = Y^m Y$ and therefore, $Y \leq^{\sim} X$. Accordingly, the second part of statement (iv) holds based on the observation that X^m is a $\{1, 3, 4\}$-inverse of Y .

From definition, $Y \leq^{\sim} X \implies Y^m Y = Y^m X$ and $Y Y^m = XY^m$ and therefore the $\sim$-symmetry of XY^m and $Y^m X$ are trivial. Further, by $Y^m Y = Y^m X$, we get

$Y^m X Y^m = Y^m Y Y^m = Y^m,$

and hence Y^m is an $\{2\} - inverse$ of X, hence confirming only a portion of assumption (v).

We now proceed to prove the converse part, by assuming that Y^m is a $\{2, 3, 4\}$-

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inverse of X. Thus, $Y^{m}X$ and $(XY^{m})^{T}$ are $\sim$- symmetric projectors onto the spaces $\mathcal{C}(Y^{\sim})$ and $\mathcal{R}(Y^{\sim})$. By the uniqueness property of the $\sim$-symmetric projector onto a given space, it follows that $Y^{m}X = Y^{m}Y$ and $XY^{m} = YY^{m}$ and hence $Y \leq^{\sim} X$, proving "if" part of assumption (v).

To establish (vi), consider matrices X and Y for which X^{m} exists and $Y \leq^{\sim} X$. With the help of (iii) $\Longrightarrow$ (iv), (v) and (vi) of Theorem 5, it follows that

$YX^{-}Z = ZX^{-}Y = 0$ and $ZX^{-}Z = Z$ for all X^{-},

similarly for X^{m}. In other words, X^{m} is a generalized inverse of Z

such that

$$YX^{m}Z = ZX^{m}Y = 0. \tag{5}$$

As we know that X^{m} is a $\{1, 3, 4\}$-inverse of Y as shown in as- sumption (iv), it follows that $X^{m}ZX^{m}$ is the minkowski inverse of Z, satisfying $ZZ^{m} = XZ^{m}$ and $Z^{m}Z = Z^{m}X$ for $Z^{m} = X^{m}ZX^{m} = X^{m} - Y^{m}$. This proves that $Z \leq^{\sim} X$. The converse part is sym- metric.

Now we focus on characterizing matrices $Y \in \mathbb{C}^{m}$, whenever they exist, that satisfy $Y \leq^{\sim} X$ and $\mathcal{C}(Y) = \mathcal{W}$, whenever they exist, where X is any rectangular matrix, does not always belongs to $\mathbb{C}^{m}$.

In general, such a matrix Y need not exist. This was inspired by the results obtained in Eagambaram et al.[5], in which it is observed that for any rectangular matrix X and a proper subspace $\mathcal{W}$ of

$\mathcal{C}(X)$, one can find a matrix Y with $\mathcal{C}(Y) = \mathcal{W}$ such that $Y \leq^{-} X$.

Moreover, there are infinitely many such matrices Y. In the same work, the authors further identified those subspaces $\mathcal{W} \subseteq \mathcal{C}(X)$ for which one can find a matrix Y with $\mathcal{C}(Y) = \mathcal{W}$ and $Y \leq^{*} X$. Unlike the previous case, whenever such a matrix Y exists, it is unique.

As in the case of star order, the next theorem shows that the matrix $Y \in \mathbb{C}^{m}$ such that $Y \leq^{\sim} X$ with $\mathcal{C}(Y) = \mathcal{W}$, if exists, then it is unique.

Theorem 13. *If $Y, Z \in \mathbb{C}^{m}$ such that $Y, Z \leq^{\sim} X$ and $\mathcal{C}(Y) =$*

$\mathcal{C}(Z)$, then $Y = Z$.

Proof. Given that $\mathcal{C}(Y) = \mathcal{C}(Z)$, by Lemma 6, the $\sim$-symmetric projectors onto $\mathcal{C}(Y)$ and $\mathcal{C}(Z)$ represented by YY^{m} and ZZ^{m}, respectively, are the same. Therefore,

$Y = YY^{m}Y$

$= (YY^{m})X \qquad \therefore Y \leq^{\sim} X$

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$= ZZ^{m}X$ $\qquad\qquad\qquad\qquad$ $\therefore YY^{m} = ZZ^{m}$

$= Z(Z^{m}Z)$ $\qquad\qquad\qquad\qquad$ $\therefore Z \le^{\sim} X$

$= Z,$

proving the theorem. □

The theorem below gives a set of necessary and sufficient conditions under which a matrix $Y \le^{\sim} X$ exists satisfying $\mathrm{C}(Y) = \mathrm{W} \subseteq \mathrm{C}(X)$.

Theorem 14. *let X be any rectangular matrix and $\mathrm{W} \subseteq \mathrm{C}(X)$, we consider the statements below:*

(i) *There exists a matrix $Y \in \mathrm{C}^{m}$ such that $Y \le^{\sim} X$ with $\mathrm{C}(Y) = \mathrm{W}$.*

(ii) *For some $J \in \mathbf{J}(\mathrm{W})$ we have $JXX^{\sim}(I - J) = 0$.*

(iii) *For some $J \in \mathbf{J}(\mathrm{W})$ we have J commutes with $XX^{\sim}$.*

(iv) *For some $J \in \mathbf{J}(\mathrm{W})$ we have $JXX^{\sim}$ is $\sim$-symmetric.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ holds. In addition, $(iv) \Rightarrow (i)$ holds if X^{m} exists.

Proof. In order to prove (i) $\Rightarrow$ (ii), let $Y \in \mathrm{C}^{m}$ as stated in (i). Now, consider $J = YY^{m}$, the $\sim$-projector onto the space W. Upon substituting, it follows that $JX = Y(Y^{m}X) = YY^{m}Y = Y$ as $Y \le^{\sim} X$. Now,

$$Y \le^{\sim} X \quad \Rightarrow \quad YY^{\sim} = YX^{\sim} \Rightarrow JXX^{\sim}J = YY^{\sim} = YX^{\sim} = JXX^{\sim}$$

which proves (ii).

(ii) $\Rightarrow$ (iii) is clear, by considering that $JXX^{\sim}J$ is $\sim$-symmetric.

Let us now consider $J \in \mathbf{J}(\mathrm{W})$ as given in (iii). Thus

$$JXX^{\sim} = XX^{\sim}J = (JXX^{\sim})^{\sim},$$

which proves (iv). Therefore (iii) $\Rightarrow$ (iv).

Now, consider J as the $\sim$-projector onto the space W which satisfies (iv), that is , $JXX^{\sim}$ is $\sim$-symmetric. Consider $Y = JX$. It can be seen that

$$\mathrm{C}(Y) \subseteq \mathrm{C}(X) \qquad \text{and} \qquad \mathrm{R}(Y) \subseteq \mathrm{R}(X), \tag{6}$$

as $\mathrm{C}(Y) = \mathrm{W} = \mathrm{C}(J) \subseteq \mathrm{C}(X)$. Additionally,

$$YY^{\sim} = JXX^{\sim}J = (JXX^{\sim})^{\sim}J = XX^{\sim}JJ = XX^{\sim}J = XY^{\sim}, \tag{7}$$

and

$$Y^{\sim}Y = X^{\sim}PPX = X^{\sim}PX = X^{\sim}Y = Y^{\sim}X, \qquad (8)$$

which show that $Y \leq^{\sim} X$.

Finally, to finish the proof of (iv) ⇒ (i), we must prove that $Y \in \mathrm{C}^{m}$, if, $X \in \mathrm{C}^{m}$. From equation (6), we get $\mathrm{C}(Y^{\sim}) \subseteq \mathrm{C}(X^{\sim})$ and using equation (7), we obtain that $X^{m}YY^{\sim} = X^{m}XY^{\sim} = Y^{\sim}$ (because $X^{\sim} \simeq X^{m}$), proving that $\mathrm{r}(YY^{\sim}) = \mathrm{r}(Y^{\sim}) = \mathrm{r}(Y)$. In the

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same way, from equations (6) and (8), we get that $\mathrm{r}(YY^{\sim}) = \mathrm{r}(Y)$. Now, by Theorem 2, we get that $Y \in \mathrm{C}^{m}$. Therefore, (iv) ⇒ (i). ☐

The next theorem provides a characterization of the subspaces W of $\mathrm{C}(X)$ under which a matrix Y exists with $\mathrm{C}(Y) = W$ and $Y \leq^{m} X$.

Theorem 15. *For any rectangular matrix X and $\mathrm{W} \subseteq \mathrm{C}(X)$, the statements below are equivalent:*

(i) For some matrix Y we have $Y \leq^{m} X$ with $\mathrm{C}(Y) = \mathrm{W}$.

(ii) For some $J \in \mathbf{J}(\mathrm{W})$ we have $JXX^{\sim}(I - J) = 0$ and $\mathrm{r}(XX^{\sim}J) = \mathrm{r}(JXX^{\sim}) = \mathrm{r}(J)$.

(iii) W is a preserved subspace of $XX^{\sim}$ having $\sim$-symmetric pro- jector (that is, $XX^{\sim}(\mathrm{W}) = \mathrm{W}$ and $\mathbf{J}(\mathrm{W}) \neq \emptyset$).

Additionally, in the above case matrix Y is given by

$$Y = V(V^{\sim}V)^{-}V^{\sim}X \qquad (9)$$

for any choice of $(V^{\sim}V)^{-}$.

Proof. Let Y be a matrix satisfying (i), the existence of $J \in \mathbf{J}(\mathrm{W})$ such that

$$JXX^{\sim}(I - J) = 0$$

is shown similar to the case of (i) =⇒ (ii) of Theorem 14, where $J = YY^{m}$. Since Y has m-inverse, for which

$$\mathrm{r}(YY^{\sim}) = \mathrm{r}(Y) = \mathrm{r}(J).$$

This proves the second half of (ii) of the theorem. Thus, (i) =⇒ (ii).

If J is $\sim$-symmetric projector which satisfies (ii) of the theorem, then taking $Y = JX$ where $\mathrm{C}(Y) = \mathrm{C}(J)$, then $Y \leq^{\sim} X$ follows as in the case of (iv) =⇒ (i) of Theorem 14. As J is an $\sim$-symmetric projector onto $\mathrm{C}(Y) = \mathrm{C}(J)$, it follows from Lemma 6 (iii) that

$$\mathrm{r}(Y^{\sim}Y) = \mathrm{r}(Y).$$

Now, $\mathrm{r}(YY^{\sim}) = \mathrm{r}(Y)$ follows from the fact that $JXX^{\sim}J = XX^{\sim}J$ and $\mathrm{r}(XX^{\sim}J) =$

r(J), as stated in (ii). Thus, Y^m exists and it follows from equation (2) that $Y \leq^m X$. Therefore, (ii) ⇒ (i).

Observe that

$$JXX^{\sim}J = XX^{\sim}J = JXX^{\sim} \quad \Longleftrightarrow \mathcal{C}(XX^{\sim}J) = \mathcal{C}(J).$$

Also, the equality on the right-hand side satisfies iff

$$\mathrm{r}(XX^{\sim}J) = \mathrm{r}(J),$$

Thus (iii) ⇐⇒ (ii) is satisfied.

Furthermore, equation (9) results from the expression for ∼- symmetric projector as shown in Lemma 6 (iii). □

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