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BEST APPROXIMATION IN SOFT METRIC SPACES

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ABSTRACT

This article examines best approximation within soft metric spaces, a framework that effectively models uncertain and imprecise data. Extending classical best approximation theory, defining soft best approximation distance and establish theorem under soft contractive and cyclic contractive mappings in complete soft metric spaces. To demonstrate applicability, we provide illustrative examples, including a store-customer model and a blood diagnostics case study, where optimal approximate solutions are needed under parameterized uncertainty. In addition, we present algorithmic procedures for identifying soft best proximity points, reinforcing the practical relevance of our results with flow chart of healthy person.

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Key words*:* Soft Metric Space, Soft Convergent Sequence, Best Approximation

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1. INTRODUCTION

Soft set theory, introduced by Molodtsov in 1999, provides a flexible mathematical framework for handling uncertainty, vagueness, and incomplete information that classical structures often fail to address. Building on this foundation, Das and Samanta (2013), introduced the notion of soft metric spaces, extending the classical metric space into the soft set setting. In this framework, each soft point is associated with parameters, and a soft metric function measures distances in a parameter-dependent way, making it particularly effective for modeling real-world imprecision.

Shabir and Naz (2011), initiated the study of soft topological spaces, showing that every soft topology induces a parameterized family of classical topologies. Their work introduced soft open and closed sets, interiors, and boundaries, which later motivated investigations into compactness, continuity, and regularity in soft topological contexts. Abbas, Murtaza, and Romaguera (2016) further contributed to this field by establishing fixed point results in soft metric spaces, thereby extending contraction principles into uncertain environments.

In parallel, the concept of best approximation (or best proximity point theory) has emerged in classical metric spaces as a natural generalization of fixed-point theory. While fixed points guarantee solutions within a set, best proximity points yield optimal approximate solutions when two sets do not intersect. Basha (2011), among others, developed best proximity point theorems under contraction mappings, highlighting their importance in optimization, nonlinear analysis, and applied sciences. This approximation theory can be studied in Menger space for researchers too, see references [7-10]. This study bridges a gap between soft set theory and best approximation theory, offering both new theoretical insights and practical models for handling imprecise data.

2.PRELIMINARIES

We begin by recalling some fundamental definitions of best proximity point theory. To enhance under- standing and practical relevance, we present real-life analogies alongside formal mathematical definitions.

Definition 1 [1]. *Let* (X, d) *be a metric space, and let* $A, B \subseteq X$ *be two nonempty subsets. A point* $x \in A$ *is called a* best proximity point *for a mapping* $T: A \to B$ *if*

$d(x, Tx) = \inf\{d(a, b) \mid a \in A,\ b \in B\} =: \text{dist.}(A, B)$.

Here, Given One Mathematical Example in support of above definition.

Example 2. *Let* $A = [0, 1]$, $B = [2, 3]$, *and define* $T: A \to B$ *by* $T(x) = 3 - x$, *For* $x = 1$,

$T(1) = 2$, *and* $d(1, T(1)) = |1 - 2| = 1$.

Since this is the minimum possible distance between elements of A and B, $x = 1$ *is a best proximity point.*

Here, Given Real-Life Example in support of above definition.

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Example 3. *Set A: The library building (closed at night).*

Set B: The area outside the library, where people can walk even at night.

You want to return a book at night, but the library (Set A) is closed -you can't go inside.

The book drop box is the best proximity point. It is not inside the library (A), but it is as close as possible to it- and still serves the purpose.

When you can't access the target set (A), the best proximity point gives you the closest working option in the other set (B).

So, you use the book drop box placed outside the library.

Definition 4 [2]**.** *Let X be a nonempty set and E be a set of parameters. A soft set over X is a pair (F, E), where F is a mapping from E into the power set of X (i.e., F: $E \to$ P(X)).*

*Let $\tilde{X}$ be a soft set over X. Then a mapping $\tilde{d}$:$\tilde{X} \times \tilde{X} \to R^{+}(E)$ is said to be a **soft metric** if it satisfies the following conditions for all $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ ϵ $\widetilde{X}$ and e ∈ E:*

(1) $\tilde{d}(\tilde{x}_E, \tilde{y}_E) \geq 0$ if and only if $\tilde{x}_E = \tilde{y}_E)$
(2) $\tilde{d}(\tilde{x}_E, \tilde{y}_E) = 0$ if and only if $\tilde{x}_E = \tilde{y}_E)$
(3) $\tilde{d}(\tilde{x}_E, \tilde{y}_E) = \tilde{d}(\tilde{y}_E, \tilde{x}_E)$
(4) $\tilde{d}(\tilde{x}_E, \tilde{z}_E) \leq \tilde{d}(\tilde{x}_E, \tilde{y}_E) + \tilde{d}(\tilde{y}_E, \tilde{z}_E)$

*Here $\tilde{d}(\tilde{x}_E, \tilde{y}_E)$ represent the parametric distance between $\tilde{x}_E$ $and \tilde{y}_E$ $.Then$ $(\tilde{X}, \tilde{d}, E)$ is called a **soft metric space***

Example 5. *Measuring Book Similarity Imagine: You have a collection of books.*

You want to compare how similar they are, based on different criteria like:

E_1: Genre, E_2:Author,E_3 : $Number\ of\ pages$

Soft Metric Space Structure.

Let U = {Book A, Book B, Book C} (our universal set) $E = E_1, E_2, E_3$(set of parameters).

Let A = Book A, B = Book B, and $F_A: A \to P(U), F_B: B \to P(U)$

Define a soft distance function $\tilde{d}(\tilde{x}_E, \tilde{y}_E)$ Measures distance between books x and y under parameter E.

Interpretation

Books A and B have same genre, but different authors. Book A and C differ in genre, author, and page count. We can now analyze similarity more flexibly, by considering one or more parameters.

Definition of Completeness of soft metric spaces.

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Definition 6 [2]. *Let($\widetilde{X}$, $\widetilde{d}, E$) be a soft metric space and $\{\tilde{x}_{E_n}\}$ be a sequence of soft metric in $\tilde{X}$.we say that $\{\tilde{x}_{E_n}\}$ converge to a soft elements $\tilde{x}_{E_0}$ in $\tilde{X}$ if for each $\tilde{\epsilon} > \tilde{0}$ there exists a natural number E_{n_0}such that $\tilde{d}(\tilde{x}_{E_n}, \tilde{x}_{E_0}) < \tilde{\epsilon}$ for all $n \geq E_{n_0}$. Then we write $\tilde{x}_{E_n} \to \tilde{x}_{E_0}$ as $n \to \infty$ and $\tilde{x}_{E_0}$ is called the limit of the sequence $\{\tilde{x}_{E_n}\}$.*

A sequence $\{\tilde{x}_{E_n}\}$ in a soft metric space ($\widetilde{X}$, $\widetilde{d}, E$) is said to be convergent if it converges to some soft element called a Cauchy sequence $\tilde{x}_{E_0}$in $\tilde{X}$.

Definition of Cauchy Sequences of soft metric spaces.

Definition 7 [2]. *A sequence $\{\tilde{x}_{E_n}\}$ in a soft metric space ($\widetilde{X}$, $\widetilde{d}, E$) is called a Cauchy sequence if for every $\tilde{\epsilon} > \tilde{0}$*, there exists a natural number E_{n_0} such that

$$\tilde{d}(\tilde{x}_{E_n}, \tilde{x}_{E_m}) < \tilde{\epsilon} \text{ for all } E_m, E_n \geq E_{n_0}.$$

Theorem 8 [2]. Every convergent sequence in a soft metric space ($\widetilde{X}$, $\widetilde{d}, E$) is a Cauchy sequence and every Cauchy sequence is bounded. A Cauchy sequence is convergent if and only if it has a convergent sub-sequence.

Theorem 9 [2]. *A soft metric space ($\widetilde{X}$, $\widetilde{d}, E$) is said to be complete if every Cauchy sequence in*

($\widetilde{\mathrm{X}}$, $\widetilde{\mathrm{d}}, \mathrm{E}$) is convergent in $\widetilde{X}$ .Now we recall theorem of Banach Fixed Point Theorem in soft metric spaces.

Theorem 10 [2]. *Let ($\widetilde{X}$, $\widetilde{d}$, E) be a complete soft metric space. If $f: (\widetilde{X}, \widetilde{d}, E) \to (\widetilde{X}, \widetilde{d}, E)$ is a soft contraction mapping (if there is positive soft real number $\tilde{t}$ with $\tilde{0} < \tilde{t} < \tilde{1}$ such that $\tilde{d}(f(\tilde{x}_E), f(\tilde{y}_E)) < \tilde{t}\, \tilde{d}(\tilde{x}_E, \tilde{y}_E))$ $\tilde{x}, \tilde{y} \epsilon \widetilde{X}$* then (f, φ) all *a unique fixed element.*

3. Main Results

Here we introduce the definition of best approximation and related required definition to proof theorem the in frame of soft metric spaces.

Let U be a universal set and E be a set of parameters. A *soft set* over U is defined as a pair

(F, A), were $A \subseteq E$ and$F: A \to P(U)$ F: A →(U), with P (U) denoting the power set of U.

Definition 11. *Let ($\widetilde{X}$, $\widetilde{d}, E$) be a soft metric space over the universe U with parameter set E. Let(F_A, E) and (F_B, E) be two nonempty soft subsets of $\widetilde{X}$. The soft best proximity distance δ between F_A and F_B is defined as:*

$$\tilde{d}\,(F_A, F_B) = inf\{\tilde{d}(x_E, y_E): x_E \epsilon\ F_A, y_E \epsilon F_B\}$$

$$F_{A_0} = \ inf\{x_E \epsilon\ F_A : \tilde{d}(x_E, y_E) = \tilde{d}\,(F_A, F_B)\ for\ some\ y_E \epsilon\ F_B\}$$

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$$F_{B_0} = inf\{y_E \epsilon\ F_B : \tilde{d}(x_E, y_E) = \tilde{d}\ (F_A, F_B)\ for\ some\ x_E \epsilon\ F_A\}$$

Definition 12. *A mapping $T:(F_A, E) \cup (F_B, E) \to (F_A, E) \cup (F_B, E)$ is said to be soft cyclic contraction mapping if there exist a positive soft real number $\tilde{k} \leq 1$, such that*

$\tilde{d}\big(T(x_E), T(y_E)\big) < \tilde{k}\tilde{d}(x_E, y_E) + (1-k)\tilde{d}(F_A, F_B)$ *holds for all $x_E \epsilon (F_A, E)$ and $y_E \epsilon (F_B, E)$*

Theorem 13. *Let $(\widetilde{X}, \widetilde{d}, E)$ be a complete soft metric space, and let (F_A, E) and (F_B, E) be two nonempty closed subset of complete soft metric space $\tilde{X}$ such that$(F_A, E) \cup (F_B, E) \to (F_A, E) \cup (F_B, E)$ is a soft cyclic contractive mapping, i.e., there exists $0 \leq k < 1$ such that for all $x_E \in (F_A, E), y_E \in (F_B, E)$.Then, there exists approximate element $x_E \in (F_A, E)$.*

Proof. Let $x_{E_0} \in (F_A, E)$ be arbitrary. Define the sequence $\{x_{E_n}\}$ by $x_{E_n+1} = T\big(x_{E_n}\big)$.

Then each $x_{E_0} \in (F_B, E)$ and

$$\tilde{d}\big(x_{E_n+1}, y_{E_n}\big) = \tilde{d}\left(T\big(x_{E_n}\big), T\big(x_{E_n+1}\big)\right) \leq k\tilde{d}\big(x_{E_n}, x_{E_n+1}\big) + (1-k)\tilde{d}(F_A, F_B)$$

By induction,

$$\tilde{d}\left(T\big(x_{E_n}\big), T\big(x_{E_n+1}\big)\right) \leq k^n\tilde{d}\big(x_{E_0}, x_{E_1}\big) + (1-k)^n\tilde{d}(F_A, F_B)$$

As $n \to \infty$, $\tilde{d}\left(T\big(x_{E_n}\big), T\big(x_{E_n+1}\big)\right) = \tilde{d}(F_A\ , F_B)$.

Here given example, Store Selection in Location-Based Decision System.

Example 14. Let $U = \{s_1, s_2, s_3, c_1, c_2\}$ *be a universal set where:*

- s_1, s_2, s_3 *represent store locations,*
- c_1, c_2 *represent customer locations.*

Let E = {nearby, accessible} be a set of parameters, and define soft sets:

$F_A\ (nearby) = \{s_1, s_2\}$, $F_A(accessible) = \{s_2, s_3\}$

$F_B\ (nearby) = \{c_1\}$, $F_B(accessible) = \{c_2\}$

Then (F_A, E) and (F_B, E) are soft sets representing store and customer groups respectively.

Assume a soft metric $\tilde{d}(\tilde{x}_E, \tilde{y}_E) = \frac{1}{|E|}\sum_{e \epsilon E} d_H(F_x(e), F_y(e))$ where d_H is the Haus Dorff distance between subsets of U under a fixed Euclidean space distance metric on an abstract 2D space (assume distance between point are known).

Let the Haus Dorff distances be given by:

$$d_H\big(F_A(nearby), F_B(nearby)\big) = 2\ ,$$

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$$\left(F_A(accessible), F_B(accessible)\right) = 1 \,.$$

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Then the soft distance:

$$\tilde{d}(A, B) = \frac{1}{2}(2 + 1) = 1.5$$

Let $\tilde{d}(F_A, F_B) = 1.5$.Then the best proximity set BP (A, B) consists of all soft store configurations in A.

that are at a soft distance δ from some customer configuration in B.

Proof. By definition:

BP $(A, B) = \{x_E \epsilon A | \exists y_E \in B \text{ such that } \tilde{d}(x_E, y_E): 1.5\}$

Here, both components of A (under the parameters "nearby" and "accessible") have respective distances of 2 and 1 from their counterparts in B. Since the average is exactly 1.5, this implies the entire soft set (F_A, E) belongs to the best proximity set.

Hence, BP $(A, B) = \{F_A, E\}$, which is non-empty.

This example models how a company could determine the best group of stores (soft set over U) that are most optimally close to a customer base (another soft set), even if no exact match exists (e.g., no store exactly at a customer location). The best proximity set gives a practical alternative in such scenarios.

Corollary 15. *If* $F_A \cap F_B \neq \emptyset$, *then* $\tilde{d}(F_A, F_B) = 0$ *and every point in* $F_A \cap F_B$ *is a best approximation element.*

Remark 16. *This result extends the classical best approximation theorem to the context of soft metric spaces, as introduced in "On Soft Metric Spaces". In that work, the structure of soft continuity and compact- ness was developed. Our theorem utilizes these foundations to assert the existence of proximity points even when the sets do not intersect.*

In comparison with the paper "Set Values with Best", this theorem supports the same idea of minimizing distances between set-valued maps under soft conditions. The compactness assumption parallels classical metric space theorems, but within a parameter-dependent soft structure.

4.APPLICATION

Application in Blood Diagnostics.

Theorem 17. *Let* U *be the set of blood samples and* E *be a set of biological parameters such as white blood cell count (WBC), hemoglobin level, and platelet count etc. Define two soft sets over* U *as follows:*

- *Let* $\tilde{A} = (F_A, E)$ *represent blood samples showing infected markers (e.g., abnormal WBC, low* hemoglobin).

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- *Let $\tilde{B} = (F_B, E)$ represent samples with non-infected (healthy) markers.*

Suppose that:

- *The soft set $\tilde{A}$ is approximately compact with respect to $\tilde{B}$,*
- *There exists a soft mapping $T: \tilde{A} \to \tilde{B}$ representing a diagnostic transformation (e.g., treatment simulation or reclassification),*
- *The mapping T is a soft proximally contraction.*

Then, by the Best Proximity Point Theorem, there exists a unique blood sample $x_E^* \in \tilde{A}$ such that

$$\tilde{d}\left(x_E^*, T(x_E^*)\right) = inf\left\{\tilde{d}\,(x_E, y_E) : x_E \epsilon \tilde{A}, y_E \epsilon \tilde{B}\right\}.$$

This identifies the most diagnostically borderline case between infection and non-infection.

We present algorithmic frameworks for computing best proximity points in soft metric spaces. The procedures are designed for practical implementation and are supported by flowcharts for clarity.

5. ALGORITHM

Algorithm 18. *Require Soft mapping $T: A \to B$, sets $A, B \subseteq$ E, soft metric $\tilde{d}$, tolerance $\epsilon > 0$,*

Ensure A best proximity point $\tilde{x} \in A$ such that $\tilde{d}(\tilde{x}, T(\tilde{x})) \approx D(A, B)$.

State Initialize:$D_{AB} \leftarrow min\left\{\tilde{d}(a, b) | a \in A, b \in B\right\}$ State best_point ← None, min_ distance ← ∞

Foreach $\tilde{x} \epsilon A$

State $T\tilde{x} \leftarrow T(\tilde{x})$

State $d \leftarrow \tilde{d}(\tilde{x}, T\tilde{x})$

If $|d - D_{AB}| \leq \epsilon$

State Return $\tilde{x}$ Comment Found a best proximity point

ElsIfd <min_ distance

State min_ distance ← d State best_point ← $\tilde{x}$

EndIf

EndFor

*State Return best_ point Comment Approximate best proximity point***ere, we have shown the flow**

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Chart for algorithm

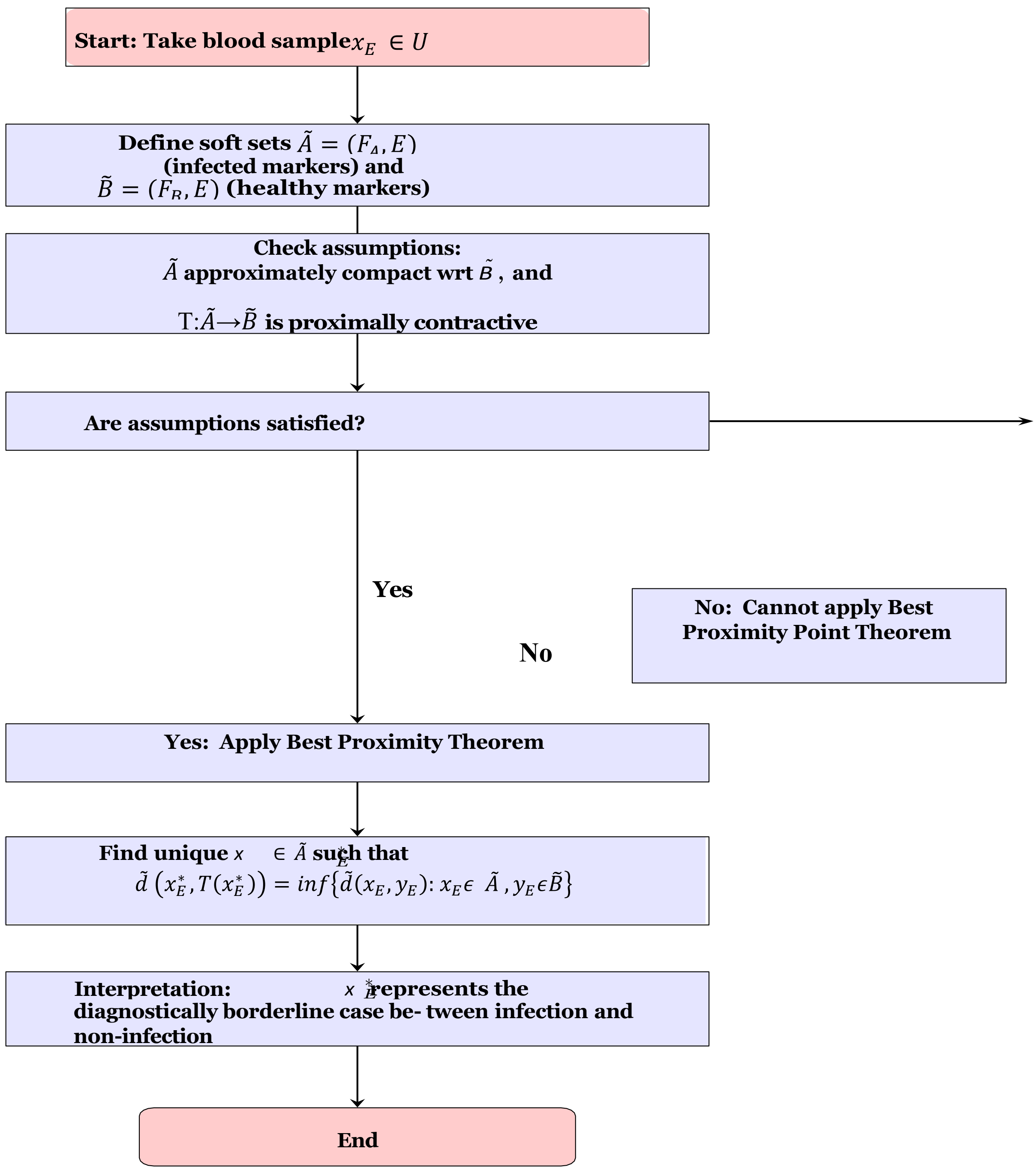

FIGURE 1. Flowchart for Algorithm 18: Know Healthy Person (Soft Set + Best Proximity Point Formulation).

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Conclusion: We seek a sample in $\tilde{A}$ (possibly abnormal) whose soft characteristics are closest to those in $\tilde{B}$ (healthy), using a mapping T:$\tilde{A}\to\tilde{B}$. The theorem guarantees a unique such sample minimizing the diagnostic distance - a "best proximity diagnosis."

- *Extended classical best proximity theory to soft metric spaces.*
- *Proved existence and uniqueness of best proximity points under soft contractive mappings.*
- *Provides a framework for approximate solutions in uncertain environments.*

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REFERENCES

[1] Basha, S. Sadiq, Best Proximity Point Theorems, *Journal of Approximation Theory*, 163(2011), 1772-1781.

[2] Das, S. and Samanta, S.K., On Soft Metric Spaces, Soft metric. *Ann. Fuzzy Math. Inform.*, 6(1) (2013), 77-94.

[3] Abbas, M., Murtaza, G., & Romaguera, S., On the fixed point theory of soft metric spaces, *Fixed Point Theory and Applications*, 17(2016), 1-11.

[4] Molodtsov, D., *Soft set theory: First results*, *Computers & Mathematics with Applications*, 37(4) (1999), 19-31.

[5] Shabir, M., & Naz, M., On soft topological spaces, *Computers & Mathematics with Applications*, 61(7) (2011), 1786-1799.

[6] Tantawy, O. A., & Hassan, R. M., Soft real analysis, *Journal of Progressive Research in Mathematics*, 8(1) (2016), 1207-1219.

[7] Chaudhary, A. K., Control Function in Menger Space, *Nonlinear Functional Analysis and Applications*, 30(1) (2025), 265-276.

[8] Chaudhary, A. K., A common fixed-point result in Menger space, *Communication of Applied Non-linear Analysis*, 31(5s) (2024), 458-465.

[9] Chaudhary, A. K., Occasionally weakly compatible mappings and common fixed points in Menger space, *Results in Non- linear Analysis*, 6(4) (2023), 47-54.

[10] Chaudhary, A. K. and Jha, K., Contraction conditions in Probabilistic Metric Space, *American Journal of Mathematics and Statistics*, 9(5) (2019), 199-202.