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Common Neighbourhood Spectrum of permutability graph of some non-abelian groups

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Abstract

This study presents a detailed analysis of the Common Neighbourhood Spectrum (CN-Spectrum) associated with two graph-theoretic structures: the cyclic permutability graph of a group and the permutability graph of cyclic subgroups of selected finite non-abelian groups. The results provide insights into the local connectivity patterns and structural properties of these graphs, contributing to the broader understanding of algebraic graph theory.

1 Introduction

Algebraic graph theory has emerged as a vibrant area of research in recent years, with a significant number of papers exploring the interplay between algebraic structures and graph-theoretic concepts. One of the foundational contributions in this domain was made by L. Verardi, M. Bianchi, and A. Gillio [2], who introduced the *permutability graph of non-normal subgroups* of a group G. In this graph, the vertices represent all proper non-normal subgroups of G, and two vertices A and B are adjacent if and only if $AB = BA$, or equivalently, if the product AB forms a subgroup of G. Their work analyzed important structural properties such as the number of connected components and the graph's diameter. Later developments extended this framework. In [10], the authors considered a more general concept: the *Planrity and permutability graphs of subgroups of a group G* , where all proper subgroups form the vertex set, and edges are drawn between any two subgroups that permute in G. Similarly, in [9], a specific variation called the *permutability graph of cyclic subgroups* was introduced. This graph takes all cyclic subgroups of a finite group as vertices, connecting two when they permute with each other. Building upon these foundations, we recently introduced a novel structure termed the *cyclic permutability graph of a group*. Unlike prior constructions, this graph uses all non-trivial elements of the group as vertices, and two elements a and b are adjacent if the product of their cyclic subgroups, $\langle a\rangle\langle b\rangle$, is a subgroup of the group. On a related front, the concept

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of *common neighbourhood energy* was introduced in [1], initiating a new direction of spectral graph analysis. Subsequently, W. N. T. Fasfous and collaborators [5] studied the *common neighbourhood spectrum of commuting graphs of finite groups*, while in [6], researchers computed both the spectrum and energy for commuting conjugacy class graph. Motivated by these studies, the present work explores the *common neighbourhood spectrum* of various algebraic graphs. Specifically, we focus on the CN-spectrum of graphs introduced in [4, 9], enriching the spectral theory of algebraically defined graphs. Throughout this paper, we utilize the *common neighbourhood matrix*, defined as follows. Let Γ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For any two distinct vertices v_i and v_j, the common neighbourhood $C(v_i, v_j)$ is the set of vertices (excluding v_i and v_j) that are adjacent to both. The *common neighbourhood matrix* $\mathrm{CN}(\Gamma)$ is the $n \times n$ matrix defined by:

$$(CN(\Gamma))_{i,j} = \begin{cases} |C(v_i, v_j)| & \text{if } i \neq j\ ; \\ 0 & \text{if } i = j \end{cases}$$

Before proceeding to the main results, we review some standard graph-theoretic notations. Let $\Gamma = (V, E)$ be a simple graph with vertex set V and edge set E. The *complement* of Γ, denoted $\overline{\Gamma}$, is the graph in which two vertices are adjacent if and only if they are non-adjacent in Γ. The *complete graph* on n vertices is denoted K_n, where every pair of vertices is adjacent. Given two simple graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, their *union* $\Gamma_1 \cup \Gamma_2$ has vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, while their *join* $\Gamma_1 + \Gamma_2$ includes all edges connecting each vertex of Γ_1 to every vertex of Γ_2.

2 CN-Spectrum of cyclic permutability graph of some finite non-abelian groups

Theorem 2.1. *For any positive odd integer $n \geq 3$, the characteristic polynomial of the CN matrix of $\Gamma(D_n)$ is given by*

$$\phi(CN(\Gamma(D_n)), x) = (x+2n-3)^{(n-2)}(x+n-1)^{(n-1)}[x^2+x(-3n^2+9n-7)+n^4-6n^3+14n^2-15n+6]$$

Proof. By indexing the rows and columns of $CN(\Gamma(D_n))$ in order by the vertices of $\Gamma(D_n)$ are a, $a^2, \cdots, a^{n-1}, b, ab, a^ib, \cdots, a^{n-1}b$, we get

$$CN(\Gamma(D_n)) = \begin{pmatrix} 0 & 2n-3 & 2n-3 & \cdots & 2n-3 & n-2 & n-2 & \cdots & n-2 \\ 2n-3 & 0 & 2n-3 & \cdots & 2n-3 & n-2 & n-2 & \cdots & n-2 \\ . & . & . & & . & . & . & & . \\ 2n-3 & 2n-3 & 2n-3 & \cdots & 0 & n-2 & n-2 & \cdots & n-2 \\ n-2 & n-2 & n-2 & \cdots & n-2 & 0 & n-1 & \cdots & n-1 \\ n-2 & n-2 & n-2 & \cdots & n-2 & n-1 & 0 & \cdots & n-1 \\ . & . & . & & . & . & . & \ddots & . \\ n-2 & n-2 & n-2 & \cdots & n-2 & n-1 & n-1 & \cdots & 0 \end{pmatrix}, \qquad (2.1)$$

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Then $\phi(CN(\Gamma(D_n)), x) =$

$$\begin{array}{ccccccccc}
x & -(2n-3) & -(2n-3) & \cdots & -(2n-3) & -(n-2) & -(n-2) & \cdots & -(n-2) \\
-(2n-3) & x & -(2n-3) & \cdots & -(2n-3) & -(n-2) & -(n-2) & \cdots & -(n-2) \\
. & . & . & & . & . & . & & . \\
-(2n-3) & -(2n-3) & -(2n-3) & \cdots & x & -(n-2) & -(n-2) & \cdots & -(n-2) \\
-(n-2) & -(n-2) & -(n-2) & \cdots & -(n-2) & x & -(n-1) & \cdots & -(n-1) \\
-(n-2) & -(n-2) & -(n-2) & \cdots & -(n-2) & -(n-1) & x & \cdots & -(n-1) \\
. & . & . & & . & . & . & \ddots & . \\
-(n-2) & -(n-2) & -(n-2) & \cdots & & -(n-1) & -(n-1) & \cdots & x
\end{array}.$$

$$= \begin{array}{cc} A & B \\ B^T & C \end{array},$$

where $A = (x+2n-3)I_{(n-1)} - (2n-3)J_{(n-1)}$, $B = -(n-2)J_{(n-1)\times(n)}$, and
$C = (x+n-1)I_n - (n-1)J_n$.

By The Schur complement formula [3],

$$\phi(CN(\Gamma(D_n)), x) = det(A) \times det(C - B^T A^{-1}B). \tag{2.2}$$

Note that $B^T A^{-1}B = -(n-2)J_{n\times(n-1)}A^{-1}(-(n-2))J_{(n-1)\times n} = (n-2)^2 sJ_n$, where s is the sum of all entries in the matrix A^{-1}. Since A is a square matrix in which the sum of the entries in each row is $x-(n-2)(2n-3)$, so by [12], the sum of the entries in each row of A^{-1} is $\frac{1}{x-(n-2)(2n-3)}$ and so $s = \frac{n-1}{x-(n-2)(2n-3)}$.

It follows that

$$C - B^T A^{-1}B = (x+n-1)I_n - kJ_n.$$

Where $k = \frac{nx-x-n^3+4n^2-5n+2}{x-(n-2)(2n-3)}$. Hence,

$$det(C - B^T A^{-1}B) = (x+n-1)^{(n-1)} \frac{x^2 + x(-3n^2+9n-7) + n^4 - 6n^3 + 14n^2 - 15n + 6}{x-(n-2)(2n-3)} . \tag{2.3}$$

and

$$det(A) = (x+2n-3)^{(n-2)}[x - 2n^2 + 7n - 6]. \tag{2.4}$$

Applying (2.2) and (2.3), (2.4), we get the result. □

Theorem 2.2. *For any positive even integer $n \geq 3$, the characteristic polynomial of the CN matrix of $\Gamma(D_n)$ is given by,*
$\phi(CN(\Gamma(D_n)), x) = (x+2n-3)^{(n-2)}(x+n-1)^{(n-1)}[x^2+x(-3n^2+9n-7)+n^4-8n^3+19n^2-18n+6]$.

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Proof. By indexing the rows and columns of $CN(\Gamma(D_n))$ in order by the vertices $a, a^2, \cdots a^{n-1}$, $b, ab, a^2b, \cdots, a^{n-1}b$, we get,

$$CN(\Gamma(D_n)) = \begin{bmatrix} 0 & 2n-3 & 2n-3 & \cdots & 2n-3 & n-1 & n-1 & \cdots & n-1 \\ 2n-3 & 0 & 2n-3 & \cdots & 2n-3 & n-1 & n-1 & \cdots & n-1 \\ . & . & . & & . & . & . & & . \\ 2n-3 & 2n-3 & 2n-3 & \cdots & 0 & n-1 & n-1 & \cdots & n-1 \\ n-1 & n-1 & n-1 & \cdots & n-1 & 0 & n-1 & \cdots & n-1 \\ n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & 0 & \cdots & n-1 \\ . & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & n-1 & \cdots & 0 \end{bmatrix}, \qquad (2.5)$$

Then

$$\phi(CN(\Gamma(D_n)), x) = \begin{vmatrix} A & B \\ B^T & C \end{vmatrix},$$

where $A = (x + 2n - 3)I_{(n-1)} - (2n - 3)J_{(n-1)}$, $B = -(n - 1)J_{(n-1)\times(n)}$, and $C = (x + n - 1)I_n - (n - 1)J_n$.

By The Schur complement formula[3],

$$\phi(CN(\Gamma(D_n)), x) = det(A) \times det(C - B^T A^{-1}B). \qquad (2.6)$$

Note that $B^T A^{-1}B = -(n-1)J_{n\times(n-1)}A^{-1}(-(n-1))J_{(n-1)\times n} = (n-1)^2 sJ_n$, where s is the sum of all entries in the matrix A^{-1}. Since A is a square matrix in which the sum of the entries in each row is $x - (n-2)(2n-3)$, so by [12], the sum of the entries in each row of A^{-1} is $\frac{1}{x-(n-2)(2n-3)}$ and so $s = \frac{n-1}{x-(n-2)(2n-3)}$.

It follows that

$$C - B^T A^{-1}B = (x + n - 1)I_n - kJ_n.$$

Where $k = \frac{nx-x-n^3+6n^2-10n+5}{x-(n-2)(2n-3)}$. Hence,

$$det(C - B^T A^{-1}B) = (x + n - 1)^{(n-1)} \frac{x^2 + x(-3n^2 + 9n - 7) + n^4 - 8n^3 + 19n^2 - 18n + 6}{x - (n-2)(2n-3)}. \qquad (2.7)$$

and

$$det(A) = (x + 2n - 3)^{(n-2)}[x - 2n^2 + 7n - 6]. \qquad (2.8)$$

Applying (2.7) and (2.8) in (2.6), we get the result. □

Theorem 2.3. *For any positive odd integer $n \geq 2$, the characteristic polynomial of the CN matrix of $\Gamma(Q_n)$ is given by,*
$\phi(CN(\Gamma(Q_n)), x) = (x + 4n - 3)^{(2n-2)}(x + 2n - 1)^{(2n-1)}[x^2 + x(-12n^2 + 18n - 7) + 16n^4 - 64n^3 + 76n^2 - 36n + 6]$

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Proof. By indexing the rows and columns of $CN(\Gamma(Q_n))$ in order by the vertices $a, a^2, \cdots, a^{2n-1}$, $b, ab, a^ib, \cdots, a^{2n-1}b$, we get

$$CN(\Gamma(Q_n)) = \begin{pmatrix} 0 & 4n-3 & 4n-3 & \cdots & 4n-3 & 2n-1 & 2n-1 & \cdots & 2n-1 \\ 4n-3 & 0 & 4n-3 & \cdots & 4n-3 & 2n-1 & 2n-1 & \cdots & 2n-1 \\ . & . & . & & . & . & . & & . \\ 4n-3 & 4n-3 & 4n-3 & \cdots & 0 & 2n-1 & 2n-1 & \cdots & 2n-1 \\ 2n-1 & 2n-1 & 2n-1 & \cdots & 2n-1 & 0 & 2n-1 & \cdots & 2n-1 \\ 2n-1 & 2n-1 & 2n-1 & \cdots & 2n-1 & 2n-1 & 0 & \cdots & 2n-1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-1 & 2n-1 & 2n-1 & \cdots & 2n-1 & 2n-1 & 2n-1 & \cdots & 0 \end{pmatrix}, \quad (2.9)$$

Then

$$\phi(CN(\Gamma(Q_n)), x) = \begin{vmatrix} A & B \\ B^T & C \end{vmatrix},$$

where $A = (x + 4n - 3)I_{(2n-1)} - (4n - 3)J_{(2n-1)}$, $B = -(2n - 1)J_{(2n-1)\times(2n)}$, and $C = (x + 2n - 1)I_{2n} - (2n - 1)J_{2n}$.

By The Schur complement formula[3],

$$\phi(CN(\Gamma(Q_n)), x) = det(A) \times det(C - B^T A^{-1}B). \quad (2.10)$$

sum of the entries in each row of C^{-1} is $\frac{1}{x-(2n-2)(4n-3)}$ and so $s = \frac{2n}{x-(2n-2)(4n-3)}$.

It follows that

$$C - B^T A^{-1}B = (x + 2n - 1)I_{2n} - kJ_{2n}.$$

Where $k = \frac{2nx-x-8n^3+24n^2-20n+5}{x-(2n-2)(4n-3)}$. Hence,

$$det(C - B^T A^{-1}B) = (x + 2n - 1)^{(2n-1)} \left(\frac{x^2 + x(-12n^2 + 18n - 7) + 16n^4 - 64n^3 + 76n^2 - 46n + 6}{x - (2n - 2)(4n - 3)}\right). \quad (2.11)$$

and

$$det(A) = (x + 4n - 3)^{(2n-2)}(x - 8n^2 + 14n - 6). \quad (2.12)$$

Applying (2.11) and (2.12) in (2.10), we get the result. □

Theorem 2.4. *For any positive even integer $n \geq 2$, the characteristic polynomial of the CN matrix of $\Gamma(Q_n)$ is given by,*

$$\phi(CN(\Gamma(Q_n)), x) = (x + 2n + 1)^{\frac{3n}{2}}(x + 4n - 3)^{(2n-2)}(x - 14n + 1)^{\frac{n}{2}-1}(x + 4n^2 - 16n + 1) \left[\frac{x^2+x(-12n^2+18n-13)+16n^4-96n^3+140n^2-120n+42}{x-4n^2+4n-7}\right].$$

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Proof. By indexing the rows and columns of $CN(\Gamma(Q_n))$ in order by the vertices $a, a^2, \cdots, a^{2n-1}$, $b, a^n b, a^{\frac{n}{2}} b, a^{\frac{3n}{2}} b, \cdots, a^{(n-1)} b, a^{(2n-1)} b, a^{\frac{3n-2}{2}} b, a^{\frac{5n-2}{2}} b$. We get, Here

$$CN(\Gamma(Q_n)) = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where $A = (4n-3)I_{(2n-1)} - (4n-3)J_{(2n-1)}$, $B = (2n+1)J_{(2n-1)\times(2n)}$,

Therefore,

$$\phi(CN(\Gamma(Q_n)), x) = \begin{vmatrix} x - A & x - B \\ x - B^T & x - C \end{vmatrix} = \begin{vmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{vmatrix}$$

where $A_1 = (x + 4n - 3)I_{2n-1} - (4n-3)J_{2n-1}$; $B_1 = -(2n+1)J_{2n-1\times 2n}$ and here

$$C_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix}, \tag{2.13}$$

Where

$$A_{ii} = \begin{bmatrix} x & -(2n+1) & -(2n+1) & -(2n+1) \\ -(2n+1) & x & -(2n+1) & -(2n+1) \\ -(2n+1) & -(2n+1) & x & -(2n+1) \\ -(2n+1) & -(2n+1) & -(2n+1) & x \end{bmatrix}, \tag{2.14}$$

where $i = 1, 2, \cdots, \frac{2n}{4}$

and $A_{ij} = -(2n-1)J_4$, for all $i \neq j$

By Schur complement formula,

$$\phi(CN(\Gamma(Q_n)), x) = det(C_1) \times det(A_1 - B_1 C_1^{-1} B_1^T) \tag{2.15}$$

Note that $B_1 C_1^{-1} B_1^T = -(2n+1)J_{(2n-1)\times 2n} C^{-1}(-(2n+1))J_{2n\times(2n-1)} = (2n+1)^2 s J_{2n-1}$, where s is the sum of all entries in the matrix C^{-1}. Since C is a square matrix in which the sum of the entries in each row is $x - 4n^2 + 4n - 7$, so by [12], the sum of the entries in each row of C^{-1} is $\frac{1}{x-4n^2+4n-7}$ and so $s = \frac{2n}{x-4n^2+4n-7}$. It follows that

$$A_1 - B_1 C_1^{-1} B_1^T = (x + 4n - 3)I_{2n-1} - kJ_{2n-1}.$$

Where $k = \frac{-8n^3+36n^2-38n+21-3x+4nx}{x-4n^2+4n-7}$ Hence,

$$det(A_1 - B_1 C_1^{-1} B_1^T) = (x+4n-3)^{(2n-2)} \frac{x^2 + x(-12n^2 + 18n - 13) + 16n^4 - 96n^3 + 140n^2 - 120n + 42}{x - 4n^2 + 4n - 7}.$$

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Since A_{ij}, for $i, j = 1, 2, \cdots, n$ in C_1 are real symmetric matrices of order 4 such that they commutes with each other.

Then by corollary 2 in [7],

$$\sigma(C_1) = \sum_{r=1}^{3} \sigma(E_{X_r}), \tag{2.16}$$

where,

$$E_{X_r} = \begin{bmatrix} a_{11}^{X_r} & a_{12}^{X_r} & a_{13}^{X_r} & \cdots & a_{1n}^{X_r} \\ a_{21}^{X_r} & a_{22}^{X_r} & a_{23}^{X_r} & \cdots & a_{2n}^{X_r} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}^{X_r} & a_{n2}^{X_r} & a_{n3}^{X_r} & \cdots & a_{nn}^{X_r} \end{bmatrix}, \tag{2.17}$$

with $a_{ij}^{X_r}$ is an eigen value of A_{ij} corresponding to same eigen vector X_r for each $i, j = 1, 2, \cdots, n$, $r = 1, 2, 3$. [Note that for every i the eigen values of A_{ii} are $3(2n+1)$, $-(2n+1)$, $-(2n+1)$, $-(2n+1)$ and for every $i \neq j$, $i, j = 1, 2, \cdots, n$, the eigen values of A_{ij} are $-4(2n-1)$, 0, 0, 0. Also the eigen values $3(2n+1)$ and $-4(2n-1)$ are have the same eigen vector.] Therefore,

$$E_{X_1} = \begin{bmatrix} 3(2n+1) & -4(2n-1) & \cdots & -4(2n-1) \\ -4(2n-1) & 3(2n+1) & \cdots & -4(2n-1) \\ \vdots & \vdots & & \vdots \\ -4(2n-1) & -4(2n-1) & \cdots & 3(2n+1) \end{bmatrix}, \tag{2.18}$$

and

$$E_{X_k} = \begin{bmatrix} -(2n+1) & 0 & \cdots & 0 \\ 0 & -(2n+1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -(2n+1) \end{bmatrix}, \tag{2.19}$$

for $k = 2, 3, 4$. Since $\phi(E_{X_1}, x) = (x-14n+1)^{\frac{n}{2}-1}(x+4n^2-16n+1)$ and $\phi(E_{X_k}, x) = (x+2n+1)^{\frac{n}{2}}$, for $k = 2, 3, 4$.

Therefore,

$$\sigma(C_1) = (x+2n+1)^{\frac{3n}{2}}(x+4n^2-16n+1)(x-14n+1)^{\frac{n}{2}-1}.$$

Apply the above computed values in (2), we get $\phi(CN(\Gamma(Q_n)), x) = (x+2n+1)^{\frac{3n}{2}}(x+4n-3)^{(2n-2)}(x-14n+1)^{\frac{n}{2}-1}(x+4n^2-16n+1)\left[\frac{x^2+x(-12n^2+18n-13)+16n^4-96n^3+140n^2-120n+42}{x-4n^2+4n-7}\right]$.

This completes the proof. □

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3 CN-Spectrum of Permutability graph of cyclic subgroups of some finite groups

Theorem 3.1. *For any positive even integer $n \geq 3$, the characteristic polynomial of the CN matrix of $\Gamma(D_n)$ is given by*
$\phi(CN(\Gamma(D_n)), x) = (x+s)^{(n-1)}(x+r)^t[x^2 + x(-rt + s - ns) - srt + nrst - ns^3]$.

Proof. By indexing the rows and columns of $CN(\Gamma(D_n))$ in order by the vertices are cyclic subgroups of D_n are $\langle a^{\frac{n}{r}}\rangle$, where r is a divisor of n, $r \neq 1$ and $\langle a^i b\rangle$, of order 2, where $i = 1, 2, \cdots, n$ i.e $b, ab, a^ib, \cdots a^{2n-1}b$, we get

$$CN(\Gamma(D_n)) = \begin{bmatrix} 0 & r & r & \cdots & r & s & s & \cdots & s \\ r & 0 & r & \cdots & r & s & s & \cdots & s \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r & r & r & \cdots & 0 & s & s & \cdots & s \\ s & s & s & \cdots & s & 0 & s & \cdots & s \\ s & s & s & \cdots & s & s & 0 & \cdots & s \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ s & s & s & \cdots & s & s & s & \cdots & 0 \end{bmatrix}, \tag{3.1}$$

Where $r = \tau(n) + n - 3$, $s = \tau(n) - 1$ and $t = \tau(n) - 2$. Then

$$\phi((CN(\Gamma(D_n)), x) = \begin{vmatrix} A & B \\ B^T & C \end{vmatrix},$$

where $A = (x + r)I_{(\tau(n)-1)} - rJ_{(\tau(n)-1)}$, $B = -sJ_{(\tau(n)-1)\times(n)}$, and
$C = (x+s)I_n - sJ_n$.
By The Schur complement formula [3],

$$\phi((CN(\Gamma(D_n)), x) = det(A) \times det(C - B^T A^{-1} B). \tag{3.2}$$

Note that $B^T A^{-1} B = -sJ_{(n)\times(\tau(n)-1)} A^{-1}(-s)J_{(\tau(n)-1)\times(n)} = s^2 U J_n$, where U is the sum of all entries in the matrix A^{-1}. Since A is a square matrix in which the sum of the entries in each row is $x - rt$, so by [12], the sum of the entries in each row of A^{-1} is $\frac{1}{x-rt}$ and so $U = \frac{\tau(n)-1}{x-rt}$.
It follows that

$$C - B^T A^{-1} B = (x+s)I_n - kJ_n.$$

where $k = \frac{s(x-rt)+s^3}{x-rt}$, Hence,

$$det(C - B^T A^{-1} B) = (x+s)^{(n-1)} \left(\frac{x^2 + x(-rt + s - ns) - srt + nsrt - ns^3}{x - rt} \right). \tag{3.3}$$

and

$$det(A) = (x+r)^t(x + r - r(\tau(n) - 1)). \tag{3.4}$$

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Applying (3.3) and (3.4) in (3.2), we get,

$$\phi\,((CN(\Gamma(D_n)), x) = (x+r)^t(x+s)^{(n-1)}[x^2 + x(-rt+s-ns) - srt + nsrt - ns^3]. \tag{3.5}$$

This completes the proof. □

Theorem 3.2. *For any positive odd integer $n \geq 3$, the characteristic polynomial of the CN matrix of $\Gamma(D_n)$ is given by*
$\phi(CN(\Gamma(D_n)), x) = (x+s)^{(n-1)}(x+r)^t[x^2 + x(-rt+s-ns) - srt + nrst - nst^2]$

Proof. By indexing the rows and columns of $CN(\Gamma(D_n))$ in order by the vertices are cyclic subgroups of D_n are $\langle a^{\frac{n}{r}}\rangle$, wherer r is a divisor of n, $r \neq 1$ and $\langle a^i b\rangle$, of order 2, where $i = 1, 2, \cdots, n$ i.e b, ab, $a^i b$, $\cdots$, $a^{2n-1}b$. we get,

$$CN(\Gamma(D_n)) = \begin{bmatrix} 0 & r & r & \cdots & r & t & t & \cdots & t \\ r & 0 & r & \cdots & r & t & t & \cdots & t \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r & r & r & \cdots & 0 & t & t & \cdots & t \\ t & t & t & \cdots & t & 0 & s & \cdots & s \\ t & t & t & \cdots & t & s & 0 & \cdots & s \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \cdots & t & s & s & \cdots & 0 \end{bmatrix}, \tag{3.6}$$

where $r = \tau(n) + n - 3, s = \tau(n) - 1$ and $t = \tau(n) - 2$. Then

$$\phi\,((CN(\Gamma(D_n)), x) = \begin{vmatrix} A & B \\ B^T & C \end{vmatrix},$$

where $A = (x+r)I_{(\tau(n)-1)} - rJ_{(\tau(n)-1)}$, $B = -tJ_{(\tau(n)-1)\times(n)}$, and $C = (x+s)I_n - sJ_n$.
By The Schur complement formula [3],

$$\phi\,((CN(\Gamma(D_n)), x) = det(A) \times det(C - B^T A^{-1} B). \tag{3.7}$$

Note that $B^T A^{-1} B = -tJ_{(n)\times(\tau\,(n)-1)} A^{-1}(-t)J_{(\tau\,(n)-1)\times(n)} = t^2 U J_n$, where U is the sum of all entries in the matrix A^{-1}. Since A is a square matrix in which the sum of the entries in each row is $x - rt$, so by [12], the sum of the entries in each row of A^{-1} is $\frac{1}{x-rt}$ and so $U = \frac{\tau(n)-1}{x-rt}$.
It follows that

$$C - B^T A^{-1} B = (x+s)I_n - kJ_n.$$

where $k = \frac{s(x-rt)+st^2}{x-rt}$. Hence,

$$det(C - B^T A^{-1} B) = (x+s)^{(n-1)} \frac{x^2 + x(-rt+s-ns) - srt + nsrt - nst^2}{x - rt}. \tag{3.8}$$

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and

$$det(A) = (x + r)^t(x + r - r(\tau(n) - 1)). \tag{3.9}$$

Applying (3.8) and (3.9) in (3.7), we get,

$$\phi\ ((CN(\Gamma(D_n)), x) = (x + r)^t(x + s)^{(n-1)}[x^2 + x(-rt + s - ns) - srt + nsrt - nst^2]. \tag{3.10}$$

This completes the proof. □

Theorem 3.3. *For any positive odd integer $n \geq 2$, the characteristic polynomial of the CN matrix of $\Gamma(Q_n)$ is given by,*
$\phi(CN(\Gamma(Q_n)), x) = (x + s)^{(2n-1)}(x + r)^t[x^2 + x(-rt + s - 2ns) - srt + 2nrst - 2nst^2]$. *where* $r = \tau(2n) + 2n - 3$, $s = \tau(2n) - 1$ *and* $t = \tau(2n) - 2$.

Theorem 3.4. *For any positive even integer $n \geq 2$, the characteristic polynomial of the CN matrix of $\Gamma(Q_n)$ is given by,*
$\phi(CN(\Gamma(Q_n)), x) = (x + s)^{(2n-1)}(x + r)^t[x^2 + x(-rt + s - 2ns) - srt + 2nrst - 2ns^3]$, *where* $r = \tau(2n) + 2n - 3$, $s = \tau(2n) - 1$ *and* $t = \tau(2n) - 2$.

Theorem 3.5. *For any integer $\alpha > 3$, the characteristic polynomial of the CN matrix of QD_{2^α} is given by,*
$\phi(CN(\Gamma(QD_{2^\alpha})), x) = (x + s)^{(t-1)}(x + r)^{s-1}[x^2 + x(-rs + r + s - ts) - rs^2 + rs + trs^2 - rst - ts^3$,
where $r = \alpha - 3 + 3 \cdot 2^{\alpha-3}$, $s = \alpha - 1$ *and* $t = 3 \cdot 2^{\alpha-3}$.

Proof. By indexing the rows and columns of CN matrix of $\Gamma(QD_{2^\alpha})$ in order by the vertices of $\Gamma(QD_{2^\alpha})$ are cyclic subgroups of QD_{2^α} are $\langle a^{\frac{2^{\alpha-1}}{r}}\rangle$, wherer r is a divisor of $2^{\alpha-1}$, $r \neq 1$; $\langle a^i b\rangle$, where $i = 2, 2^2, \cdots, 2^{\alpha-2}$ and $\langle a^i b\rangle$ of order 4, where $i = 1, 3, \cdots, 2^{\alpha-2} - 1$. We get,

$$CN(\Gamma(QD_{2^\alpha})) = \begin{bmatrix} 0 & r & r & \cdots & r & s & s & \cdots & s \\ r & 0 & r & \cdots & r & s & s & \cdots & s \\ \multicolumn{9}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ r & r & r & \cdots & 0 & s & s & \cdots & s \\ s & s & s & \cdots & s & 0 & s & \cdots & s \\ s & s & s & \cdots & s & s & 0 & \cdots & s \\ \vdots & . & . & & . & . & . & \ddots & \vdots \\ s & s & s & \cdots & s & s & s & \cdots & 0 \end{bmatrix}, \tag{3.11}$$

where $r = \alpha - 3 + 3 \cdot 2^{\alpha-3}$, $s = \alpha - 1$ and $t = 3 \cdot 2^{\alpha-3}$. Then,

$$\phi(CN(\Gamma(QD_{2^\alpha})), x) = \begin{vmatrix} A & B \\ B^T & C \end{vmatrix},$$

where $A = (x + r)I_s - rJ_s$, $B = -sJ_{s\times t}$, and $C = (x + s)I_t - sJ_t$.
By The Schur complement formula [3],

$$\phi(CN(\Gamma(QD_{2^\alpha})), x) = det(A) \times det(C - B^T A^{-1}B). \tag{3.12}$$

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Note that $B^T A^{-1}B = -sJ_{t\times s}A^{-1}(-s)J_{s\times t} = s^2UJ_t$, where U is the sum of all entries in the matrix A^{-1}. Since A is a square matrix in which the sum of the entries in each row is $x - r(s-1)$, so by [12], the sum of the entries in each row of A^{-1} is $\frac{1}{x-r(s-1)}$ and so $U = \frac{s}{x-r(s-1)}$.
It follows that

$$C - B^T A^{-1}B = (x+s)I_t - kJ_t.$$

where $k = \frac{sx-rs^2+rs+s^3}{x-rs+r}$. Hence,

$$det(C - B^T\, A^{-1}B) = (x+s)^{(t-1)}\frac{x^2 + x(-rs+r+s-ts) - rs^2 + rs + trs^2 - trs - ts^3}{x-rs+r}. \quad (3.13)$$

and

$$det(A) = (x+r)^{s-1}(x+r-rs). \quad (3.14)$$

Applying (3.13) and (3.14) in (3.12), we get,

$$\phi(CN(\Gamma(QD_{2^\alpha})), x) = (x+r)^{s-1}(x+s)^{(t-1)}[x^2 + x(-rs+r+s-ts) - rs^2 + rs + trs^2 - rst - ts^3(]3.\ .15)$$

This completes the proof.

□

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