International Journal of Applied Mathematics
Volume 38 No. 9s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

A Game-Theoretic Perspective on Soft Topology: Linking Banach-Mazur Games with Soft Spaces

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Abstract

This paper introduces a novel link between soft topology and game topology by exploring Banach-Mazur-type games in the context of soft topological spaces. After presenting foundational concepts of both soft and game topology, we define game-theoretic operations on soft sets and propose soft Banach-Mazur games. The convergence, openness, and strategy-based characterizations in soft topological spaces are investigated through player interactions. This work opens new pathways for uncertainty modeling and decision analysis through the lens of game dynamics.

Received: August 06, 2025

International Journal of Applied Mathematics
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1 Introduction

Soft topology provides a framework for dealing with uncertainties based on parameterized families of subsets. On the other hand, game topology explores the dynamic behavior of sets via infinite games. This manuscript investigates their intersection by defining a version of the Banach-Mazur game in soft topological spaces.

2 Preliminaries

2.1 Soft Set and Soft Topology

Definition 2.1. *Let X be a universe and E be a set of parameters. A soft set over X is a pair (F, E) where $F : E \to \mathcal{P}(X)$.*

Definition 2.2. *A collection τ of soft sets over X is a soft topology if:*

- $\tilde{\emptyset}, \tilde{X} \in \tau$
- *Closed under soft union and finite soft intersection*

Then (X, τ, E) is called a soft topological space.

2.2 Game Topology and Banach-Mazur Game

Definition 2.3. *In a topological space (X, τ), the Banach-Mazur game involves two players who alternately choose nested open sets:*

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$$

Player I wins if $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

3 Soft Banach-Mazur Game

Definition 3.1. *Let (X, τ, E) be a soft topological space. Players I and II alternately choose soft open sets $(U_n, e_n) \in \tau$ such that:*

$$(U_1, e_1) \supseteq (U_2, e_2) \supseteq \cdots$$

Player I wins if:

$$\bigcap_{n=1}^{\infty} F(e_n) \neq \emptyset.$$

Example 3.1. *Let $X = \mathbb{R}$, $E = \{e\}$, and define soft open sets using usual open intervals. Suppose Player I chooses intervals shrinking around an irrational point $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Player II responds with smaller intervals, but the nested sequence always contains α. Hence Player I wins, demonstrating that irrational points can serve as winning targets.*

Example 3.2. *Let $X = \mathbb{Q}$, $E = \{e\}$, with soft open sets induced from the subspace topology of $\mathbb{R}$. Here, every nested sequence of rational intervals converges to a point in $\mathbb{R}$ but not necessarily in $\mathbb{Q}$. Thus Player II can ensure that the final intersection is empty within X, so Player II wins.*

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4 Main Theorems

Theorem 4.1. *If Player I has a winning strategy in the soft Banach-Mazur game, then the soft space is non-meager (not soft nowhere dense).*

Proof. Assume the space is soft meager:

$$X = \bigcup_{n=1}^{\infty} (F_n, E), \quad \text{each } (F_n, E) \text{ soft nowhere dense.}$$

Then Player II can always choose $(V_n, e_n) \subseteq (U_n, e_n)$ avoiding (F_n, E). This leads to an empty intersection, contradicting the assumption that Player I has a winning strategy. □

Theorem 4.2. *If Player II has no winning strategy, the space is soft Baire.*

Proof. Suppose the space is not soft Baire. Then $X = \bigcup_{n=1}^{\infty}(F_n, E)$ with (F_n, E) nowhere dense. Player II can avoid each F_n using nested soft open sets, which forms a winning strategy. This contradicts the assumption. □

5 Propositions and Detailed Proofs

Proposition 5.1. *Let (F, E) be soft nowhere dense. Then Player I cannot guarantee a win when restricted to (F, E).*

Proof. By definition of soft nowhere dense, for every nonempty soft open set $(U, E) \in \tau$ there exists a nonempty soft open $(V, E) \in \tau$ with $(V, E) \subseteq (U, E)$ and $(V, E) \cap (F, E) = \widetilde{\emptyset}$. Suppose the play is restricted to (F, E) (so Player I must always choose soft opens contained in (F, E)). After Player I chooses any nonempty soft open $(U_1, E) \subseteq (F, E)$, Player II can reply with a nonempty soft open $(V_1, E) \subseteq (U_1, E)$ that avoids (F, E) (possible because (F, E) has empty interior). Once Player II plays (V_1, E), no future move contained in (V_1, E) can intersect (F, E). Hence Player II can force the intersection of all chosen parameter-wise sets to be empty, so Player I has no guaranteed winning strategy when confined to (F, E). □

Proposition 5.2. *In a soft Baire space, the union of any countable collection of soft nowhere dense sets is not soft dense.*

Proof. A soft Baire space is, by definition, one in which the countable union of soft nowhere dense sets cannot be the whole space (equivalently, any countable union of nowhere dense sets has empty interior). Let

$$X = \bigcup_{n=1}^{\infty} (F_n, E),$$

with each (F_n, E) soft nowhere dense. By the Baire property the interior of $\bigcup_n (F_n, E)$ is empty, so this union cannot be dense (a dense set has closure equal to X, and in particular its interior would not be empty in the presence of the Baire property). Thus the countable union of soft nowhere dense sets is not soft dense. □

Proposition 5.3. *Every soft open set in a soft Baire space contains a soft dense G_δ subset.*

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Proof. Let (U, E) be a nonempty soft open set in the soft Baire space (X, τ, E). Equip the subspace (U, τ_U, E) with the subspace soft topology. The Baire property is hereditary to nonempty open subspaces: (U, τ_U, E) is itself a soft Baire space. A standard consequence of the Baire category theorem (applied in the soft setting) is that every nonempty Baire space contains a dense G_δ set (a countable intersection of dense open sets). Concretely, one constructs a sequence of dense soft open sets $(G_n, E) \subseteq (U, E)$ whose intersection $\bigcap_{n=1}^{\infty}(G_n, E)$ is a G_δ subset of (U, E) and is dense. Viewing this set as a soft G_δ in X contained in (U, E) gives the desired result. □

Proposition 5.4. *Player II cannot avoid a soft dense G_δ set in a soft Baire space.*

Proof. Let $D = \bigcap_{n=1}^{\infty}(G_n, E)$ be a soft dense G_δ set (each (G_n, E) is soft open and dense). Consider any play of the soft Banach–Mazur game producing a nested sequence of nonempty soft open sets

$$(U_1, e_1) \supseteq (U_2, e_2) \supseteq \cdots .$$

Because each (G_n, E) is dense, for each n the set $(U_n, e_n) \cap (G_n, E)$ is nonempty; therefore one can choose subsequent moves inside these intersections. Passing to the limit, every point in the intersection $\bigcap_{n=1}^{\infty} F(e_n)$ belongs to each (G_n, E), so the final intersection meets D. Thus Player II cannot force the play to avoid D. □

Proposition 5.5. *If every soft open set is meager, then Player II has a winning strategy in the soft Banach-Mazur game.*

Proof. If every nonempty soft open set is meager (i.e. a countable union of soft nowhere dense sets), then in particular the whole space X is meager:

$$X = \bigcup_{n=1}^{\infty}(F_n, E),$$

with each (F_n, E) nowhere dense. Using the avoidance strategy: whenever Player I plays (U_n, E), Player II selects a nonempty soft open $(V_n, E) \subseteq (U_n, E)$ such that $(V_n, E) \cap (F_n, E) = \widetilde{\emptyset}$ (possible because F_n is nowhere dense). Doing this for each n yields a nested sequence whose parameter-wise intersection is contained in $\bigcap_n (X \setminus F_n(e_n)) = \varnothing$. Hence Player II forces an empty intersection and wins. □

Proposition 5.6. *If Player I has a winning strategy, the intersection set is dense in the soft space.*

Proof. Let S denote the set of points obtainable as limits of plays following Player I's winning strategy. To show S is dense, take any nonempty soft open set $(U, E) \in \tau$. Consider the play in which Player I makes the first move inside (U, E) according to her winning strategy (this is always possible since the strategy prescribes a legal first move and one may choose that move contained in (U, E)). Because the strategy is winning, the resulting infinite play has nonempty intersection; that intersection lies inside (U, E). Thus every nonempty soft open set meets S, so S is dense in (X, τ, E). □

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6 Applications

- **Decision Systems:** Agent-based strategy modeling under uncertainty.
- **AI and Robotics:** Game-like motion planning in soft environments.
- **Planning:** Convergent selections under soft constraints.

7 Conclusion

This paper bridges soft topology and game theory via Banach-Mazur games. Strategic convergence, soft Baire properties, and meagerness are redefined in soft terms. This opens avenues for modeling dynamic decision-making under uncertainty.

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Received: August 06, 2025