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DYNAMICS OF PERTURBATIONS OF THE ROSSBY-HAURWITZ WAVES

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Abstract

The stability of meteorologically important Rossby-Haurwitz (RH) waves and the dynamics of their perturbations are analyzed. The waves are solutions of the nonlinear vorticity equation for ideal flows on a rotating sphere. They belong to subspaces $\mathbf{H}_1 \oplus \mathbf{H}_n$ $(n \geq 2)$ where $\mathbf{H}_k$ is the subspace of homogeneous spherical polynomials of degree k. A conservation law is derived that divides all perturbations of the wave into invariant disjoint sets $\mathbf{M}_-^n$, $\mathbf{M}_0^n$ and $\mathbf{M}_+^n$. The set $\mathbf{M}_0^n$ contains the invariant subspace $\mathbf{H}_n$ of neutral (stable) perturbations. A factor space of perturbations is introduced, and the relationship between the enstrophy norm, the energy norm, and the factor norm of perturbations is demonstrated. A hyperbolic law between the energy and average spectral number of perturbations in $\mathbf{M}_-^n$ and

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$\mathbf{M}_+^n$ is obtained. A geometric interpretation of the change in perturbation energy is given. The Lyapunov instability of nonstationary non-zonal RH waves is proved, and the instability mechanism is explained. A necessary condition for linear instability is obtained. For Legendre polynomials and zonal RH waves it complements the Rayleigh–Kuo and Fjörtoft conditions. The bounds on the growth rate of unstable modes are given, and the orthogonality of their amplitudes to the basic wave is demonstrated.

Math. Subject Classification: 76B47, 35B10

Key Words and Phrases: Rossby-Haurwitz wave, conservation laws for perturbations, invariant sets and linear and nonlinear stability of perturbations

1 Introduction

The vorticity equation describing the motion of an unforced ideal incompressible fluid on a rotating sphere takes into account nonlinear interaction, dispersion and rotation of the sphere. Its solutions obey an infinite number of integral conservation laws. The ideal fluid model has been an important object of application of mathematical methods for studying nonlinear hydrodynamics for decades.

Within the framework of this model, a large number of research works have been carried out on such important problems as nonlinear interaction of waves [2, 10, 20, 33], turbulence and vortex dynamics [4, 19, 22, 32], construction and stability of solutions [3, 7, 12, 14, 15, 28, 37], etc. In particular, the dynamics and stability of meteorologically significant Rossby-Haurwitz (RH) waves are of considerable interest for a better understanding of the nature of low-frequency variability of large-scale atmospheric processes [19, 21, 24, 25, 32, 33, 34], as well as for the development of data assimilation methods [30].

The RH wave belongs to the set $\mathbf{H}_1 \oplus \mathbf{H}_n$ $(n \geq 2)$ where $\mathbf{H}_k$ is the subspace of homogeneous spherical polynomials of degree k. In this paper, we show that any perturbations of this wave are divided into

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invariant sets $\mathbf{M}_-^n$, $\mathbf{M}_+^n$, $\mathbf{M}_0^n$ and $\mathbf{H}_n$, so the dynamics and stability of the wave can be studied independently in each invariant set.

Since the solutions of the vorticity equation on the sphere S are defined up to a constant, we consider only spaces of functions orthogonal to any constant on S. In this regards, in Section 2 we introduce modified Hilbert spaces $\mathbb{H}^0$, $\mathbb{H}^1$ and $\mathbb{H}^2$ corresponding to the classical spaces $\mathbb{L}^2(S)$, $\mathbb{W}_2^1(S)$ and $\mathbb{W}_2^2(S)$. The properties of the Jacobian (the nonlinear advection term of the vorticity equation) are discussed in Section 3, and the conservation laws of vortex dynamics are considered in Section 4. These laws impose restrictions on the motion of the fluid and are useful for checking the accuracy of numerical calculations or even for constructing adequate numerical schemes. In Section 5 we derive the conservation law for arbitrary (and not just infinitesimal) perturbations of the RH wave, and in Section 6 we discuss invariant sets, factor space and norms for these perturbations. In Section 7 we derive a hyperbolic law for perturbations from the invariant sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$. An explanation of the reason for the changes in the perturbation energy is given in Section 8 using projections in Hilbert spaces, and some examples of the dynamics of RH wave perturbations are considered in Section 9. The Lyapunov instability of any nonstationary non-zonal RH wave from $\mathbf{H}_1 \oplus \mathbf{H}_n$ $(n \geq 2)$ in the invariant set $\mathbf{M}_-^n$ is proved in Section 10, and the mechanism of the instability caused by the difference in the velocities of two RH waves is explained in Section 11.

A new necessary condition for the linear (exponential) instability of the normal modes of the stationary RH wave is obtained in Section 12 using the conservation law of perturbations. The condition imposes a constraint on the spectral distribution of the energy of unstable modes: the average spectral Fjörtoft number [10] of the amplitude of any growing mode of a stationary RH wave from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ must be equal to $\sqrt{n(n+1)}$. This means that linear (exponential) instability of the RH wave is possible only in the set $\mathbf{M}_0^n$ (but not in the subspace $\mathbf{H}_n$). The new instability condition depends only on the degree n of the homogeneous spherical polynomial forming the RH wave. The maximum growth rate of

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unstable modes is also estimated and the orthogonality of the amplitude of any unstable mode to the RH wave in the Hilbert spaces $\mathbb{H}^0$ and $\mathbb{H}^1$ is shown. The results specify the spectral structure of growing disturbances and are especially useful for testing computational algorithms and programs used in numerical studies of the linear stability of fluid flows [32, 34]. In the case of flows determined by Legendre polynomials and zonal RH waves, the new instability condition complements the well-known Rayleigh–Kuo and Fjörtoft conditions [9, 17, 26]. It should also be noted that the average spectral Fjörtoft number [10] is included both in the instability condition and in the estimates of the growth rate of unstable modes. In this regard, the Fjörtoft number is of primary importance in the study of the linear stability of ideal flows on a sphere.

The conclusions are given in Section 13, where, in particular, it is noted that the stability of the RH wave in the invariant set of perturbations $\mathbf{M}_+^n$ is still unsolved problem.

2 Hilbert spaces

We denote by $\mathbb{C}^\infty(S)$ a set of infinitely differentiable functions defined on the unit sphere $S = \{x \in \mathbb{R}^3 : |x| = 1\}$, and by $\mathbb{L}^2(S)$ the Hilbert space of square integrable functions on S with the inner product

$$\langle f, g\rangle = \int_S f(x)\,\overline{g(x)}\,dS \tag{2.1}$$

and the norm

$$\|f\| = \langle f,\, f\rangle^{1/2} \tag{2.2}$$

We will use x and (λ, μ) indiscriminately to denote a point on the sphere S. Also, $dS = d\lambda d\mu$ is an infinitesimal element of the sphere's surface, $\mu = \sin\phi$; $\mu \in [-1, 1]$, ϕ is latitude, $\lambda \in [0, 2\pi)$ is longitude, and $\overline{g}$ is the complex conjugate of g.

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As is known [22, 34], spherical harmonics

$$Y_n^m(\lambda,\mu) = c_{nm} P_n^m(\mu)\, e^{im\lambda}\ , \quad n \geq 0, \quad |m| \leq n, \tag{2.3}$$

form an orthonormal basis in $\mathbb{L}^2(S)$: $\left\langle Y_n^m, Y_l^k \right\rangle = \delta_{mk}\delta_{nl}$, where δ_{mk} is the Kronecker delta,

$$P_n^m(\mu) = \frac{\left(1-\mu^2\right)^{m/2}}{2^n n!} \frac{d^{n+m}}{d\mu^{n+m}} \left(\mu^2 - 1\right)^n$$

is the associated Legendre function of degree n and zonal number m, and

$$c_{nm} = \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}\right]^{1/2}$$

For every integer $n \geq 0$, the set

$$\mathbf{H}_n = \{\psi : -\Delta\psi = \chi_n \psi\} \tag{2.4}$$

is the $(2n+1)$-dimensional eigenspace of homogeneous spherical polynomials of degree n corresponding to the eigenvalue

$$\chi_n = n\,(n+1) \tag{2.5}$$

of multiplicity $2n+1$ of the symmetric and positive definite spherical Laplace operator $-\Delta$ [27]. Moreover,

$$\psi(x) = \sum_{n=0}^{\infty} \mathrm{Y}_n(\psi(x)) \tag{2.6}$$

for any $\psi \in \mathbb{L}^2(S)$ [13, 27, 34], where

$$\mathrm{Y}_n(\psi(x)) = \sum_{m=-n}^{n} \psi_n^m Y_n^m(x) \tag{2.7}$$

is the orthogonal projection of ψ from $\mathbb{L}^2(S)$ onto the subspace $\mathbf{H}_n$ [13, 34]. For simplicity, we will often write (2.6) as

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$$\psi = \sum_{n=0}^{\infty} \mathrm{Y}_n(\psi) \tag{2.8}$$

In this paper we consider only functions ψ orthogonal to a constant on S:

$$\mathrm{Y}_0(\psi) = 0 \tag{2.9}$$

Therefore, below $\mathbb{H}^0$, $\mathbb{H}^1$ and $\mathbb{H}^2$ denote the subspaces of functions of the Sobolev spaces $\mathbb{L}^2(S)$, $\mathbb{W}_2^1(S)$ and $\mathbb{W}_2^2(S)$ on S that satisfy condition (2.9). The inner product and the norm in $\mathbb{H}^k$ $(k = 0, 1, 2)$ are defined as

$$\langle f, g\rangle_k = \langle(-\Delta)^k f, g\rangle = \sum_{n=1}^{\infty} \chi_n^k \,\langle \mathrm{Y}_n(f), \mathrm{Y}_n(g)\rangle \tag{2.10}$$

and

$$\|f\|_k = \langle f,\, f\rangle_k^{1/2} = \left\{ \sum_{n=1}^{\infty} \chi_n^k \,\|\mathrm{Y}_n(f)\|^2 \right\}^{1/2} \tag{2.11}$$

where $\chi_n = n(n+1)$ (see (2.5)) [34]. We will also use $\langle f, g\rangle$ instead of $\langle f, g\rangle_0$.

Lemma 1. *[31]. Let $\psi \in \mathbb{H}^k$, $k = 1, 2$. Then*

$$\|\psi\|_{k-1} \le 2^{-1/2} \|\psi\|_k \ , \tag{2.12}$$

If $\psi \in \mathbb{H}^1$ and $f \in \mathbb{H}^2$ then

$$\|\psi\|_1 = \|\nabla\psi\| \quad \textit{and} \quad \|f\|_2 = \|\Delta f\| \tag{2.13}$$

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3 Jacobian determinant

Let ψ and h be complex-valued functions defined on S, and let $\overrightarrow{n}$ be the unit normal to the surface of the sphere at a point x. The term

$$J(\psi, h) = (\overrightarrow{n} \times \nabla\psi) \cdot \nabla h = \frac{\partial\psi}{\partial\lambda}\frac{\partial h}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial h}{\partial\lambda} \tag{3.1}$$

can be considered as the Jacobian determinant. In Section 4, the term $J(\psi, \Delta\psi)$ (where ψ and $\Delta\psi$ are the streamfunction and potential vorticity, respectively) will represent the advective nonlinear term of the vorticity equation describing the motion of an ideal non-divergent fluid (see (4.1)).

Obviously,

$$J(\psi, h) = -J(h, \psi) \tag{3.2}$$

and

$$J(\psi, \psi) = 0 \tag{3.3}$$

Moreover, $J(\psi, h) = 0$ if both $\psi(\mu)$ and $h(\mu)$ are zonal functions.

It is well known [8, 13, 34] that

$$\int_S J(\psi, h)\ dS = 0 \tag{3.4}$$

Let ψ, g and h be continuously differentiable complex-valued functions on S. Integrating the identity $J(h\psi, g) = hJ(\psi, g) - \psi J(g, h)$ over S and using (3.2) and (3.4) we get

$$\langle J(\psi,\ g), h\rangle = \left\langle J(g, \overline{h}), \overline{\psi}\right\rangle = -\left\langle J(\psi,\ \overline{h}),\ \overline{g}\right\rangle \tag{3.5}$$

Lemma 2. *[32]. Let k be a natural, and let ψ and h be continuously differentiable complex-valued functions on S. Then*

$$\left\langle J(\psi,\ \mu), \overline{\Delta^k\psi}\right\rangle = 0 \tag{3.6}$$

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and

$$Re\,\langle J(\psi,\mu),\psi\rangle = 0\ , \qquad Re\,\langle J(\psi,\mu),\Delta\psi\rangle = 0 \tag{3.7}$$

In particular, if ψ is a real function then

$$Re\,\langle J(h,\psi),h\rangle = 0 \ \ \textit{and} \quad \langle J(\psi,h),\psi\rangle = 0 \tag{3.8}$$

4 Conservation laws for an ideal incompressible fluid on a sphere

The dynamics of an inviscid incompressible fluid on a rotating sphere obeys the nonlinear vorticity equation

$$\frac{\partial}{\partial t}\Delta\psi + J\left(\psi, \Delta\psi + 2\mu\right) = 0 \tag{4.1}$$

describing the conservation of absolute vorticity $\Omega = \Delta\psi + 2\mu$ in each material particle of the fluid [6, 12].

Since every real solution $\psi(t,x)$ of equation (4.1) is defined up to a constant, all functions in this work are assumed to be orthogonal to a constant on the sphere, that is,

$$\mathrm{Y}_0(\psi) = 0 \tag{4.2}$$

(see (2.9)).

Integrating equation (4.1) with respect to S and taking into account (3.4) we obtain that the integral relative vorticity is preserved over time

$$\frac{d}{dt}\int_S \Delta\psi \; dS = 0$$

In addition, equation (4.1) has the following important invariants of motion [6, 7, 10]:

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$$\frac{d}{dt}\int_S \mu\Delta\psi dS = 0 \tag{4.3}$$

$$\frac{d}{dt}\mathbf{K} = \frac{1}{2}\frac{d}{dt}\|\nabla\psi\|^2 = 0 \tag{4.4}$$

$$\frac{d}{dt}\boldsymbol{\eta} = \frac{1}{2}\frac{d}{dt}\|\Delta\psi\|^2 = 0 \tag{4.5}$$

where

$$\mathbf{K} = \frac{1}{2}\|\nabla\psi\|^2 = \frac{1}{2}\|\psi\|_1^2 = \frac{1}{2}\sum_{n=1}^{\infty}\chi_n\|\mathrm{Y}_n(\psi)\|^2 \tag{4.6}$$

and

$$\boldsymbol{\eta} = \frac{1}{2}\|\Delta\psi)\|^2 = \frac{1}{2}\|\psi\|_2^2 = \frac{1}{2}\sum_{n=1}^{\infty}\chi_n^2\|\mathrm{Y}_n(\psi)\|^2 \tag{4.7}$$

are the total kinetic energy and enstrophy of the solution ψ, respectively. Thus the energy norm $\|\nabla\psi\|$ is related to the kinetic energy of the flow, and the enstrophy norm $\|\Delta\psi)\|$is related to the distribution of vorticity (a measure of local rotation). According to Lemma 1, the enstrophy norm is stronger than the energy norm. The energy $\mathbf{K}$ and enstrophy η can be written as

$$\mathbf{K}(t) = \sum_{n=1}^{\infty}\mathbf{K}_n(t), \qquad \boldsymbol{\eta}(t) = \sum_{n=1}^{\infty}\chi_n\mathbf{K}_n(t) \tag{4.8}$$

where $\mathbf{K}_n = \frac{1}{2}\chi_n\|\mathrm{Y}_n(\psi)\|^2$ is the portion of the energy concentrated in the subspace $\mathbf{H}_n$ (see (2.7)) and $\chi_n = n(n+1)$.

Due to (4.4) and (4.5), the average spectral number

$$\chi_a = \boldsymbol{\eta}(t)/\mathbf{K}(t) = Const = \boldsymbol{\eta}(0)/\mathbf{K}(0) \tag{4.9}$$

is also invariant of motion, and $\sqrt{\chi_a}$ is known as the mean Fjörtoft spectral number [10]. According to the law (4.9), during motion the energy $\mathbf{K}(t)$ cannot be accumulated only on small scales (i.e.

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in the subspaces $\mathbf{H}_n$ with $\chi_n > \chi_a$) or only on large scales (in the subspaces $\mathbf{H}_n$ with $\chi_n < \chi_a$).

Lemma 3. *[34]. Let ψ be a real solution of equation (4.1). Then $\mathbf{K}_1 = Const$, that is, the part of the solution energy concentrated in the subspace $\mathbf{H}_1$ does not change with time.*

Corollary 1. *[34]. The Fourier coefficients ψ_1^0 , ψ_1^1 and ψ_1^{-1} (see (2.7)) of any real solution ψ of equation (4.1) do not change in time.*

5 Conservation laws for Rossby-Haurwitz wave perturbations

It is well known that non-stationary complex-valued Thompson solutions of equation (4.1) on S [36] include meteorologically important real Rossby-Haurwitz (RH) waves from the subspaces $\mathbf{H}_1 \oplus \mathbf{H}_n$ ($n \geq 2$) [3, 12, 28]:

$$\psi(t,\lambda,\mu) = -\omega\mu + \mathrm{Y}_n(\psi(t,\lambda,\mu)) \equiv -\omega\mu + \sum_{m=-n}^{n} \psi_n^m Y_n^m\left(\lambda - C_n t, \mu\right) \tag{5.1}$$

where $\omega\mu \in \mathbf{H}_1$ is the Legendre polynomial of the first degree, and the second term $\mathrm{Y}_n(\psi) \in \mathbf{H}_n$ is a homogeneous spherical polynomial of degree n (see (2.7)). Since $Y_n^{-m}(x) = (-1)^m \overline{Y_n^m(x)}$, the wave (5.1) is real only if $\psi_n^{-m} = (-1)^m \overline{\psi_n^m}$. This is the solution of the vorticity equation (4.1) provided that

$$C_n = \omega - 2\left(\omega + 1\right)/\chi_n \tag{5.2}$$

where $\chi_n = n(n+1)$ (see (2.5)).

If $\widetilde{\psi}(t,x)$ is another real solution of (4.1), then the perturbation

$$\psi' = \widetilde{\psi} - \psi \tag{5.3}$$

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of the wave (5.1) is a solution to the initial value problem

$$\Delta\psi'_t + J(\psi, \Delta\psi') + J(\psi', \Delta\psi + 2\mu) + J(\psi', \Delta\psi') = 0 \qquad (5.4)$$

$$\psi'(0, x) = \varphi(x) \equiv \widetilde{\psi}(0, x) - \psi(0, x) \qquad (5.5)$$

Using the formula $\Delta\psi + 2\mu = -\chi_n\psi - \chi_n C_n\mu$ equation (5.4) can be written as

$$\Delta\psi'_t + J(\psi + \psi', \Delta\psi' + \chi_n\psi') - \chi_n C_n J(\psi', \mu) = 0 \qquad (5.6)$$

or

$$\Delta\psi'_t + J(\widetilde{\psi}, \Delta\psi' + \chi_n\psi') - \chi_n C_n J(\psi', \mu) = 0 \qquad (5.7)$$

[34]. Equation (5.6) reduces to

$$\Delta\psi'_t + J(\psi + \psi', \Delta\psi' + \chi_n\psi') = 0 \qquad (5.8)$$

if (5.1) is a stationary wave ($C_n = 0$).

By virtue of Szeptycki's theorem on the existence and uniqueness of a solution of equation (4.1) [35], an adequate norm for estimating the perturbations of the solution must be equivalent to the enstrophy norm of perturbations.

Theorem 1. *[32, 34]. The kinetic energy $K(t)$ and enstrophy $\eta(t)$ of any perturbation ψ' of the RH wave (5.1) decrease, remain constant or increase simultaneously, according to the law*

$$\frac{d}{dt}\eta(t) = \chi_n \frac{d}{dt}K(t) \qquad (5.9)$$

where $\chi_n = n(n+1)$,

$$K(t) = \frac{1}{2}\|\nabla\psi'(t)\|^2 = \frac{1}{2}\|\psi'(t)\|_1^2 \qquad (5.10)$$

and

$$\eta(t) = \frac{1}{2}\|\Delta\psi'(t)\|^2 = \frac{1}{2}\|\psi'(t)\|_2^2 \qquad (5.11)$$

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The growth of the perturbation enstrophy is $n(n+1)$ times faster than the growth of the perturbation energy. Theorem 1 generalizes the results obtained by Gill [11] and Karunin [16] for infinitesimal perturbations of a planetary wave on the beta plane to arbitrary (and not just infinitesimal) perturbations of any RH wave from $\mathbf{H}_1 \oplus \mathbf{H}_n$ on the sphere ($n \geq 2$).

Theorem 1 notes an important property of the dynamics of the RH wave perturbations. Indeed, despite the fact that, according to Lemma 1, the $\mathbb{H}^2$-norm (the enstrophy norm) is stronger than the $\mathbb{H}^1$-norm (the energy norm), Theorem 1 states that the RH wave (5.1) has the same stability properties in both norms.

The average spectral number

$$\chi \equiv \chi(\psi') = \eta \,/K = \|\psi'\|_2^2 \,/\, \|\psi'\|_1^2 \tag{5.12}$$

of the perturbation ψ' depends on time, but, by (5.9),

$$U[\psi'] \equiv \eta - \chi_n K = \{\chi - \chi_n\}\, K = const \tag{5.13}$$

i.e. the functional $U[\psi']$ does not change with time for an arbitrary (and not just infinitesimal) perturbation ψ' of the RH wave ($n \geq 1$):

$$\frac{d}{dt}U[\psi'] = \frac{d}{dt}\{[\chi(t) - \chi_n]K(t)\} = 0 \tag{5.14}$$

Using (5.10), the conservation law $U[\psi'] = L_0 = const$ can be written as

$$\sum_{k=n+1}^{\infty} \chi_k(\chi_k - \chi_n)\left\|\mathrm{Y}_k\left(\psi'\right)\right\|^2 = \sum_{k=1}^{n-1} \chi_k(\chi_n - \chi_k)\left\|\mathrm{Y}_k\left(\psi'\right)\right\|^2 + L_0 \tag{5.15}$$

where

$$L_0 = 2U[\varphi] = \sum_{k=n+1}^{\infty} \chi_k(\chi_k - \chi_n)\left\|\mathrm{Y}_k(\varphi)\right\|^2 - \sum_{k=1}^{n-1} \chi_k(\chi_n - \chi_k)\left\|\mathrm{Y}_k(\varphi)\right\|^2 \tag{5.16}$$

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is a constant determined by the initial value (5.5) of the perturbation ψ'.

In particular, laws (5.9) and (5.15) are valid for arbitrary perturbations of the flow in the form of the Legendre polynomial $aP_n(\mu)$ and the zonal RH wave $-\omega\mu + aP_n(\mu)$. Note that for infinitesimal perturbations of a harmonic wave on the β-plane, law (5.15) was introduced by Petroni et al. [24, 25] (see also [29]).

Conservation of angular momentum $\langle \Delta\psi, \mu \rangle$ for any solution $\psi(t,x)$ of equation (4.1) (see (4.3)) gives another conservation law for arbitrary perturbations of the RH wave:

$$\langle \Delta\psi', \mu \rangle = -2 \langle \psi', \mu \rangle = Const \tag{5.17}$$

Thus (see also Lemma 3), the projection $\mathrm{Y}_1(\psi')$ of any perturbation ψ' onto the subspace $\mathbf{H}_1$ is also invariant, and hence

$$\|\mathrm{Y}_1(\psi')\|_1 = const \qquad \text{and} \qquad \|\mathrm{Y}_1(\psi')\|_2 = const \tag{5.18}$$

6 Factor space of RH wave perturbations

According to (5.14), all perturbations of the RH-wave (5.1) from $\mathbf{H}_1 \oplus \mathbf{H}_n$ are divided into the three invariant (disjoint) sets:

$$\begin{aligned} \mathbf{M}_+^n &= \{\psi' : \chi(\psi'(t)) > \chi_n \text{ and } U[\psi'] > 0\} \\ \mathbf{M}_0^n &= \{\psi' : \chi(\psi'(t)) = \chi_n \text{ and } U[\psi'] = 0\} \\ \mathbf{M}_-^n &= \{\psi' : \chi(\psi'(t)) < \chi_n \text{ and } U[\psi'] < 0\} \end{aligned} \tag{6.1}$$

where $\chi_n = n(n+1)$, $n \geq 1$. Thus, if a perturbation at some point in time belongs to one of the sets (6.1), it will always belong to this set. Note that the average spectral number (5.12) is equal to χ_n for any perturbation from $\mathbf{M}_0^n$ and is bounded for all perturbations from $\mathbf{M}_-^n$. The set $\mathbf{M}_0^n$ separates the sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$ of perturbations of larger and smaller scales compared to the scale characterized by the

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number n. This allows us to independently analyze the dynamics of RH wave perturbations in each invariant set (6.1) [38].

It should be emphasized that the set $\mathbf{M}_0^n$ contains the subspace $\mathbf{H}_n$ of homogeneous spherical polynomials of degree n, which is another invariant set of perturbations of RH-waves. In this case, by (5.6), any perturbation from $\mathbf{H}_n$ develops according to the linear equation

$$\psi'_t + C_n \psi'_\lambda = 0 \tag{6.2}$$

that preserves its form, and hence its energy and enstrophy. This leads to the following statement:

Theorem 2. *All perturbations of the RH wave* (5.1) *are divided into invariant (disjoint) sets* $\mathbf{M}_-^n$, $\mathbf{M}_+^n$, $\mathbf{M}_0^n$, *besides the linear subspace* $\mathbf{H}_n \subset \mathbf{M}_0^n$ *is one more invariant set of neutral perturbations.*

By virtue of Theorem 2, we can introduce a factor space $\mathbb{H}^2/\mathbf{H}_n$ of classes of perturbations of the RH wave (5.1), in which the 0-class is $\mathbf{H}_n$, and two elements g and h belong to the same class only when $g-h \in \mathbf{H}_n$. As a result, the entire space of perturbations is divided into disjoint classes. For any perturbation ψ', the corresponding class is $\{\psi' + h : \ h \in \mathbf{H}_n\}$. The representative element of the class is the element $\psi'_\perp$, which has a zero projection onto $\mathbf{H}_n : \mathrm{Y}_n(\psi'_\perp) = 0$ (see (2.7)). The functional

$$\|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\| \tag{6.3}$$

is a seminorm in the space $\mathbb{H}^2$, but a norm in the factor space $\mathbb{H}^2/\mathbf{H}_n$. Moreover, by (6.1), all classes of perturbations in $\mathbb{H}^2/\mathbf{H}_n$ are divided into three invariant disjoint sets which we will again call $\mathbf{M}_-^n$, $\mathbf{M}_+^n$ and $\mathbf{M}_0^n$.

Note that

$$\|\Delta\psi' + \chi_n\psi'\|^2 + \chi_n \|\nabla\psi'\|^2 = 2U[\psi'] + \chi_n^2 \|\psi'\|^2$$

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where $U[\psi']$ is defined by formula (5.13), and therefore $\|\Delta\psi' + \chi_n\psi'\|^2 + \chi_n \|\nabla\psi'\|^2 = \chi_n^2 \|\psi'\|^2$ in the invariant set $\mathbf{M}_0^n$ of perturbations of the RH wave (5.1), and

$$\frac{d}{dt}(\|\Delta\psi' + \chi_n\psi'\|^2 + \chi_n \|\nabla\psi'\|^2) = \chi_n^2 \frac{d}{dt} \|\psi'\|^2$$

for any perturbation of this wave.

Theorem 3. *Let $\mathrm{Y}_n(\psi')$ be the projection of the perturbation ψ' onto the subspace $\mathbf{H}_n$. Then*

$$\|\Delta\psi'\|^2 = \|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\|^2 + \chi_n \|\mathrm{Y}_n(\psi')\|_1^2 \qquad (6.4)$$

for any perturbation of the RH wave (5.1).

Indeed,

$$\|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\|^2 = \langle \Delta\psi' + \chi_n \mathrm{Y}_n(\psi') \, , \, \Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\rangle$$
$$= \|\Delta\psi'\|^2 + 2\chi_n \, \langle \Delta\psi', \mathrm{Y}_n(\psi')\rangle + \chi_n^2 \|\mathrm{Y}_n(\psi')\|^2$$
$$= \|\Delta\psi'\|^2 - \chi_n^2 \, \|\mathrm{Y}_n(\psi')\|^2 = \|\Delta\psi'\|^2 - \chi_n \, \|\mathrm{Y}_n(\psi')\|_1^2$$

By virtue of (6.4), the enstrophy norm $\|\Delta\psi'\|$ is a combination of the factor norm $\|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\|$ and the energy norm $\|\mathrm{Y}_n(\psi')\|_1 = \chi_n \|\mathrm{Y}_n(\psi')\|$. We will show later (Theorem 7) that the enstrophy norm is suitable for studying the stability of RH waves in the $\mathbb{H}^2$ space, since the energy norm $\|\mathrm{Y}_n(\psi')\|_1$ controls the evolution of the perturbation ψ' in the subspace $\mathbf{H}_n$, and the factor norm $\|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\|$ controls the evolution of its orthogonal part $\psi'_\perp$ (i.e., the representative element of the factor class $\{\psi' + h : \ h \in \mathbf{H}_n\}$).

7 Hyperbolic law of evolution of perturbations in sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$

Let us consider the behavior of perturbations of the RH wave (5.1) in the invariant sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$. Using the notation

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$$\rho(t) = \chi(t) - \chi_n \tag{7.1}$$

law (5.14) can be rewritten as

$$\rho(t)K(t) = L_0 \tag{7.2}$$

where $L_0 = \rho(0)K(0)$ is the constant (5.16) determined by the values $\chi(0)$ and $K(0)$ of the perturbation at the initial moment of time (L_0 is negative in $\mathbf{M}_-^n$ and positive in $\mathbf{M}_+^n$). On the plane (ρ, K), a point $(\rho(t), K(t))$ moving along some hyperbola (7.2) represents the current state of the kinetic energy $K(t)$ and the average spectral number $\chi(t)$ of the perturbation.

Theorem 4. *Let $\mathbf{K}_\psi$ be the energy of the RH wave* (5.1). *The hyperbolic relation* (7.2) *additionally splits the RH wave perturbations from the invariant sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$ into a family of invariant subsets depending on the parameter L_0. Let F_α be a set of those solutions $\widetilde{\psi}$ of equation* (4.1) *that belong to the energy surface $\mathbf{K}_{\widetilde{\psi}} = \alpha^2\mathbf{K}_\psi$, $\alpha > 0$, and $\mathbf{K}_a = \mathbf{K}_\psi + \mathbf{K}_{\widetilde{\psi}} = (1+\alpha^2)\mathbf{K}_\psi$. Then the kinetic energy $K(t)$ of the perturbations $\psi'(t,x) = \widetilde{\psi}(t,x) - \psi(t,x)$, generated by the solutions from the set F_α, can vary only within the following limits:*

$$K_{\min} \equiv (\alpha - 1)^2\mathbf{K}_\psi \leq K(t) \leq (\alpha + 1)^2\mathbf{K}_\psi \equiv K_{\max} \tag{7.3}$$

Inequalities (7.3) follow directly from (5.10). Thus, the changes in the perturbation energy are bounded by α and $\mathbf{K}_\psi$:

$$\Delta K = K_{\max} - K_{\min} \leq 4\alpha\mathbf{K}_\psi$$

They increase with increasing both α and $\mathbf{K}_\psi$ and are small in at least two cases: when $\alpha \ll 1$ (and, therefore, $\mathbf{K}_{\widetilde{\psi}} \ll 1$), and then $\mathbf{K}_\psi \ll 1$. Both $K(t)$ and ΔK decrease with decreasing energy

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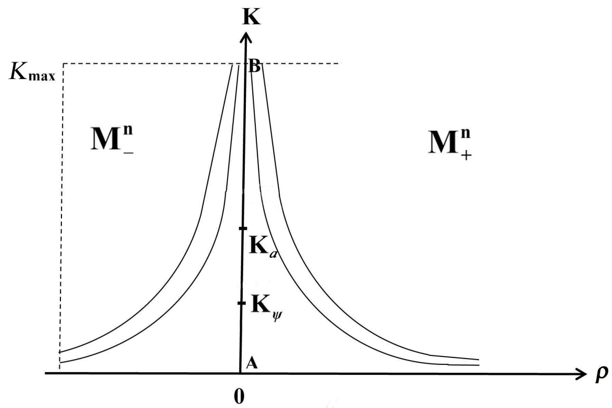

Figure 1: A hyperbolic relation between $K(t)$ and $\rho(t)$ for perturbations of RH wave ($\alpha = 1$, $K_{\min} = 0$)

$\mathbf{K}_\psi$ of the RH wave (5.1). In the limit when $\mathbf{K}_\psi = 0$ (i.e., $\psi = 0$, and $\psi'(t, x) \equiv \widetilde{\psi}(t, x)$) the zero solution is stable, since by virtue of (4.4) and (4.5) the energy K and enstrophy η of perturbations are constant.

Note that according to (7.3), the absolute minimum of $K(t)$ is achieved at $\alpha = 1$ (when $\widetilde{\psi}$ belongs to the same energy level as ψ, and the line $K = K_{\min} = 0$ coincides with the ρ-axis). This case is shown in Fig.1.

Note also that the point $(\rho(t), K(t))$ describing the evolution of the perturbation can move along the corresponding hyperbola (7.2) only within the limits (7.3). If $\alpha \neq 1$ then $\mathbf{K}_{\min} \neq 0$ and

$$|\chi(t) - \chi_n| \leq |L_0| \; / \left[(\alpha - 1)^2 \mathbf{K}_\psi\right] \tag{7.4}$$

that is, the average spectral number of each perturbation is also limited in its changes.

The interval $[A, B]$ is the locus for perturbations from the invariant set $\mathbf{M}_0^n$ ($\rho = 0$ or $\chi = \chi_n$), and any perturbation from $\mathbf{H}_n$ is represented by a fixed point of the interval $[A, B]$.

From (5.12) and (5.9) it follows that

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$$\frac{d}{dt}\chi(t) = \frac{1}{K(t)} \{\chi_n - \chi(t)\} \frac{d}{dt}K(t) \tag{7.5}$$

that is, the average spectral number $\chi(\psi')$ of a growing (or decaying) perturbation grows (decays) in $\mathbf{M}_-^n$ and decays (grows) in $\mathbf{M}_+^n$. Thus, the following statement is true:

Theorem 5. *The energy cascade of growing (or decaying) perturbations of the RH wave* (5.1) *has opposite directions in the invariant sets* $\mathbf{M}_-^n$ *and* $\mathbf{M}_+^n$.

By virtue of (7.1) and (7.2), the perturbation energy increases in both $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$ as the average spectral number $\chi(t)$ approaches χ_n ($\rho(t) \to 0$). Later we will see (Theorem 9) that the average spectral number of linearly unstable disturbances of the RH wave (5.1) is equal to χ_n.

8 Geometric explanation of changes in perturbation energy

In works devoted to the stability of ideal flows, the phrase that "the perturbation draws its energy from the basic flow or wave" is often encountered. In my opinion, this phrase is not entirely correct. After all, by virtue of (4.4), the energy of any solution of equation (4.1) is constant. Consequently, the perturbation cannot "draw energy from the basic flow." Let us now give a geometric interpretation of the changes in the kinetic energy of perturbations of the RH wave (5.1). According to (5.5), any perturbation $\psi'(t,x)$ of this wave is equal to $\psi'(t,x) = \widetilde{\psi}(t,x) - \psi(t,x)$, where $\widetilde{\psi}(t,x)$ is another solution of equation (4.1). By (5.10), the energy $K(t)$ of this perturbation is represented as

$$K = \frac{1}{2}\|\psi'\|_1^2 = \frac{1}{2}\left\|\widetilde{\psi} - \psi\right\|_1^2 = \mathbf{K}_{\widetilde{\psi}} + \mathbf{K}_{\psi} - \left\langle \widetilde{\psi}, \psi \right\rangle_1 \tag{8.1}$$

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where $\langle\cdot,\cdot\rangle_1$ is the inner product (2.11) in the Hilbert space $\mathbb{H}^1$. According to (4.4), $\mathbf{K}_{\widetilde{\psi}} + \mathbf{K}_{\psi} = \mathbf{K}_a = Const$, and the perturbation energy changes only when the inner product $\left\langle \widetilde{\psi}, \psi \right\rangle_1$ (i.e., the projection of $\widetilde{\psi}$ onto ψ) changes. In other words, the perturbation energy changes in time only due to the change in the relative spatial position of the solutions $\widetilde{\psi}$ and ψ:

$$\frac{d}{dt}K = -\frac{d}{dt}\left\langle \widetilde{\psi}, \psi \right\rangle_1 = -\frac{d}{dt}\left\langle \psi', \psi \right\rangle_1 = \frac{d}{dt}\left\langle \psi', \Delta\psi \right\rangle \tag{8.2}$$

Using the waveform (5.1) and the motion invariant (5.17), equation (8.2) can be written as

$$\begin{aligned}\frac{d}{dt}K = \frac{d}{dt}\left\langle \psi', \Delta\psi \right\rangle &= \frac{d}{dt}\left\langle \psi', \Delta \mathrm{Y}_n(\psi) \right\rangle \\ &= -\chi_n \frac{d}{dt}\left\langle \psi', \mathrm{Y}_n(\psi) \right\rangle \end{aligned}\tag{8.3}$$

where $\mathrm{Y}_n(\psi) \in \mathbf{H}_n$ is the polynomial part of the wave (5.1). Also, using the first equality (8.2), the inner product (2.10) and the conservation law (5.17) we obtain

$$\begin{aligned}\frac{d}{dt}K &= -\frac{d}{dt}\left\langle \widetilde{\psi}, \psi \right\rangle_1 = -\frac{d}{dt}\left\langle \widetilde{\psi}, -\omega\mu + \mathrm{Y}_n(\psi) \right\rangle_1 \\ &= -\frac{d}{dt}\left\langle \widetilde{\psi}, \mathrm{Y}_n(\psi) \right\rangle_1 = -\chi_n \frac{d}{dt}\left\langle \widetilde{\psi}, \mathrm{Y}_n(\psi) \right\rangle \end{aligned}\tag{8.4}$$

On the other hand, by (5.9),

$$\frac{d}{dt}\eta = -\chi_n^2 \frac{d}{dt}\left\langle \psi', \mathrm{Y}_n(\psi) \right\rangle = -\chi_n^2 \frac{d}{dt}\left\langle \widetilde{\psi}, \mathrm{Y}_n(\psi) \right\rangle \tag{8.5}$$

Thus, the following statement is true:

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Theorem 6. *The energy and enstrophy of the perturbation ψ' of the RH wave (5.1) change only when the projection of the perturbation ψ' (or the projection of the solution $\widetilde{\psi} = \psi + \psi'$) onto $\mathrm{Y}_n(\psi)$ changes.*

For example, if $\widetilde{\psi}(t, x)$ is an RH wave from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_k$, where $k > n$ or $k < n$, then $\left\langle \widetilde{\psi}, \mathrm{Y}_n(\psi) \right\rangle = 0$ at any moment of time, and according to (8.4) and (8.5), $\frac{d}{dt}K = 0$ and $\frac{d}{dt}\eta = 0$.

The following two statements were proved earlier:

Lemma 4. *[34]. The superrotation component $-\omega\mu$ of the RH wave (5.1) does not affect its stability either in the energy norm $\|\psi'\|_1 = \|\nabla\psi'\| = \sqrt{2K}$ or in the enstrophy norm $\|\psi'\|_2 = \|\Delta\psi'\| = \sqrt{2\eta}$, i.e. in both norms the RH wave has the same stability properties as its polynomial part $\mathrm{Y}_n(\psi) \in \mathbf{H}_n$.*

Lemma 5. *[33]. Let $n = 1$. Then the RH wave (5.1), belonging to the subspace $\mathbf{H}_1$, is stable to any perturbations in the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, since $K = Const$ and $\eta = Const$.*

By Theorem 6, the energy $K(t)$ of a perturbation ψ' is conserved if ψ' is orthogonal to $\mathrm{Y}_n(\psi)$, and hence if ψ' is orthogonal to $\mathbf{H}_n$. Thus, the energy norm $\|\mathrm{Y}_n(\psi')\|_1$ in (6.4) is insensitive to variations of ψ' in the subspace orthogonal to $\mathbf{H}_n$. In contrast, the factor norm $\|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\|$ in (6.4) is insensitive to variations of ψ' in $\mathbf{H}_n$, but changes with variations of ψ' in the subspace orthogonal to $\mathbf{H}_n$. Since any perturbation of the RH wave (5.1) can be written in the form $\psi' = \psi'_\perp + \mathrm{Y}_n(\psi')$, the following statement is true:

Theorem 7. *Let $\psi' = \psi'_\perp + \mathrm{Y}_n(\psi')$ be the perturbation of the RH wave (5.1). Then the energy norm $\|\mathrm{Y}_n(\psi')\|_1$ and the factor norm $\|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\|$, which constitute the enstrophy norm according to (6.4), control the evolution of the orthogonal components $\mathrm{Y}_n(\psi')$ and $\psi'_\perp$ of the perturbation ψ', respectively.*

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9 Examples of the dynamics of RH wave perturbations

Let us now consider several examples demonstrating the importance of Lemma 4 and the conservation law (5.15). In certain situations, this law can provide useful information about the distribution of the perturbation energy over subspaces $\mathbf{H}_n$ (over the spectrum of scales).

Example 1. *Let n, $n \geq 2$. Consider the Legendre polynomial*

$$\psi(\mu) = \psi_n^0 P_n(\mu) \equiv \psi_n^0 P_n^0(\mu) \tag{9.1}$$

(see (2.3)) or the zonal RH wave

$$\psi(\mu) = -\omega\mu + \psi_n^0 P_n(\mu) \tag{9.2}$$

as the basic solution $\psi(\mu)$ and the RH wave (5.1) as another solution $\widetilde{\psi}(t, \lambda, \mu)$ of the vorticity equation (4.1). Then the perturbation $\psi'(t, \lambda, \mu) = \widetilde{\psi}(t, \lambda, \mu) - \psi(\mu)$ takes the form

$$\psi'(t, \lambda, \mu) = -\omega\mu + \sum_{0<|m|\leq n} \psi_n^m Y_n^m(\lambda - C_n t, \mu) \tag{9.3}$$

for the flow (9.1), or the form

$$\psi'(t, \lambda, \mu) = \sum_{0<|m|\leq n} \psi_n^m Y_n^m(\lambda - C_n t, \mu) \tag{9.4}$$

for the flow (9.2). Since (9.3) and (9.4) are themselves RH waves and solutions of equation (4.1) from the subspaces $\mathbf{H}_n$ and $\mathbf{H}_1 \oplus \mathbf{H}_n$, respectively, the energy of these perturbations does not change with time. This is completely consistent with Theorem 6, since perturbations (9.3) and (9.4) are orthogonal to $\mathrm{Y}_n(\psi) = \psi_n^0 P_n(\mu)$.

Of course, we get the same result if we take the wave

$$\psi(t, \lambda, \mu) = -\omega\mu + \sum\nolimits_{m=-k}^{k} \psi_n^m Y_n^m(\lambda - C_n t, \mu)$$

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as the basic solution $(1 \leq k < n)$ *and the RH wave* (5.1) *as the solution* $\widetilde{\psi}(t, \lambda, \mu)$.

Example 2. *Consider the RH wave*

$$\psi\left(t, \lambda, \mu\right) = -\omega\mu + \sum\nolimits_{m=-2}^{m=2} \psi_2^m Y_2^m \left(\lambda - C_2 t, \mu\right) \qquad (9.5)$$

from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_2$. *This wave is an exact solution of the vorticity equation* (4.1) *provided that* $C_2 = \omega - \left(\omega + 1\right)/3$ *(see* (5.1) *and* (5.2)*).*

We now show that the growth of perturbations of wave (9.5) *is possible only in the subspace* $\mathbf{H}_2$. *Indeed,* (5.15) *takes the following form:*

$$\sum\nolimits_{k=3}^{\infty} \chi_k(\chi_k - \chi_2) \left\|\mathrm{Y}_k(\psi')\right\|^2$$

$$= \chi_1(\chi_2 - \chi_1) \left\|\mathrm{Y}_1(\psi')\right\|^2 + L_0 \qquad (9.6)$$

Since, according to (5.18), *the norm* $\left\|\mathrm{Y}_1(\psi')\right\|$ *does not change over time, it follows from (9.6) that*

$$\sum\nolimits_{k=3}^{\infty} \chi_k(\chi_k - \chi_2) \left\|\mathrm{Y}_k(\psi')\right\|^2 = Const \qquad (9.7)$$

Therefore,

$$\frac{d}{dt}\left\{\sum\nolimits_{k=3}^{\infty} \chi_k(\chi_k - \chi_2) \left\|\mathrm{Y}_k(\psi')\right\|^2 + \chi_1^2 \left\|\mathrm{Y}_1(\psi')\right\|^2\right\} = 0 \qquad (9.8)$$

Let's define

$$W\left[\psi'\right] \equiv \sum\nolimits_{k=3}^{\infty} \chi_k(\chi_k - \chi_2) \left\|\mathrm{Y}_k(\psi')\right\|^2 + \chi_2^2 \left\|\mathrm{Y}_2(\psi')\right\|^2 + \chi_1^2 \left\|\mathrm{Y}_1(\psi')\right\|^2$$

and denote $V\left[\psi'\right] = \sqrt{W\left[\psi'\right]}$. *With*

$$\eta = \frac{1}{2}\left\|\psi'\right\|_2^2 = \frac{1}{2}\sum\nolimits_{k=1}^{\infty} \chi_k^2 \left\|\mathrm{Y}_k(\psi')\right\|^2$$

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for the perturbation enstrophy, we obtain $W[\psi'] \leq 2\eta$. *On the other side,* $\chi_k(\chi_k - \chi_2) = \chi_k^2(1-\chi_2/\chi_k) \geq \frac{1}{2}\chi_k^2$ *for* $k \geq 3$, *and hence* $W[\psi'] \geq \eta$, *i.e.* $\sqrt{\eta} \leq V[\psi'] \leq \sqrt{2\eta}$, *or* $\frac{1}{\sqrt{2}}\|\psi'\|_2 \leq V[\psi'] \leq \|\psi'\|_2$

Thus, the positive functional $V[\psi'(t)]$ *is equivalent to the enstrophy norm* $\|\psi'\|_2$. *From* (9.8) *it follows that*

$$\frac{d}{dt}W[\psi'] \equiv \chi_2^2 \frac{d}{dt}\|\mathrm{Y}_2(\psi')\|^2 \tag{9.9}$$

i.e., any change in the enstrophy norm $\|\psi'\|_2$ *and the energy norm* $\|\psi'\|_1$ *(Theorem 1) of the perturbation* ψ' *is possible only through a change in the norm* $\|\mathrm{Y}_2(\psi')\|$, *where* $\mathrm{Y}_2(\psi')$ *is the projection of* ψ' *onto the subspace* $\mathbf{H}_2$ *of homogeneous spherical polynomials of the second degree (see also [4, 34]). A necessary condition for Lyapunov instability is* $\frac{d}{dt}\|\mathrm{Y}_2(\psi')\| > 0$ *for some* $\psi'(t)$.

Example 3. *Let us consider the RH wave* (9.5) *once again. If at the initial moment* $\chi_1(\chi_2-\chi_1)\|\mathrm{Y}_1(\varphi)\|^2 + L_0 = const \ll 1$ *then from (9.6) it follows that*

$$\sum_{k=3}^{\infty}\chi_k(\chi_k-\chi_2)\|\mathrm{Y}_k(\psi')\|^2 = \chi_1(\chi_2-\chi_1)\|\mathrm{Y}_1(\psi')\|^2 + L_0 = Const \ll 1$$

at any moment in the future.

Example 4. *Consider the RH wave* (5.1). *If at the initial moment* $L_0 = 0$ *(i.e. the perturbation* $\psi'(0) = \varphi$ *belongs to the invariant set* $\mathbf{M}_0^n$*) and* $\|\mathrm{Y}_k(\varphi)\| = 0$ *for* $k = 1, 2, ..., n-1$, *then, according to* (5.16), $\|\mathrm{Y}_k(\varphi)\| = 0$ *for all* $k \geq n+1$, *and the perturbation* $\psi'(t)$ *belongs to the invariant set* $\mathbf{H}_n$ *for all* $t \geq 0$ *and, therefore, is neutral.*

Example 5. *Consider the RH wave* (5.1). *If at the initial moment* $L_0 = 0$ *(i.e. the perturbation* $\psi'(0) = \varphi$ *belongs to the set* $\mathbf{M}_0^n$*) and* $\|\mathrm{Y}_k(\varphi)\| = 0$ *for all* $k \geq n+1$, *then, according to* (5.16), $\|\mathrm{Y}_k(\varphi)\| = 0$ *for* $k = 1, 2, ..., n-1$, *and the perturbation* $\psi'(t)$ *belongs to the invariant set* $\mathbf{H}_n$ *for all* $t \geq 0$ *and is neutral.*

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10 Lyapunov instability of RH wave to perturbations from $\mathbf{M}_{-}^{n}$

According to Lemma 5, any RH wave (5.1) from the subspace $\mathbf{H}_1$ is Lyapunov stable in the energy norm $\|\cdot\|_1$ and the enstrophy norm $\|\cdot\|_2$. The instability of any nonstationary non-zonal RH wave (5.1) for $n \geq 2$ was proved in [34] by constructing a perturbation that does not satisfy the definition of Lyapunov stability. However, we will again give a brief proof here, and in Section 11, we will show that the mechanism of this instability has nothing to do with the orbital (Poincaré) instability. We call the RH wave (5.1) non-zonal if $\psi_n^m \neq 0$ for at least one m ($1 \leq |m| \leq n$).

Theorem 8. *[34]. Let $n \geq 2$. Then any non-zonal non-stationary (periodic) RH wave* (5.1) *from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ is Lyapunov unstable in the enstrophy norm $\|\cdot\|_2$ with respect to perturbations from the invariant set $\mathbf{M}_{-}^{n}$.*

Proof. By Lemma 4, without loss of generality, as the basic solution of equation (4.1) we take the RH wave

$$\psi(t,\lambda,\mu) = \sum_{m=-n}^{n} \psi_n^m Y_n^m(\lambda - C_n t, \mu) \tag{10.1}$$

from the subspace $\mathbf{H}_n$ where $\omega = 0$, $C_n = -2/\chi_n$ and $\chi_n = n(n+1)$. Thus, $\psi \equiv \mathrm{Y}_n(\psi)$.

At the end of Section 7, it was noted that the average spectral number of the unstable perturbation is expected to be close to χ_n. We will also show later (Theorem 9) that the amplitude of the unstable modes of the stationary RH wave (5.1) belongs to the set $\mathbf{M}_0^n$. Finally, according to Theorem 6, an increase in the energy of the perturbation ψ' of the RH wave (5.1) is possible only with a decrease in the projection of the perturbation ψ' on $\mathrm{Y}_n(\psi)$ (or the projection of the solution $\widetilde{\psi} = \psi + \psi'$ on $\mathrm{Y}_n(\psi)$) (see (8.3) and (8.4)). Therefore, as another solution of equation (4.1) we take the periodic RH wave

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$$\widetilde{\psi}(t,x) = -\frac{\delta}{2} Y_1^0(\mu) + \sum_{m=-n}^{n} \psi_n^m Y_n^m(\lambda - \widetilde{C}_n t, \mu) \tag{10.2}$$

from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ where $\delta > 0$ is arbitrary small, $Y_1^0(\mu) = a\mu$, $a = \sqrt{3/(4\pi)}$ and, according to (5.2), $\widetilde{C}_n = \frac{\delta}{2}a - (\delta a + 2)/\chi_n$. The waves (10.1) and (10.2) have the same coefficients ψ_n^m and slightly different velocities C_n and $\widetilde{C}_n$. The perturbation $\psi' = \widetilde{\psi} - \psi$ belongs to $\mathbf{H}_1 \oplus \mathbf{H}_n$ and has the form

$$\psi'(t,x) = -\frac{\delta}{2} Y_1^0(\mu)$$

$$+ \sum_{m=-n}^{n} \psi_n^m Y_n^m(\lambda, \mu) \exp(-im\widetilde{C}_n t) \; [1 - \exp\{im(\widetilde{C}_n - C_n)t\}] \tag{10.3}$$

It belongs to the invariant set $\mathbf{M}_-^n$ (see (6.1)), since its first constant term belongs to the subspace $\mathbf{H}_1$, and the second to the subspace $\mathbf{H}_n$. Incidentally, perturbation (10.3) shows that a nonzero projection $\mathrm{Y}_n(\psi')$ of the perturbation onto $\mathbf{H}_n$ can arise even if it is zero at the initial time $t = 0$. We see that the projection $\mathrm{Y}_n(\psi')$ of the perturbation (10.3) onto $\mathbf{H}_n$ is the only one that changes with time. Therefore, according to Theorems 6 and 7, the factor norm

$$\|\Delta\psi' + \chi_n \mathrm{Y}_n(\psi')\| = \left\| \frac{\delta}{2} Y_1^0(\mu) \right\| = \frac{\delta}{2} = Const$$

of perturbation (10.3) does not change with time, and, according to (6.4), the evolution of perturbation (10.3) is estimated only by the energy norm $\|\psi'\|_1 = \sqrt{2K}$.

The kinetic energy $K(t)$ of this perturbation is equal to

$$K(t) = \frac{\delta^2}{4} + \frac{\chi_n}{2} \sum_{m=-n}^{n} |\psi_n^m|^2 \, [1 - \exp\{im(\widetilde{C}_n - C_n)t\}]^2 \tag{10.4}$$

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where

$$\widetilde{C}_n - C_n = \frac{\delta a}{2\chi_n}(\chi_n - 2) \tag{10.5}$$

Since (10.1) is a non-zonal wave, there exists k $(1 \leq k \leq n)$ such that $\psi_n^k \neq 0$. According to the definition of Lyapunov stability, the number ε is taken to be

$$\varepsilon = 2\sqrt{\chi_n}\left|\psi_n^k\right| > 0 \tag{10.6}$$

At the initial time $t = 0$, the energy norm of the perturbation (10.3) is equal to

$$\|\psi'(0)\|_1 = \sqrt{2K(0)} = \delta/\sqrt{2} < \delta$$

As already mentioned, the number $\delta > 0$ in (10.2) can be chosen arbitrarily small. However, we will now show that, no matter how small δ is, there is no moment $t_1 > 0$ such that $\|\psi'(t)\|_1 < \epsilon$ for all $t > t_1$. This means that the wave (10.1) is unstable in the sense of Lyapunov 's definition.

Indeed, it follows from (10.5) that $\widetilde{C}_n - C_n \neq 0$ since $\chi_n \geq 6$ for $n \geq 2$. Moreover, the RH waves are real, and hence $\psi_n^{-k} = (-1)^k\overline{\psi_n^k}$, that is, $\left|\psi_n^{-k}\right| = \left|\psi_n^k\right|$.Despite the fact that $\|\psi'(0)\|_1 < \delta$ at the initial moment $t = 0$, we obtain that $K(\tau_j) \geq \delta^2/4 + 4\chi_n\left|\psi_n^k\right|^2 = \delta^2/4 + \varepsilon^2$ whenever

$$\tau_j = \frac{\pi(2j+1)}{k\ (\widetilde{C}_n - C_n)} \tag{10.7}$$

and, therefore, $\|\psi'(\tau_j)\|_1 = \{2K(\tau_j)\}^{1/2} > \varepsilon$ for any moment $t = \tau_j$ $(j = 0, 1, 2, \dots)$. Since $\tau_j \to \infty$ as $j \to \infty$, it is impossible to find a time $t_1 \geq 0$ such that $\|\psi'(t)\|_1 < \varepsilon$ for all $t \geq t_1$. Thus, strictly speaking, wave (10.1) (and, consequently, any nonstationary non-zonal RH wave (5.1)) is unstable according to Lyapunov's definition.

Based on Theorem 2, the stability of the RH wave (5.1) can be analyzed independently in each of the invariant sets $\mathbf{M}_-^n$, $\mathbf{M}_+^n$

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and $\mathbf{M}_0^n$. Since perturbation (10.3) belongs to $\mathbf{M}_-^n$, Theorem 8 demonstrates the Lyapunov instability of nonstationary non-zonal RH waves to perturbations from the invariant set $\mathbf{M}_-^n$. □

11 Mechanism of instability of RH wave

It should be noted that the mechanism of Lyapunov instability of nonstationary non-zonal RH waves described in Section 10 has nothing in common with the orbital (Poincaré) instability [18]. Indeed, if we consider a thin tube (of small radius δ) around the orbit of the basic RH wave (10.1) in the phase space of solutions, then the orbit of the solution (10.2) will always be inside this tube, that is, for any moment of time t_1 there exists a moment of time t_2 such that $\left\|\widetilde{\psi}(t_1,x)-\psi(t_2,x)\right\| < \delta$. Indeed,

$$\begin{aligned}
&\widetilde{\psi}(t_1,x)-\psi(t_2,x)\\
&=-\frac{\delta}{2}Y_1^0(\mu)+\sum_{m=-n}^{n}\psi_n^m Y_n^m(\lambda,\mu)[\exp(-im\widetilde{C}_n t_1)-\exp(-imC_n t_2)]\\
&=-\frac{\delta}{2}Y_1^0(\mu)\\
&\quad+\sum_{m=-n}^{n}\psi_n^m Y_n^m(\lambda,\mu)\exp(-imC_n t_2)[\exp\{im(C_n t_2-\widetilde{C}_n t_1)\}-1]
\end{aligned}$$

Thus, whenever $C_n t_2 = \widetilde{C}_n t_1$, or $t_1 = (C_n/\widetilde{C}_n)\,t_2$, the distance $\left\|\widetilde{\psi}(t_1,x)-\psi(t_2,x)\right\|$ between the periodic solutions (10.1) and (10.2) is equal to $\delta/2$.

In fact, the instability mechanism is due to the difference in the velocities of the waves (10.1) and (10.2) ($\widetilde{C}_n \neq C_n$). This is reminiscent of the instability mechanism of periodic solutions of the equation of a nonlinear pendulum near the center (equilibrium point)

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during small-amplitude oscillations without bifurcations (without the destruction of periodic orbits, or transitions to chaotic behavior) [1] (Fig.2).

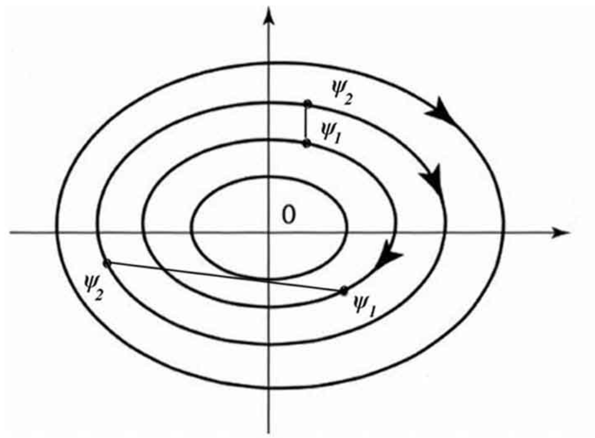

Figure 2: Due to different speeds of motion in the phase space, the distance between the periodic solutions ψ_1 and ψ_2 changes over time.

Note that if $n = 1$ then $\widetilde{C}_n = C_1$ by (10.5), and, consequently, $K(t) = Const$ in full accordance with Lemma 5. Note also that if $\delta = 0$ then, according to (10.5), $\widetilde{C}_n = C_n$, i.e., the perturbation (10.3) belongs to $\mathbf{H}_n$, and its norm is constant as it should be (see (6.2)). And finally, by (8.3) and Theorem 6, $K_t(t) = -\chi_n \langle \psi', \mathrm{Y}_n(\psi) \rangle_t$, i.e. the energy of perturbation ψ' changes only if its projection on the polynomial part $\mathrm{Y}_n(\psi)$ of the RH wave changes. This is exactly what happens with perturbation (10.3).

It is also important to note that the instability in Theorem 8 was proved for the periodic (nonstationary) wave (10.1). The situation changes if this wave is stationary (note that the stationary wave (10.1) has a non-zero projection onto $Y_1^0(\mu)$, but by Lemma 4 this additional term does not affect its stability). Indeed, since $C_n = 0$, the kinetic energy of the perturbation

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$$\psi'(t,x) = -\frac{\delta}{2}Y_1^0(\mu) + \sum_{m=-n}^{n} \psi_n^m Y_n^m(\lambda,\mu)\exp(-im\widetilde{C}_n t) \qquad (11.1)$$

is constant:

$$K = \frac{\delta^2}{4} + \frac{\chi_n}{2}\sum_{m=-n}^{n} |\psi_n^m|^2 = Const$$

This means that the behavior of perturbations (11.1) near the stationary RH wave (the zero point in Fig.2) is similar to the behavior of the trajectories of a nonlinear system near the center (equilibrium point) in phase space. Therefore, the choice of perturbation (10.3) does not allow us to answer the question of the instability of the stationary RH wave. Moreover, it is highly probable that the stationary wave (10.1) is Lyapunov stable. The results obtained here explain the fact that meteorologically important RH waves are regularly observed on Earth. Indeed, not every solution to the equations of motion, even if it is exact, can be realized in nature. RH waves observed in nature must not only satisfy the hydrodynamic equations, but also be stable.

12 Linear instability of stationary RH Wave

According to Lemma 5, every RH wave (5.1) from the subspace $\mathbf{H}_1$ is Lyapunov stable. Therefore, we present the main results obtained in [34] on the linear (exponential) stability of a real stationary RH wave

$$\psi(\lambda,\mu) = -\omega\mu + \sum_{m=-n}^{n} \psi_n^m Y_n^m(\lambda,\mu) \qquad (12.1)$$

from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ where $n \geq 2$, and $C_n = 0$. In addition, it is a stationary solution of the vorticity equation (4.1) if $\omega =$

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$2/(\chi_n-2)$. Considering that $\widetilde{\psi} = \psi+\psi'$, infinitesimal perturbations of this wave are described by the linearized equation (5.8):

$$\Delta\psi'_t + J(\psi, \Delta\psi' + \chi_n\psi') = 0 \tag{12.2}$$

An exponentially growing (or decaying) solution to equation (12.2) is sought in the form of a normal mode

$$\psi'(t, \lambda, \mu) = \Psi(\lambda, \mu) \exp\{\nu t\} \tag{12.3}$$

where $\nu = \nu_r + i\nu_i$, $i = \sqrt{-1}$, and the amplitude $\Psi(\lambda, \mu)$ is the solution of the spectral problem

$$J(\psi, \Delta\Psi + \chi_n\Psi) = -\nu\Delta\Psi \tag{12.4}$$

Theorem 9. *[34]. Let $n \geq 2$. The normal mode* (12.3) *of the wave* (12.1) *from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ may be unstable only under the condition*

$$\chi_\Psi = \chi_n = n(n+1) \tag{12.5}$$

where $\chi_\Psi = \eta_\Psi/K_\Psi$ is the average spectral number of the mode amplitude $\Psi(\lambda, \mu)$.

Theorem 9 states that an infinitesimal perturbation (12.3) can grow exponentially with time only if its amplitude $\Psi(\lambda, \mu)$ belongs to the subset $\mathbf{M}_0^n$ (however $\Psi(\lambda, \mu)$ cannot completely belong to $\mathbf{H}_n$, since such disturbances are neutral).

Thus, if the amplitude $\Psi(\lambda, \mu)$ of mode (12.3) belongs to the sets $\mathbf{M}_-^n$, $\mathbf{M}_+^n$ or $\mathbf{H}_n$ then the mode is neutral. Since $\mathbf{M}_0^n$ is the boundary between the sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$, the probability of the realization of condition (12.5) and exponential instability is very small in practice. Moreover, the instability of the RH wave (Theorem 8) was proved only for non-stationary non-zonal waves and is not orbital. These results also appear to be consistent with the fact that RH waves are regularly observed in daily meteorological observations.

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We note that in the case of zonal solutions (flows determined by Legendre polynomials and zonal RH waves), the instability condition (12.5) complements the well-known Rayleigh–Kuo and Fjörtoft conditions.

Theorem 9 has important practical applications in the numerical study of the linear stability of wave (12.1) using the spectral method and triangular truncation

$$\psi' = \sum_{n=0}^{N} \mathrm{Y}_n(\psi')$$

of the Fourier series (see (2.8)). Indeed, in this case the truncation number N of the Fourier series for disturbances must be greater than n, since according to Theorem 9 none of the normal modes can be unstable if $N \le n$.

Theorem 10. *[32, 34]. The growth rate of unstable modes of the RH wave* (12.1) *is limited:*

$$|\nu_r| \le \sqrt{n(n+1)} \max_S |\nabla\psi(x)| = \sqrt{n(n+1)} \max_S \left|\vec{U}\right| \qquad (12.6)$$

where $\left|\vec{U}\right| = \left|\vec{k} \times \nabla\psi\right|$ *is the wave velocity.*

Thus, the growth rate of unstable normal modes decreases in direct proportion to the maximum velocity and the degree n of the RH wave (12.1).

Theorem 11. *[32]. The amplitude* $\Psi(x)$ *of each unstable, decaying or non-stationary mode of the RH wave* (12.1) *is orthogonal to the polynomial* $\mathrm{Y}_n(\psi)$ *in the space* $\mathbb{H}^0$, *and to the wave* ψ *in the spaces* $\mathbb{H}^0$ *and* $\mathbb{H}^1$*:*

$$\langle \Psi, \mathrm{Y}_n(\psi) \rangle = 0\,, \qquad \langle \Psi, \psi \rangle = 0, \qquad \langle \Psi, \psi \rangle_1 = 0 \qquad (12.7)$$

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The last equation in (12.7) means that integrally (over the entire sphere), the velocity $\overrightarrow{u} = \overrightarrow{k} \times \nabla\Psi$ generated by the mode amplitude is orthogonal to the velocity $\overrightarrow{U} = \overrightarrow{k} \times \nabla\psi$ of the basic wave:

$$\int_S \overrightarrow{u} \cdot \overrightarrow{U} \, dS = 0 \tag{12.8}$$

The analytical results (12.5)-(12.8) are also useful for quality control of calculations in numerical studies of the linear stability of the RH wave.

13 Conclusions

In this paper, we investigate the stability of meteorologically important Rossby-Haurwitz (RH) waves and the dynamics of their perturbations. We derive a conservation law for arbitrary (and not just infinitesimal) perturbations of the RH wave from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ (Theorem 1) and introduce the invariant disjoint sets $\mathbf{M}_-^n$, $\mathbf{M}_+^n$, $\mathbf{M}_0^n$ of perturbations (Theorem 2). It is shown that the subspace $\mathbf{H}_n$ belonging to the set $\mathbf{M}_0^n$ is another invariant subspace of neutral perturbations, which can be considered as the zero class of the factor space $\mathbb{H}^2/\mathbf{H}_n$ of perturbations. A number of norms for estimating perturbations are defined, and the relationship between the enstrophy norm, the energy norm, and the factor norm of perturbations is shown (Theorems 3 and 7). A hyperbolic law is established for perturbations of the RH wave (5.1) from the invariant sets $\mathbf{M}_-^n$ and $\mathbf{M}_+^n$ (Theorem 4), and it is shown that the energy cascades of growing (or decaying) perturbations have opposite directions in these sets (Theorem 5). A geometric explanation of the change in the perturbation energy is given (Theorem 6), and the Lyapunov instability of nonstationary nonzonal RH waves from $\mathbf{H}_1 \oplus \mathbf{H}_n$ ($n \geq 2$) in the invariant set $\mathbf{M}_-^n$ is proved (Theorem 8). However, it is shown that the mechanism of Lyapunov instability has nothing in common with orbital (Poincaré) insta-

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bility and is caused by the difference in the velocities of the two RH waves, resembling the instability mechanism of periodic solutions of the equation of a nonlinear pendulum near the center (equilibrium point) during small-amplitude oscillations without bifurcations (without the destruction of periodic orbits or transitions to chaotic behavior). It is also noted that in the case of a stationary RH wave, the method of proving Theorem 8 is inapplicable.

The law of conservation of arbitrary perturbations is also used to derive a new necessary condition for the instability of normal modes of a stationary RH wave from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ $(n \geq 2)$. This condition imposes a constraint on the spectral distribution of the energy of unstable modes: the mean spectral Fjörtoft number of the amplitude of any unstable normal mode must be equal to $\sqrt{n(n+1)}$. Thus, linear (exponential) instability of the wave is possible only in the set $\mathbf{M}_0^n$ (but not in the subspace $\mathbf{H}_n$), and the instability condition depends only on the degree n of the homogeneous spherical polynomial representing the RH wave (Theorem 9). In the case of zonal solutions (flows in the form of Legendre polynomials and zonal RH waves), the new instability condition complements the well-known Rayleigh–Kuo and Fjörtoft conditions. Estimates for the growth rate of unstable modes are obtained (Theorem 10), and the orthogonality of the amplitude of any unstable mode to the basic wave in the Hilbert spaces $\mathbb{H}^0$ and $\mathbb{H}^1$ is demonstrated (Theorem 11). The obtained results clarify the spectral structure of growing disturbances. They are particularly useful for testing computational algorithms (and, ultimately, software packages) used for numerical studies of linear stability. It should be noted that both the instability conditions and the estimates of the maximum growth rate of unstable modes are specified in terms of the mean spectral Fjörtoft number. Therefore, this parameter is of primary importance in the problem of linear stability of ideal flows on a sphere.

Finally, we note that the stability of the RH wave from the subspace $\mathbf{H}_1 \oplus \mathbf{H}_n$ $(n \geq 2)$ in the invariant set of perturbations $\mathbf{M}_+^n$, containing relatively small-scale perturbations (compared to the scale of the RH wave), still remains an unsolved problem of

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considerable interest.

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