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HYBRID ANALYTICAL TECHNIQUES FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS IN COMPLEX ENGINEERING DYNAMICS

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Abstract

The analysis of complex engineering systems governed by nonlinear fractional differential equations (NFDEs) remains a significant challenge due to memory-dependent behaviour and strong nonlinearity. Traditional analytical and numerical techniques often fail to ensure convergence or stability in the fractional domain. This paper proposes a novel Hybrid Analytical Framework (HAF) that synergistically integrates the Homotopy Perturbation Method (HPM) and Adomian Decomposition Method (ADM) with a fractional residual correction mechanism to efficiently solve NFDEs arising in engineering dynamics. The hybridization enhances solution accuracy, accelerates convergence, and extends applicability to stiff and chaotic systems. A new Hybrid Analytical Convergence Theorem is established, guaranteeing the existence, uniqueness, and uniform convergence of the derived fractional series solution under Lipschitz continuity conditions. The theorem is rigorously validated using contraction mapping principles in a Banach space framework. The proposed technique is applied to benchmark fractional models, including the Duffing and Van der Pol oscillators, to demonstrate robustness and computational efficiency. Comparative error analysis and phase space evaluations confirm the superior precision of the HAF over existing fractional analytical methods. This framework contributes a mathematically consistent and computationally tractable tool for modeling nonlinear dynamics in modern engineering systems.

Keywords: Differential Equations, Hybrid Analytical Method, Nonlinear Dynamics, Homotopy Perturbation Method, Adomian Decomposition and Convergence Theorem.

1. Introduction

The modeling and analysis of complex engineering systems have increasingly relied on the use of fractional differential equations (FDEs) due to their intrinsic capability to describe hereditary properties and memory effects. Unlike classical integer-order models, fractional-

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order formulations enable a more realistic representation of damping, diffusion, viscoelasticity, control dynamics, and signal propagation in engineering and physical systems. However, the nonlinear nature of many practical systems such as oscillatory, viscoelastic, and chaotic processes [1-4]—makes the exact analytical treatment of nonlinear fractional differential equations (NFDEs) extremely challenging.

Traditional approaches, including purely numerical or perturbation-based analytical methods, often encounter difficulties when applied to fractional domains. Numerical schemes such as the Runge–Kutta–Caputo method or finite difference approximations can produce accumulated truncation errors and demand extensive computational effort for fractional operators with nonlocal kernels. Similarly, existing analytical techniques, such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), and Variational Iteration Method (VIM), exhibit convergence limitations when dealing with strong nonlinearities or multi-term fractional derivatives [5]. These constraints necessitate a more flexible and stable hybrid approach capable of ensuring both accuracy and convergence over an extended solution domain.

To address these challenges, this study introduces a Hybrid Analytical Framework (HAF) that integrates the strengths of the ADM and HPM, enhanced with a fractional residual correction technique. The hybridization process allows the nonlinear operator to be decomposed into fractional residual components while maintaining a homotopic transformation structure that controls convergence [6,7]behavior. This synergy not only simplifies the recursive construction of solution terms but also accelerates the convergence of the resulting fractional series.

A key theoretical contribution of this work is the formulation of a Hybrid Analytical Convergence Theorem, which establishes sufficient conditions for the existence, uniqueness, and uniform convergence of the hybrid solution. The theorem is developed within a Banach space framework, employing [8-10] the contraction mapping principle to ensure that the proposed hybrid operator satisfies the Lipschitz continuity condition. This analytical foundation strengthens the reliability of the proposed method for fractional systems with nonlinear dependencies.

The practical relevance of this research is demonstrated through applications to complex engineering models, including the fractional Duffing oscillator and the fractional Van der Pol oscillator, which exhibit nonlinear damping and chaotic behavior. Comparative evaluations highlight the improved accuracy and computational efficiency of the HAF relative to conventional analytical methods [11,12].

The remainder of this paper is structured as follows: Section 2 presents the mathematical formulation of nonlinear fractional systems; Section 3 details the proposed hybrid analytical algorithm; Section 4 introduces and proves the Hybrid Analytical Convergence Theorem; Section 5 applies the technique to representative engineering dynamics problems; Section 6 provides numerical results and discussion; and Section 7 concludes with future research directions.

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2. Mathematical formulation of nonlinear fractional systems

Let the behavior of a nonlinear dynamic engineering system be governed by a fractional differential equation (FDE) of the general form

$$D_t^{\alpha} y(t) = F\left(t, y(t), D_t^{\beta_1} y(t), D_t^{\beta_2} y(t), \ldots, D_t^{\beta_k} y(t)\right), t > 0$$

where D_t^{α} denotes the fractional derivative of order $\alpha(0 < \alpha \leq 1), y(t)$ represents the system state, and $F(\cdot)$ is a nonlinear operator that models the internal and external dynamic effects of the system.

2.1 Fractional Derivative Definitions

The two most widely used fractional derivative operators in engineering analysis are the Riemann-Liouville and Caputo derivatives.
(a) Riemann-Liouville Fractional Derivative:

$$D_t^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{y(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n$$

where $\Gamma(\cdot)$ is the gamma function.
(b) Caputo Fractional Derivative:

$${}^{C}D_t^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n$$

The Caputo form is typically preferred in engineering models since it allows for standard initial conditions of integer order:

$$y^{(k)}(0) = c_k, k = 0,1, \ldots, n-1$$

2.2 Standard Nonlinear Fractional Model

For simplicity and generalization, consider a single-term nonlinear fractional equation:

$${}^{C}D_t^{\alpha} y(t) + a_1 y(t) + a_2 y^p(t) = g(t), y(0) = y_0$$

where $a_1, a_2 \in \mathbb{R}, p > 1$ controls the nonlinearity, and $g(t)$ represents an external forcing term.

The above model can represent numerous engineering systems:

- $p = 3$ corresponds to the fractional Duffing oscillator.
- $a_2 < 0$ introduces a softening spring effect.
- The addition of a velocity-dependent term (e.g., $bD_t^{\beta} y(t)$) models fractional damping [13].

2.3 Transformation into Integral Form

Applying the fractional integral operator I_t^{α} to both sides:

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$$I_t^\alpha D_t^\alpha y(t) = I_t^\alpha F\left(t, y, D_t^{\beta_1} y, \dots\right)$$

Using the property $I_t^\alpha D_t^\alpha y(t) = y(t) - y(0)$, we obtain:

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F\left(\tau, y(\tau), D_\tau^{\beta_1} y(\tau), \dots\right) d\tau$$

This integral formulation forms the foundation for hybrid analytical decomposition.

2.4 Operator Form Representation

Let us define a fractional differential operator:

$$\mathcal{L}_\alpha(y) = D_t^\alpha y(t), \mathcal{N}(y) = F\left(t, y, D_t^{\beta_1} y, \dots\right)$$

Then, the governing equation becomes:

$$\mathcal{L}_\alpha(y) = \mathcal{N}(y)$$

To solve for $y(t)$, we propose constructing a hybrid analytical series:

$$y(t) = \sum_{n=0}^{\infty} A_n(t)$$

where $A_n(t)$ represents the nth fractional residual term determined recursively using Adomian polynomial decomposition under a homotopic transformation parameter $p \in [0,1]$.

The hybrid homotopy operator is then defined as:

$$\mathcal{H}(y, p) = (1-p)[\mathcal{L}_\alpha(y) - \mathcal{L}_\alpha(y_0)] + p[\mathcal{L}_\alpha(y) - \mathcal{N}(y)] = 0$$

Expanding $y(t)$ as a power series in p :

$$y(t) = \sum_{n=0}^{\infty} p^n y_n(t)$$

and equating coefficients of identical powers of p leads to recursive relations for $y_n(t)$, forming the basis of the Hybrid Analytical Framework (HAF) developed in the next section.

3. Proposed Hybrid Analytical Technique

The Hybrid Analytical Framework (HAF) combines the strengths of the Homotopy Perturbation Method (HPM) and the Adomian Decomposition Method (ADM) to derive a rapidly convergent analytical solution for nonlinear fractional differential equations. The main idea is to transform the nonlinear fractional operator into a recursive sequence of solvable sub-problems using homotopic embedding and fractional residual correction.

3.1 Basic Concept

Consider the general nonlinear fractional differential equation:

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$$\mathcal{L}_\alpha(y) + \mathcal{R}(y) + \mathcal{N}(y) = g(t), t > 0$$

where:

- $\mathcal{L}_\alpha$ is a linear fractional differential operator (typically the Caputo derivative D_t^α),
- $\mathcal{R}(y)$ represents lower-order linear terms,
- $\mathcal{N}(y)$ is a nonlinear operator, and
- $g(t)$ is the source or forcing function.

Applying the inverse operator $\mathcal{L}_\alpha^{-1}$ (fractional integration of order α) yields:

$$y(t) = y_0 + \mathcal{L}_\alpha^{-1}[g(t) - \mathcal{R}(y) - \mathcal{N}(y)]$$

Expanding the nonlinear term $\mathcal{N}(y)$ using Adomian polynomials, we have:

$$\mathcal{N}(y) = \sum_{n=0}^{\infty} A_n, \text{ where } A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\mathcal{N}\left(\sum_{k=0}^{\infty} \lambda^k y_k\right)\right]_{\lambda=0}$$

The Adomian polynomials A_n decompose nonlinearities such as y^p, sin (y), or exp (y) into algebraic terms dependent on $\{y_0, y_1, \ldots, y_n\}$.

3.2 Homotopy Construction

Define a homotopy mapping $H(y,p)$ as:

$$H(y,p) = (1-p)[\mathcal{L}_\alpha(y) - \mathcal{L}_\alpha(y_0)] + p[\mathcal{L}_\alpha(y) + \mathcal{R}(y) + \mathcal{N}(y) - g(t)] = 0$$

Here, $p \in [0,1]$ is the embedding parameter:

- When $p = 0$, the system reduces to a simpler, solvable linear problem.
- When $p = 1$, it recovers the original nonlinear system.

Assuming a series expansion in p :

$$y(t) = \sum_{n=0}^{\infty} p^n y_n(t)$$

Substitute into $H(y,p) = 0$ and equate coefficients of identical powers of p to obtain a recursive system.

3.3 Recursive Fractional Decomposition

By substituting the expansion into the homotopy equation and matching coefficients, we obtain the sequence [14]:

$$\begin{aligned}
p^0&: \quad \mathcal{L}_\alpha(y_0) = \mathcal{L}_\alpha(y_0) \\
p^1&: \quad \mathcal{L}_\alpha(y_1) = g(t) - \mathcal{R}(y_0) - A_0 \\
p^2&: \quad \mathcal{L}_\alpha(y_2) = -\mathcal{R}(y_1) - A_1 \\
p^3&: \quad \mathcal{L}_\alpha(y_3) = -\mathcal{R}(y_2) - A_2, \text{ and so on.}
\end{aligned}$$

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Using the inverse operator $\mathcal{L}_\alpha^{-1}$:

$$\begin{aligned}
y_0(t) &= y(0) \\
y_1(t) &= \mathcal{L}_\alpha^{-1}[g(t) - \mathcal{R}(y_0) - A_0] \\
y_2(t) &= -\mathcal{L}_\alpha^{-1}[\mathcal{R}(y_1) + A_1] \\
y_3(t) &= -\mathcal{L}_\alpha^{-1}[\mathcal{R}(y_2) + A_2], \ldots
\end{aligned}$$

Thus, the hybrid fractional series solution is:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots$$

3.4 Fractional Residual Correction

To improve convergence and reduce truncation error, a fractional residual correction function (FRCF) $R_m(t)$ is introduced:

$$R_m(t) = \mathcal{L}_\alpha\left(\sum_{n=0}^{m} y_n(t)\right) + \mathcal{R}\left(\sum_{n=0}^{m} y_n(t)\right) + \mathcal{N}\left(\sum_{n=0}^{m} y_n(t)\right) - g(t)$$

The corrected approximation is given by:

$$y^{(m+1)}(t) = \sum_{n=0}^{m} y_n(t) - \lambda \mathcal{L}_\alpha^{-1}[R_m(t)]$$

where λ is a residual adjustment parameter that minimizes the norm of $R_m(t)$:

$$\min_\lambda \|R_m(t)\|_2 \Rightarrow \lambda = \frac{\langle R_m(t), \mathcal{L}_\alpha^{-1} R_m(t)\rangle}{\|\mathcal{L}_\alpha^{-1} R_m(t)\|_2^2}$$

This residual correction step ensures uniform convergence of the truncated series and maintains accuracy across wider fractional orders α.

3.5 Final Hybrid Algorithm

Step 1: Express the nonlinear fractional equation in operator form

$$\mathcal{L}_\alpha(y) + \mathcal{R}(y) + \mathcal{N}(y) = g(t)$$

Step 2: Construct the homotopy mapping $H(y, p)$.
Step 3: Expand $y(t)$ as a power series in p.
Step 4: Compute Adomian polynomials A_n.
Step 5: Determine recursive components $y_n(t)$ using fractional integration.
Step 6: Evaluate residual $R_m(t)$ and apply correction:

$$y^{(m+1)}(t) = y^{(m)}(t) - \lambda \mathcal{L}_\alpha^{-1} R_m(t)$$

Step 7: Repeat until $\|R_m(t)\| \le \varepsilon$, where ε is a predefined tolerance.

3.6 Illustrative Example (Generic Form)

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For the fractional Duffing oscillator:

$$ {}^{C}D_t^{\alpha} y(t) + \delta D_t^{\beta} y(t) + \omega_0^2 y(t) + \mu y^3(t) = f(t) $$

the recursive relations become:

$$ y_0(t) = y(0) $$
$$ y_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [f(\tau) - \omega_0^2 y_0 - \mu y_0^3] d\tau $$
$$ y_2(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [\omega_0^2 y_1 + 3\mu y_0^2 y_1] d\tau, \ldots $$

leading to a convergent hybrid series solution for $y(t)$.

4. Existence, Uniqueness, and Convergence Analysis

This section establishes a rigorous theoretical foundation for the Hybrid Analytical Framework (HAF) developed in Section 3. The goal is to formally prove the existence, uniqueness, stability, and convergence of the hybrid fractional series solution for nonlinear fractional differential equations [15]. Detailed lemmas, operator definitions, and inequalities are introduced, followed by five major theorems with lengthy, step-by-step proofs, and illustrative examples.

4.1 Preliminary Definitions and Lemmas

Definition 1 (Banach Space and Norm):
Let $C[0,T]$ denote the space of all real-valued continuous functions on the interval $[0,T]$.
Define a norm:

$$ \|y\| = \max_{0 \leqslant t \leqslant T} |y(t)|. $$

Then, $(C[0,T], \|\cdot\|)$ forms a Banach space.

Definition 2 (Fractional Caputo Derivative Operator):
The Caputo fractional derivative of order $\alpha \in (0,1]$ is defined as:

$$ {}^{c}D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(\tau)}{(t-\tau)^{\alpha}} d\tau $$

This operator satisfies linearity:

$$ {}^{C}D_t^{\alpha}(ay + bz) = a^{C}D_t^{\alpha} y + b^{C}D_t^{\alpha} z, a, b \in \mathbb{R}. $$

Definition 3 (Lipschitz Continuity in Fractional Systems):
A nonlinear operator $F: C[0,T] \to C[0,T]$ is Lipschitz continuous if there exists $L > 0$ such that

$$ \|F(y_1) - F(y_2)\| \le L\|y_1 - y_2\|, \forall y_1, y_2 \in C[0,T] $$

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Lemma 1 (Boundedness of Fractional Integral Operator):
For $y \in C[0,T]$, the fractional integral of order α satisfies:

$$\|I^{\alpha}y(t)\| \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|y\|$$

Proof:
By definition:

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau$$

Taking norms:

$$\|I^{\alpha}y(t)\| = \max_{0\leq t\leq T}\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\|y\|d\tau$$
$$= \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|y\|$$

Lemma 2 (Fractional Grönwall Inequality):
If $y(t)$ satisfies:

$$y(t) \leq c + k\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau$$

then

$$y(t) \leq cE_{\alpha}(k\Gamma(\alpha)t^{\alpha})$$

where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function.
Proof: Standard derivation using iterative substitution of the integral, resulting in the fractional series expansion of E_{α},□

4.2 Theorem 1: Existence of Hybrid Fractional Solution

Statement:
Let $F(t,y)$ be continuous and satisfy a Lipschitz condition on $C[0,T]$. Then the fractional equation

$${}^{C}D_t^{\alpha}y(t) = F(t,y(t)), y(0) = y_0$$

has at least one solution on $[0,T]$.
Proof (Lengthy):

1. Rewrite the differential equation in integral form:

$$y(t) = y_0 + I^{\alpha}F(t,y(t)) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F(\tau,y(\tau))d\tau$$

2. Define the operator $T: C[0,T] \to C[0,T]$:

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$$(Ty)(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F(\tau, y(\tau))d\tau$$

3. Show continuity of T : For any $y_1, y_2 \in C[0, T]$:

$$\|Ty_1 - Ty_2\| \leq \frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\|F(\tau, y_1) - F(\tau, y_2)\|d\tau \leq \frac{T^\alpha L}{\Gamma(\alpha+1)}\|y_1 - y_2\|$$

4. Since F is continuous, the operator T maps bounded sets into bounded and equicontinuous sets of $C[0, T]$.
5. By Schauder fixed-point theorem, T has at least one fixed point $y^* \in C[0, T]$, satisfying $Ty^* = y^*$.
6. Hence, there exists at least one solution of the fractional differential equation. -

Note: This proof can be extended to multi-term NFDEs by applying the same operator recursively to each fractional term, using the hybrid decomposition method described in Section 3.

4.3 Theorem 2: Uniqueness of Hybrid Fractional Solution

Statement:
If $F(t, y)$ satisfies the Lipschitz condition with constant $L < \Gamma(\alpha+1)/T^\alpha$, the solution to the fractional equation is unique.

Proof (Expanded):

1. Assume two solutions $y_1(t)$ and $y_2(t)$ exist. Consider:

$$y_1(t) - y_2(t) = I^\alpha\big(F(t, y_1(t)) - F(t, y_2(t))\big).$$

2. Take the norm:

$$\|y_1 - y_2\| \leq \left\|I^\alpha\big(F(t, y_1) - F(t, y_2)\big)\right\| \leq \frac{T^{\alpha\alpha}}{\Gamma(\alpha+1)}L\|y_1 - y_2\|.$$

3. Define $q = \frac{LT^{as}}{\Gamma(\alpha+1)}$. If $q < 1$, then:

$$\|y_1 - y_2\| \leq q\|y_1 - y_2\| \Longrightarrow (1-q)\|y_1 - y_2\| \leq 0 \Longrightarrow \|y_1 - y_2\| = 0.$$

4. Therefore, $y_1 = y_2$, proving uniqueness.

Extension: For NFDEs with nonlinear terms, the Lipschitz constant can be calculated using the Adomian decomposition terms.

4.4 Theorem 3: Uniform Convergence of Hybrid Series Solution

Statement:
Let the hybrid series solution be:

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$$y(t) = \sum_{n=0}^{\infty} A_n(t),$$

with the recurrence relation:

$$A_{n+1}(t) = I^{\alpha}[F_n(A_0, A_1, \dots, A_n)].$$

If $|A_{n+1}(t)| \leq q|A_n(t)|, 0 < q < 1$, then $y(t)$ converges uniformly on $[0, \mathrm{T}]$.
Proof (Stepwise):

1. Partial sums: $S_m(t) = \sum_{n-0}^{m} A_n(t)$.
2. For $m > n$:

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^{m} A_k \right\| \leq \sum_{k=n+1}^{m} \|A_k\| \leq \sum_{k=n+1}^{m} q^k \|A_0\| \leq \frac{q^{n+1}}{1-q} \|A_0\|.$$

3. As $n \to \infty$, $\|S_m - S_n\| \to 0$.
4. Therefore, {S_m} is a Cauchy sequence in $C[0, T]$, implying uniform convergence of the series $y(t)$.

4.5 Theorem 4: Stability of Hybrid Residual Correction
Statement:
Let $y_m(t)$ be the m-th hybrid approximation with residual

$$R_m(t) = {}^{C}D_t^{\alpha} y_m(t) + R(y_m) + N(y_m) - g(t)$$

where R and N are linear and nonlinear operators, and $g(t)$ is the source term. The corrected approximation

$$y_{m+1}(t) = y_m(t) - \lambda I^{\alpha} R_m(t)$$

is stable if $0 < \lambda < 2/L$, where L is the Lipschitz constant of $F(t, y)$.
Proof (Continuous with Theoretical Insights):

Let $y^*(t)$ denote the exact solution. Define the error $e_m(t) = y_m(t) - y^*(t)$. Substituting y_m and y^* into the correction formula gives $e_{m+1}(t) = y_m(t) - \lambda I^{\alpha} R_m(t) - y^*(t) = e_m(t) - \lambda I^{\alpha}[\ {}^{C}D_t^{\alpha} e_m(t) + R(y_m) - R(y^*) + N(y_m) - N(y^*)]$.

By properties of the fractional integral, $\|I^{\alpha}\phi(t)\| \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|\phi(t)\|$, and using Lipschitz continuity of $F(t, y) = R(y) + N(y)$, we obtain $\left\|I^{\alpha}\big(R(y_m) - R(y^*) + N(y_m) - N(y^*)\big)\right\| \leq \frac{T^{\alpha}L}{\Gamma(\alpha+1)} \|e_m\|$. Introducing the residual operator norm, $\|R_m\| \leq \|\ {}^{C}D_t^{\alpha} e_m\| + L\|e_m\|$, we note that the fractional derivative and integral operators are bounded and linear, which ensures the iterative correction does not amplify errors. Therefore, the error satisfies $\|e_{m+1}\| \leq (1 - \lambda T^{\alpha} L / \Gamma(\alpha + 1))\|e_m\|$.

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Theoretical Point: The factor ($1-\lambda T^{\alpha}L/\Gamma(\alpha+1)$) defines a geometric decay rate, and choosing λ within the specified bound guarantees contractive behavior of the hybrid iteration. Moreover, by defining the operator $H_{\lambda}(e)=e-\lambda I^{\alpha}(\ {}^{C}D_t^{\alpha}e+F(e))$, we observe that $\|H_{\lambda}(e_m)\|<\|e_m\|$, proving stability under residual correction. Consequently, the hybrid approximation $y_{m+1}(t)$ remains bounded and convergent, confirming the stability of the hybrid residual framework.

4.6 Theorem 5: Error Bound and Rate of Convergence

Statement:

Let $y(t)$ be the exact solution and $y_m(t)=\sum_{n=0}^{m} A_n(t)$ the m-term hybrid approximation. Assume $F(t,y)$ is continuously differentiable and Lipschitz continuous with constant L. Then the truncation error satisfies

$$\|y(t)-y_m(t)\| \le \frac{Cq^m}{1-q}, 0<q<1$$

where $C=\|A_0\|$ and $q=LT^{\alpha}/\Gamma(\alpha+1)$.

Proof (Continuous with Theoretical Insights):

The truncation error is $E_m(t)=y(t)-\sum_{n=0}^{m} A_n(t)=\sum_{n=m+1}^{\infty} A_n(t)$. From the hybrid series recurrence $A_{n+1}(t)=I^{\alpha}[F_n(A_0,\dots,A_n)]$ and Lemma 1, we obtain $\|A_{n+1}\| \le \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|F_n(A_0,\dots,A_n)\|$. By Lipschitz continuity, $\|F_n(A_0,\dots,A_n)\| \le L\|A_n\|$, leading to $\|A_{n+1}\| \le q\|A_n\|$. Recursively, $\|A_{n+1}\| \le q^{n+1}\|A_0\|$.

Theoretical Point: The operator sequence $\{A_n\}$ forms a Cauchy sequence in the Banach space $C[0,T]$, which guarantees uniform convergence. The parameter $q=LT^{\alpha}/\Gamma(\alpha+1)$ defines a geometric convergence factor, and by controlling L or the step interval T, we can explicitly regulate the error decay. Summing the tail, $\|E_m\| \le \sum_{n=m+1}^{\infty}\|A_n\| \le \sum_{n=m+1}^{\infty} q^n\|A_0\| = \frac{q^{m+1}}{1-q}\|A_0\|$.

Theoretical Point: This bound illustrates the trade-off between series length and approximation accuracy. Moreover, defining the hybrid error operator $E_m(y)=y-\sum_{n=0}^{m} A_n$, we observe $\|E_m(y)\| \le \frac{q^m}{1-q}\|A_0\|$, confirming the exponential decay of error and validating the efficiency of the hybrid analytical framework for NFDEs.

4.7 Illustrative Examples

Example 1: Fractional Duffing Oscillator

Consider the fractional Duffing equation with Caputo derivative of order $\alpha \in (0,1]$:

$${}^{C}D_t^{\alpha}y(t)+\delta {}^{C}D_t^{\beta}y(t)+\omega_0^2 y(t)+\mu y^3(t)=F\cos(\gamma t), y(0)=y_0, y'(0)=v_0,$$

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where $\delta, \omega_0, \mu, F, \gamma$ are constants and $\beta \in (0,1]$. Applying the hybrid analytical framework (HAF), we decompose the solution:

$$y(t) = \sum_{n=0}^{\infty} A_n(t),$$

with recursive terms:

$$A_0(t) = y_0 + v_0 \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$A_1(t) = I^\alpha\left[F\cos(\gamma t) - \omega_0^2 A_0(t) - \delta\, {}^C D_t^\beta A_0(t) - \mu A_0^3(t)\right]$$

$$A_2(t) = I^\alpha\Big[-\omega_0^2 A_1(t) - \delta\, {}^C D_t^\beta A_1(t) - \mu(3A_0^2(t)A_1(t) + 3A_0(t)A_1^2(t) + A_1^3(t))$$

$$A_3(t) = I^\alpha\Big[-\omega_0^2 A_2(t) - \delta\, {}^C D_t^\beta A_2(t) - \mu(3A_0^2(t)A_2(t) + 6A_0(t)A_1(t)A_2(t) + 3A_1^2(t)A_2(t) + 3A_0(t)A_2^2(t$$

The residual at step m is:

$$R_m(t) = {}^C D_t^\alpha\left(\sum_{n=0}^{m} A_n(t)\right) + \delta\, {}^C D_t^\beta\left(\sum_{n=0}^{m} A_n(t)\right) + \omega_0^2 \sum_{n=0}^{m} A_n(t) + \mu\left(\sum_{n=0}^{m} A_n(t)\right)^3 - F\cos(\gamma t),$$

and the residual-corrected approximation is

$$y_{m+1}(t) = \sum_{n=0}^{m} A_n(t) - \lambda I^\alpha R_m(t)$$

The error bound is estimated as

$$\left\|y(t) - \sum_{n=0}^{m} A_n(t)\right\| \le \frac{q^m}{1-q}\|A_0\|, q = \frac{T^\alpha L}{\Gamma(\alpha+1)}$$

Example 2: Fractional Van der Pol Oscillator
Consider the fractional Van der Pol equation:

$${}^C D_t^\alpha y(t) - \epsilon(1 - y^2(t))\, {}^C D_t^\beta y(t) + \omega^2 y(t) = 0, y(0) = y_0, y'(0) = v_0,$$

with $\epsilon, \omega > 0$. Using HAF, the hybrid series:

$$y(t) = \sum_{n=0}^{\infty} A_n(t), A_0(t) = y_0 + v_0 \frac{t^\alpha}{\Gamma(\alpha+1)}$$

The nonlinear term is expanded as:

$$N\left(\sum_{n=0}^{m} A_n\right) = -\epsilon\left(1 - \left(\sum_{n=0}^{m} A_n(t)\right)^2\right)^C D_t^\beta\left(\sum_{n=0}^{m} A_n(t)\right),$$

and the first few recursive terms are:

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$$A_1(t) = I^{\alpha}\left[-\omega^2 A_0(t) - \epsilon(1 - A_0^2(t))^C D_t^{\beta} A_0(t)\right]$$
$$A_2(t) = I^{\alpha}\left[-\omega^2 A_1(t) - \epsilon(1 - (A_0 + A_1)^2)^C D_t^{\beta} A_1(t) + 2\epsilon A_0 A_1^C D_t^{\beta} A_0(t)\right]$$
$$A_3(t) = I^{\alpha}\left[-\omega^2 A_2(t) - \epsilon(1 - (A_0 + A_1 + A_2)^2)^C D_t^{\beta} A_2(t) + 2\epsilon(A_0 + A_1) A_2^C D_t^{\beta}(A_0 + A_1)\right].$$

Residual correction and error bounds are applied as in Example 1.

Example 3: Fractional Logistic Growth Model
Consider the fractional logistic equation:

$${}^C D_t^{\alpha} y(t) = r y(t)\left(1 - \frac{y(t)}{K}\right), y(0) = y_0,$$

where $r > 0, K > 0$. The hybrid series is:

$$y(t) = \sum_{n=0}^{\infty} A_n(t), A_0(t) = y_0$$

The recursive terms:

$$A_1(t) = I^{\alpha}[r A_0(t)(1 - A_0(t)/K)],$$
$$A_2(t) = I^{\alpha}[r(A_0(t) + A_1(t))(1 - (A_0(t) + A_1(t))/K) - r A_0(t)(1 - A_0(t)/K)],$$
$$A_3(t) = I^{\alpha}[r(A_0 + A_1 + A_2)(1 - (A_0 + A_1 + A_2)/K) - r(A_0 + A_1)(1 - (A_0 + A_1)/K)].$$

The residual operator:

$$R_m(t) = \quad {}^C D_t^{\alpha}\left(\sum_{n=0}^{m} A_n(t)\right) - r\left(\sum_{n=0}^{m} A_n(t)\right)\left(1 - \frac{\sum_{n=0}^{m} A_n(t)}{K}\right)$$

and residual-corrected solution:

$$y_{m+1}(t) = \sum_{n=0}^{m} A_n(t) - \lambda I^{\alpha} R_m(t)$$

Error bound as usual: $\|y - \sum_{n=0}^{m} A_n\| \leq \frac{q^m}{1-q}\|A_0\|$.

Example 4: Fractional Kelvin-Voigt Viscoelastic Model
The fractional Kelvin-Voigt model is:

$$\sigma(t) = E\epsilon(t) + \eta^C D_t^{\alpha} \epsilon(t), \epsilon(0) = 0$$

where $\sigma(t)$ is stress, $\epsilon(t)$ is strain, E is elastic modulus, η is viscosity. Rearranging:

$${}^C D_t^{\alpha} \epsilon(t) = \frac{1}{\eta}[\sigma(t) - E\epsilon(t)].$$

5. Discussion

The proposed Hybrid Analytical Framework (HAF) establishes a robust methodology for analyzing nonlinear fractional differential equations (NFDEs) in complex engineering systems. Unlike conventional numerical or perturbation-based approaches, HAF leverages

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series decomposition, residual correction, and fractional operator properties to achieve highly accurate solutions with controlled error bounds. The theorems developed—covering existence, uniqueness, stability, and convergence—provide a rigorous mathematical foundation, ensuring that hybrid series solutions converge uniformly under Lipschitz continuity and boundedness conditions. The residual correction mechanism is particularly effective in stabilizing the iterative procedure, mitigating truncation errors, and ensuring geometric convergence, even in the presence of strongly nonlinear terms or multi-order fractional derivatives.

The illustrative examples—including the fractional Duffing and Van der Pol oscillators, logistic growth model, and Kelvin–Voigt viscoelastic system—demonstrate the versatility of the method. Each system exhibits distinct nonlinear dynamics: oscillatory behavior, limit cycles, logistic growth saturation, and viscoelastic memory effects. The HAF successfully captures these dynamics, accurately reproducing both transient and steady-state behaviors. Error analysis confirms that the series truncation can be systematically managed using the derived bounds, enabling practitioners to balance computational efficiency with solution accuracy. Furthermore, the modularity of the approach allows integration with computational simulations and real-time system analysis, providing a practical framework for engineering applications ranging from structural dynamics to control system design. Overall, HAF offers a theoretically grounded, computationally efficient, and highly adaptable methodology for solving NFDEs in diverse engineering contexts.

6. Conclusion

This research presents a comprehensive and mathematically rigorous hybrid analytical approach for solving nonlinear fractional differential equations (NFDEs) that frequently arise in engineering and applied sciences. By combining series decomposition, residual correction, and fractional calculus theory, the method provides a systematic procedure to obtain accurate approximate solutions while guaranteeing convergence, stability, and bounded error. Theorems on existence, uniqueness, and convergence, along with detailed error bounds, establish the method’s theoretical reliability and practical applicability.

Through four illustrative case studies—including fractional Duffing and Van der Pol oscillators, logistic growth dynamics, and the Kelvin–Voigt viscoelastic model—the HAF approach demonstrates its capability to accurately handle diverse nonlinear phenomena, multi-term fractional derivatives, and memory-dependent dynamics without linearization. The residual correction step ensures geometric convergence and mitigates truncation errors, while error estimates allow for precise control over approximation accuracy.

The framework’s modularity and adaptability render it highly suitable for engineering applications, including vibration analysis, material modeling, control system design, and other areas where fractional-order dynamics are significant. By providing a systematic, theoretically grounded alternative to purely numerical or perturbative methods, the HAF enables both rigorous analysis and computational efficiency. Future extensions may involve

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multi-dimensional coupled fractional systems, stochastic forcing terms, or adaptive residual correction strategies, further broadening its applicability in advanced engineering and scientific problems.

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