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STUDY ON INVERSE THERMOELASTIC RESPONSE OF A SEMI-INFINITE SOLID CYLINDER EMPLOYING THE TIME-FRACTIONAL HEAT CONDUCTION EQUATION

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Abstract:
The ability to evaluate material integrity and qualities without inflicting damage is made possible by the inverse thermoelastic process, which is crucial in sectors like manufacturing and aerospace design. Additionally, by looking at temperature and stress patterns, inverse thermoelastic problems can check the state of structures and help avoid possible failures in various designs. So far, most of this research has concentrated on modelling the direct thermoelasticity problem with a fractional approach. Using a quasi-static approach, the authors aim to address this research gap by exploring how a semi-infinite solid cylinder reacts to heat within the framework of the time fractional heat conduction equation. We estimate the temperature distribution and unknown heating temperatures on the curved surface of the circular cylinder. The lower surface of the cylinder is maintained at zero temperature. In an unsteady state, the circular cylinder is exposed to a temperature that is arbitrarily known. Further, using the integral transform technique, solutions to the heat conduction equation are obtained. Finally, a numerical analysis was performed on copper, and the behaviour of temperature and stress is examined and graphically represented for different fractional time values.

Keywords: Fractional calculus; Inverse thermoelastic problem; Thermal stresses; Integral transform; Mittag-Leffler functions

Introduction:

The inverse transient heat conduction problem involves situations where certain boundary conditions are unspecified and need to be inferred, relying on the known or measured temperature changes over time at internal points. Several methods have been proposed to tackle this problem of inverse thermoelastic problems.

Determining unknown heat or mechanical parameters, such as temperature patterns or heat flows, by observing how materials move or experience stress under those circumstances is known as an inverse thermoelastic issue. There are numerous real-world uses for inverse thermoelastic issues in materials research, engineering, and structural diagnostics. By examining observed mechanical responses, such as pressures or motions, these problems assist in determining unknown thermal or mechanical inputs, such as heat fluxes or surface temperatures.

As an extension of classical thermoelasticity, fractional thermoelasticity incorporates fractional calculus a branch of mathematics concerned with non-whole number derivatives and integrals into the equations that characterize the transfer of heat and the deformation of solids. Classical integer-order models (based on Fourier's law) do not adequately depict anomalous heat transport phenomena, non-local interactions, memory effects, and other such phenomena; however, this framework does capture them. Time fractional-order thermoelasticity, which extends classical

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thermoelastic theory, finds applications in diverse areas such as heat transfer modeling, stress analysis, and the study of material responses to thermal loads. This approach is especially valuable for systems where heat transfer and stress responses exhibit memory effects and are not immediate, as seen in viscoelastic materials or structures with complex geometries. PID controllers, bio-mathematics, fluid mechanics, signal processing, viscoelasticity, and electrochemistry are just a few of the engineering disciplines that have recently included fractional calculus into their techniques for a variety of applications.

The study of fractional-order calculus provides an intriguing extension of classical calculus to real or complex orders. It is still difficult to give fractional-order calculus a concrete meaning, nevertheless. By contrasting fractional integration with its physical interpretation, Podlubny [1] provided a geometric perspective. A key advantage of fractional-order differential equations is their ability to capture nonlocal characteristics, like complicated chaotic behaviour and long-lasting memory effects. The Riemann-Liouville fractional derivative has greatly aided the development of fractional calculus and its uses in pure mathematics. However, the advancement of modern technology necessitates a re-examination of the traditional pure mathematical framework. Researchers are actively developing and examining different methods for defining and managing fractional-order derivatives. The ability of fractional theory to explain delayed reactions to physical stimuli a characteristic commonly seen in natural systems lays the groundwork for the theory. The generalized theory of thermoelasticity, on the other hand, assumes an instantaneous reaction.

The researchers have investigated problems related to the inverse thermoelastic problem using a variety of methods that take into account distinct material properties: The work of Tikhe and Deshmukh [2] involves the estimation of unknown heating temperatures and temperature distributions on the upper surface of a thin circular plate, accounting for transient inverse thermoelastic deformation. Deshmukh et al. [3] examined the thermoelastic behavior and the inverse problem of heat conduction by taking into account a semi-infinite cylinder. A semi-infinite thin circular plate was examined by Deshmukh et al. [4], who also explained how thermal deflection behaves on the outer curved surface. In their study of the thermoelastic analysis of the tribo-couple between two elastic cylinders, the work of Kushnir et al. [5] focuses on solving the inverse heat conduction problem to evaluate the unknown temperature at the boundary of the inner or outer cylindrical layers. In static thermoelasticity, Krageorghis et al. [6] obtained a stable numerical solution of the linear coupled inverse problem and observed the solution of the inverse problem on a portion of the boundary. Using the Laplace transform space and an actual space statement, Nedin et al. [7] have found the solution to the inverse problem for an inhomogeneous thermoelastic rod. Solutions derived from the theory of residues and an inversion of the Laplace transform space. The inverse transient thermoelastic problem has been solved by Ashida et al. [8] obtained the unknown transient heating temperature distribution on the surface of a transversely isotropic layer, given the electric potential difference distribution across the piezoceramic layer. By taking into account one-dimensional heat conduction and the elastic wave equation, Vikulov [9] has examined thermoelasticity and developed the concept of uniqueness of a solution. He has also identified the instability of the inverse problem solution. The transient gradient method has been used by Huange and Wang [10] to address the three-dimensional transient inverse heat conduction problem.

Different definitions and approaches to fractional-order derivatives are of interest to many scholars. The fractional theory was introduced to explain the delayed reactivity to physical stimuli seen in nature, in contrast to the generalized theory of thermoelasticity, which presumes an instantaneous reaction. Sherief et al. created the fractional-order theory of thermoelasticity [11]. Povstenko [12,13] used the quasi-static theory to study a few fractional thermoelasticity problems. Warbhe et al. [14,15] used a quasi-static technique to study various fractional-order thermoelasticity problems. The study by Warbhe [16] focuses on evaluating thermal stresses in a simply supported rectangular plate using thermal bending moments and a time-dependent fractional derivative approach. Ezzat and El-Karmany [17] examined fractional-order thermoelasticity problem. El-Sayed and Gaber [18]

used the Adomian decomposition approach to determine the precise solution of the fractional diffusion partial differential equation and analyzed certain characteristics of Caputo derivatives. Tripathi et al. [19] examined the deflection in a thin circular plate under fractional order thermoelastic conditions with a constant temperature distribution. Tripathi et al. [20] employed the time fractional-order thermoelasticity theory to analyze a half-space problem involving a periodically varying heat source, aiming to regulate wave propagation speed. El-Karamany and Ezzat [21] developed a model for thermoelastic diffusion in both isotropic and anisotropic solids using a novel generalized theory that incorporates a memory-dependent derivative. A comprehensive analysis of the two-temperature theory within the framework of G.N. generalized thermoelasticity, incorporating fractional phase-lag heat transfer, is presented in the work by Ezzat et al. [22]. Ezzat and El-Bary [23] developed a new mathematical model for two-temperature electro-thermo-viscoelasticity, incorporating a novel approach to heat conduction based on a memory-dependent derivative. Ezzat and El-Bary [24] studied the impact of fractional-order heat transfer and variable thermal conductivity on an infinitely long, perfectly conducting hollow cylinder. Ezzat et al. [25] developed a new mathematical model of generalized thermoelasticity that incorporates memory-dependent derivatives, formulated within the framework of the dual-phase-lag heat conduction law. The study by Warbhe and Gujarkar [26] focuses on the thermoelastic properties of a thick hollow cylinder, formulated through the time-fractional heat conduction equation and evaluated via a quasi-static approach.
The majority of this study to date has focused on using a fractional technique to simulate the direct thermoelasticity problem. The authors plan to address this research gap by examining how a long solid cylinder responds to heat using a slow-changing method and the heat conduction equation with a time-fractional derivative. This article uses a mathematical formula that describes how heat flows in materials that have time variations. Memory effects are represented through the time-fractional differential operator. Given the practical application of fractional calculus, we aimed to construct a mathematical model incorporating the time fractional differential operator using a quasi-static approach, and further, we examined its inverse thermoelastic effect. Finally, the model is specified for pure copper and analyzed through computations performed in Mathcad Prime 1.0.

Formulation of the problem:

Consider a semi-infinite solid cylinder in the time domain, taking into account fractional order effect with dimensions $0 \le r \le b,\ 0 \le z < \infty,\ t > 0$. Let the interior of the cylinder be exposed to a specified arbitrary temperature $f(z,t)$ within the region $0 \le r \le b$, with the lower surface $(z=0)$ of the cylinder is maintained at zero temperature. Under these more realistic prescribed conditions, it becomes necessary to determine the unknown temperature on the curved surface of the cylinder at $r=b$, as well as the quasi-static thermal stresses arising from the unknown temperature $g(z,t)$. The integral transform technique is used to determine the solution of the problem.

The Caputo fractional derivative is defined in [12]

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = \frac{1}{\Gamma(n-\beta)} \int_0^t t-\tau)^{n-\beta-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \qquad n-1 < \beta < n \tag{1}$$

The Laplace transform for Caputo derivative is given as [15]

$$L\left\{\frac{\partial^\beta f(t)}{\partial t^\beta}\right\} = s^\beta f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\beta-1-k}, \qquad n-1 < \beta < n. \tag{2}$$

The equations of displacement potential $\phi(r,z,t)$ and the stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given in [3] as,

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$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = (1+v)a_t T \tag{3}$$

$$\sigma_{rr} = -\frac{2\mu}{r}\frac{\partial \phi}{\partial r} \tag{4}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \phi}{\partial r^2} \tag{5}$$

In the plane state of stress within the solid cylinder

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0 \,. \tag{6}$$

The governing time fractional-order heat conduction equation for a semi-infinite solid cylinder subjected to arbitrary temperature in the domain is defined as $0 \le r \le b$, $0 \le z < \infty$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial^\beta T}{\partial t^\beta} \tag{7}$$

with boundary conditions

$T = f(z,t)$ (known) at $r = \xi$, (8)

$T = g(z,t)$ (unknown) at $r = b$, (9)

$T = 0$ at $z = 0$, (10)

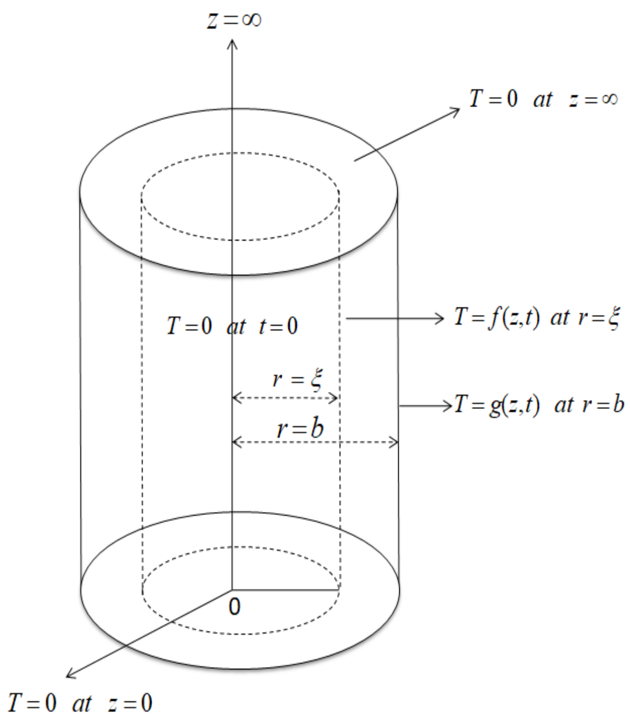
$T = 0$ at $z = \infty$,

initial conditions

$T = 0$ when $t = 0$, $0 < \beta < 1$ (11)

$\frac{\partial T}{\partial t} = 0$ when $t = 0$, $1 < \beta < 2$ (12)

Geometry of the problem:



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Solution of the problem:

By applying the Fourier sine transform, the Laplace transform technique, and their respective inverses as defined in [26, 27] to the system of Eqs. (7)–(12), the temperature distribution function and the unknown temperature are obtained as:

$$T(r,z,t)=\sqrt{\frac{2}{\pi}}\left(\frac{2k}{\beta\xi}\right)\left\{\int_0^\infty \sin(\eta z)\left(\sum_{m=1}^{\infty}\left[-k\left(\lambda_m^2+\eta^2\right)\right]^{(1-\beta)}\times p\left[\frac{J_0(pr)}{J_0(p\xi)}\right]\right.\right.$$

$$\left.\left.\times\int_{t_0=0}^{t}\overline{f}(\eta,t_0)E_{\beta,\beta}\left[-k\left(\lambda_m^2+\eta^2\right)(t-t_0)^\beta\right]dt_0\right)d\eta\right\} \quad (13)$$

$$g(z,t)=\sqrt{\frac{2}{\pi}}\left(\frac{2k}{\beta\xi}\right)\left\{\int_0^\infty \sin(\eta z)\left(\sum_{m=1}^{\infty}\left[-k\left(\lambda_m^2+\eta^2\right)\right]^{(1-\beta)}\times p\left[\frac{J_0(pb)}{J_0(p\xi)}\right]\right.\right.$$

$$\left.\left.\times\int_0^{t}\overline{f}(\eta,t_0)E_{\beta,\beta}\left[-k\left(\lambda_m^2+\eta^2\right)(t-t_0)^\beta\right]dt_0\right)d\eta\right\} \quad (14)$$

where $p=\sqrt{\eta^2+(-k)^{\beta-1}\left(\lambda_m^2+\eta^2\right)^\beta}$

And $\lambda_1,\lambda_2,\lambda_3,...$ are the positive roots of the transcendental equation $J_0(\lambda_m\xi)=0$.

The Fourier transform of is $f(z,t)$ defined as

$$\overline{f}(\eta,t_0)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(z,t)\sin(\eta z)dz$$

Here $E_{\beta,\beta}(.)$are the Mittag-Leffler function.

Determination of thermal stresses

Using Eq. (13) in Eq. (3), we find the displacement function as

$$\phi(r,z,t)=\frac{-(1+\nu)a_t}{p}\sqrt{\frac{2}{\pi}}\left(\frac{2k}{\beta\xi}\right)\left\{\int_0^\infty \sin(\eta z)\left(\sum_{m=1}^{\infty}\left[-k\left(\lambda_m^2+\eta^2\right)\right]^{(1-\beta)}\times p\left[\frac{J_0(pr)}{J_0(p\xi)}\right]\right.\right.$$

$$\left.\left.\times\int_0^{t}\overline{f}(\eta,t_0)E_{\beta,\beta}\left[-k\left(\lambda_m^2+\eta^2\right)(t-t_0)^\beta\right]dt_0\right)d\eta\right\} \quad (15)$$

Now, using Eq. (15) in Eqs. (4) and (5), the expressions of radial and angular stresses function obtained respectively as

$$\sigma_{rr}=\frac{-2\mu(1+\nu)a_t}{p}\sqrt{\frac{2}{\pi}}\left(\frac{2k}{\beta\xi}\right)\left\{\int_0^\infty \sin(\eta z)\left(\sum_{m=1}^{\infty}\left[-k\left(\lambda_m^2+\eta^2\right)\right]^{(1-\beta)}\times \frac{p}{r}\left[\frac{J_1(pr)}{J_0(p\xi)}\right]\right.\right.$$

$$\left.\left.\times\int_0^{t}\overline{f}(\eta,t_0)E_{\beta,\beta}\left[-k\left(\lambda_m^2+\eta^2\right)(t-t_0)^\beta\right]dt_0\right)d\eta\right\} \quad (16)$$

$$\sigma_{\theta\theta}=\frac{-2\mu(1+\nu)a_t}{p^2}\sqrt{\frac{2}{\pi}}\left(\frac{2k}{\beta\xi}\right)\left\{\int_0^\infty \sin(\eta z)\left(\sum_{m=1}^{\infty}\left[-k\left(\lambda_m^2+\eta^2\right)\right]^{(1-\beta)}\times p\left[\frac{-J_1(pr)}{J_0(p\xi)}\times\frac{1}{rp}+\frac{J_0(pr)}{J_0(p\xi)}\right]\right.\right.$$

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$$\left.\left.\times\int_0^t \overline{f}(\eta,t_0)E_{\beta,\beta}\left[-k\left(\lambda_m^2+\eta^2\right)(t-t_0)^\beta\right]dt_0\right)d\eta\right\} \quad (17)$$

Numerical Computation:

To formulate the mathematical model for various parameters and functions of pure copper in order to analyze the fractional-order thermal effect, the following values, given in Warbhe [14], are considered:

$b=2\,m, z=0.4\,m, t=5\,\text{sec.}, \mu=26.67\,GPa, \nu=0.35, k=112.34\times10^{-6}\,m^2s^{-1}, a_t=16.5\times10^{-6}\,/K$.

Setting $f(z,t)=(1-\omega t)\,z\,e^{-z^2}$ with $\omega>0$.

In order to examine the impacts of the time fraction in a solid cylinder, all of these numerical studies were conducted for various variables. These computations are shown in the following figures. The computational results for the dimensionless functions with fixed time are displayed in Figures 1–4, respectively.

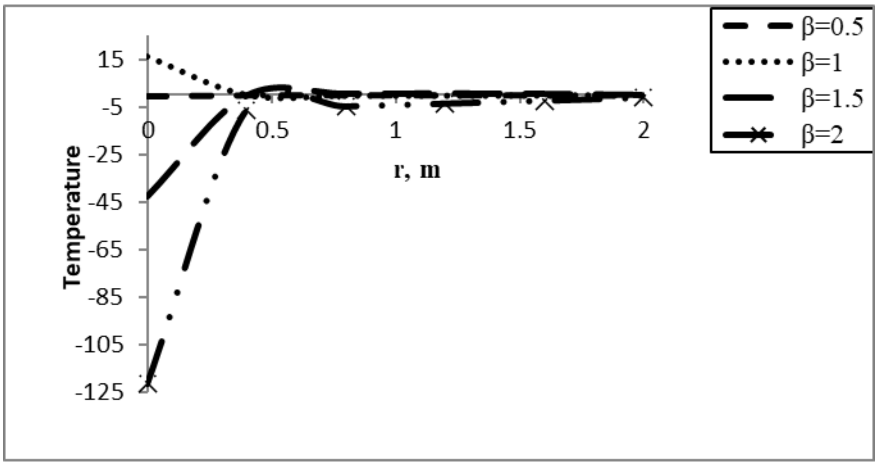

Figure 1. Temperature distribution function.

Figure 1 displays the temperature distribution along the radial direction for a range of fractional order parameter values. At first, the temperature values for $r=0$ begin to rise for $\beta=0.5, \beta=1.5,$ and $\beta=2$, but for $\beta=1$, the temperature begins to fall in the direction of the outer radii. Furthermore, as you proceed along the solid cylinder's radius, the temperature variations display an erratic flow pattern determined by the fractional parameters. Additionally, for weak, normal, and superconductivity, temperature exhibit asymmetric flow behaviour depending on the radius. Additionally, the graphic representation of the diffusion equation and wave equation for $\beta=1$ and $\beta=2$ makes it evident that they are similar to interpolating the classical heat conduction equation. More specifically, it can be shown that the influence of heat disturbances in the area fluctuates as the parameter values rise, revealing an irregular pattern with respect to the radius. Whereas the region along the axes exhibits negativity, steadily declining towards the fixed circular border, fractional parameter values show positive temperature changes. Notably, the region's temperature is below the ambient temperature for some parametric values, making the model unfeasible for the given fractional order parameter. Furthermore, the temperature equilibrium is reached more quickly for a range of β values, which is consistent with the traditional heat conduction equation for $\beta=1, \beta=2$, which stands for the diffusion and wave equations. The study's findings are consistent with the research conducted in [1], when classical heat conduction takes place, following the Fourier law of heat conduction. As a result, it can be said that curve plotting indicates a considerable

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influence of time-fractional along the radial direction for thermal variation. This kind of analysis is helpful in describing the material properties that are applicable in a variety of solid structure designs using aircraft design.

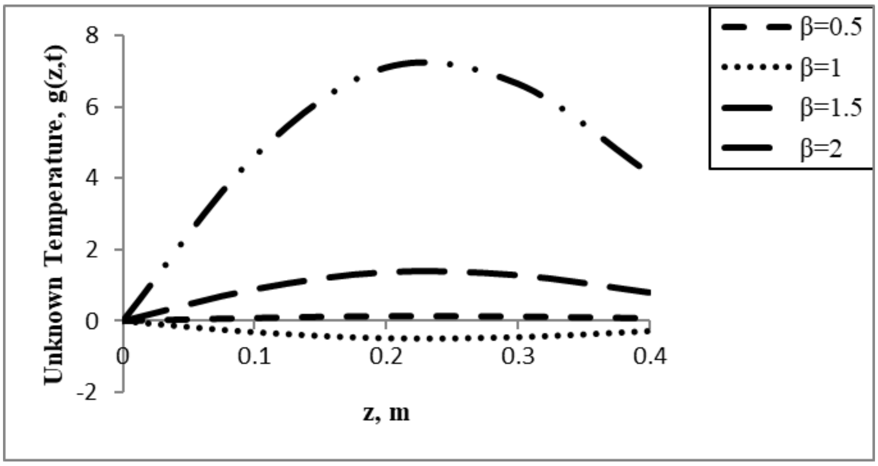

Figure 2. Unknown temperature distribution function.

A graph of the unknown temperature distribution for the distinct fractional order parameter β is shown in Figure 2. The temperature distribution first appears relatively quickly when going along the axial direction. As time passes, the effect of the temperature distribution increases, peaks in the middle, and then begins to decrease in the outer axial direction. Additionally, the discriminating in the curve variations shows considerable effects of the unknown temperature distribution for the various values of the fractional parameters. Physically, the temperature increases as thickness increases, demonstrating the temperature's strong dependence on various fractional parameter values and the ability to design a variety of solid structural designs with substantial thickness that are useful in the thermal industry. This approach is mostly helpful for thermal diagnostics of structures and component design optimization, where material behavior under heat and thermal stresses are crucial.

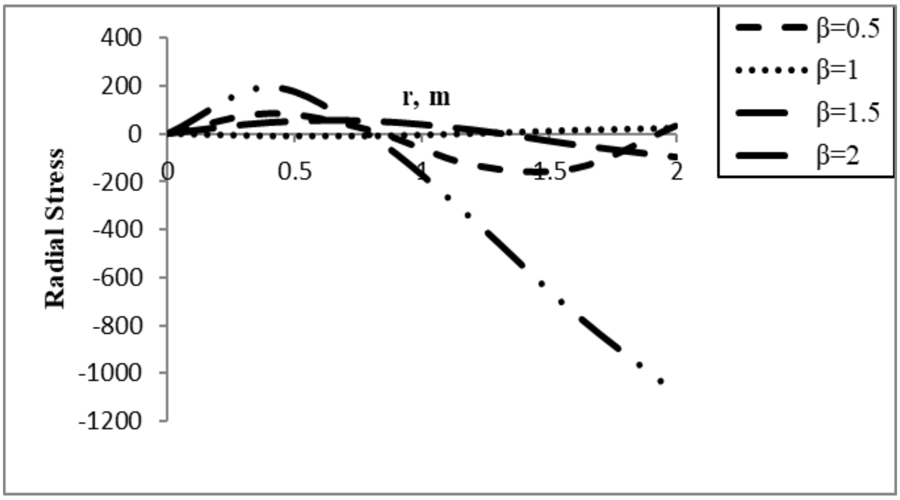

Figure 3. **Radial stress function.**

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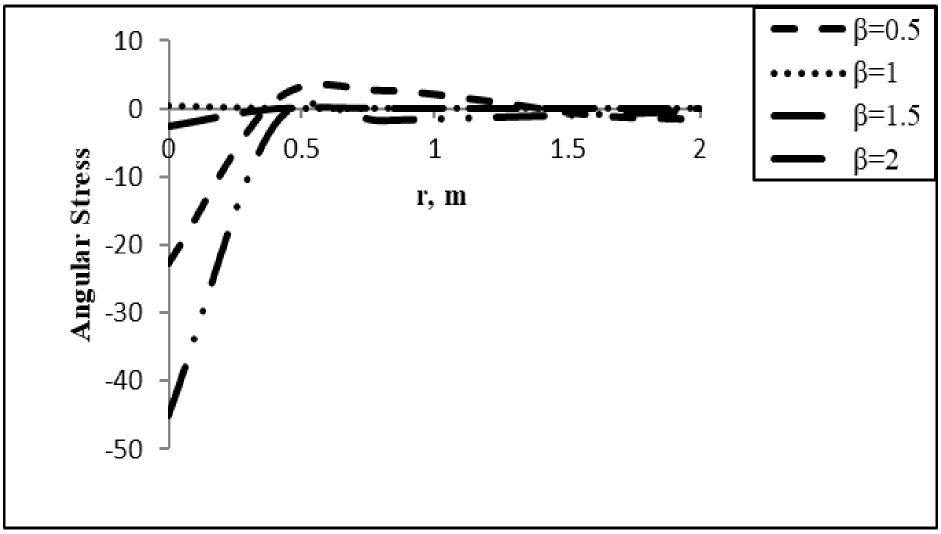

Figure 4. Angular stress function.

For varying values of fractional order parameters, the radial and angular stress distribution functions along the radial directions are displayed in Figures 3 and 4. In Figure 3, the radial stresses first rise as one move in a radial direction. Stresses continue to decline negatively as they quickly and roughly span half of the area. In contrast, as seen in Figure 4, the angular stresses first rise and then exhibit fluctuating behaviour toward the outer radii as time passes in the radial direction. The overall fluctuation of stress curves shows the overall effect of the fractional parameters, and the curves discrimination for different fractional parameter values adds to its significance and is crucial for determining how different structural designs behave before they are actually put into practice. To put it more precisely, the angular stresses exhibit a wave-like structure and are tensile throughout the region as the fractional order parameter increases.

Limiting Cases

The analytical component above makes it easy to study the solution and thermal fluctuation within predetermined boundaries of the Laplace equation, which is described by the heat equation when $\beta = 0$ is fixed.

The sub-diffusion $(0 < \beta < 1)$ implies anomalous heat conduction with divergent thermal conductivity, and the super-diffusion $(1 \le \beta \le 2)$ implies anomalous heat conduction with convergent thermal conductivity when $\beta = 1$, classical heat conduction, occurs, which is prescribed by Fourier's law of heat conduction.

Conclusion

This study investigates the response of a long solid cylinder to temperature changes over time using a particular approach called quasi-static, which is based on the heat conduction equation with a time-fractional derivative. It is founded on the time fractional heat equation and includes Caputo fractional derivatives. The graphs show how temperature, thermal stresses in the radial direction, and unknown temperature in the axial direction change with different values of time fractional order. The influence of the heat disturbance region varies throughout region $1 \le \beta \le 2$, illustrating the difference between fractional-order thermoelasticity and classical thermoelasticity. For weak, moderate, and superconductivity, the fractional-order thermoelasticity affects how fast heat waves move in materials with time-dependent heat conduction. Furthermore, this issue demonstrates a delayed reaction to physical stimuli, which is a characteristic of nature. In the context of fractional-order thermoelasticity, the study looks at the inverse thermoelastic problem in a long solid cylinder to show how traditional uncoupled thermoelasticity is adjusted for different levels of conductivity. Various time fractional parameters show a notable effect on the distribution of temperature, radial stress, and angular stress as time, thickness, and radius vary. Additionally, a graphical analysis of the thermal stress variation reveals weak, moderate, and superconductivity for various time fractional parameters. Thus, time fractional order parameters can be advantageous in physical processing and could play a significant role in the classification of material qualities.

Nomenclature

r	–	radius, m
z	–	thickness, m
k	–	thermal diffusivity
a_t	–	coefficient of linear thermal expansion
T	–	temperature distribution function
μ	–	Lamé constant
v	–	Poisson ratio
E	–	Young's modulus
ϕ	–	displacement potential function
b	–	radius of the disk, m
t	–	time
σ_{rr}	–	radial stress function
$\sigma_{\theta\theta}$	–	angular stress function
β	–	fractional order parameter for time
s	–	parameter of Laplace transform
η	–	Fourier transform variable
δ	–	Dirac-delta function,

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