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SECURE INTEGER DOMINATION FOR FORBIDDEN GRAPHS

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Abstract

An Integer dominating function on a graph G is a function $f: V(G) \to W$ such that for every vertex $v \in V(G)$, $\Sigma_{v \in V(G)}(N[v]) \geq k$. For any function $f: V(G) \to W$ and any pair of adjacent vertices with $f(v) = 0$ and $u > 0$, the function g_{uv} is defined by $g_{uv}(l) = 1$, $g_{uv}(l) = f(u) - 1$ and $g_{uv}(l) = f(l)$ if $l \in V - \{u, v\}$. A secure integer dominating function (SIDF) on a graph G is defined as an integer dominating function g which satisfies the condition that for every vertex v with $f(v) = 0$, $\exists$ a neighbor u with $f(u) > 0$ is such that g_{uv} is an integer dominating function. The weight of f is $w(f) = \Sigma_{v \in V(G)} f(v)$. The minimum weight among all the dominant secure integer functions in G is the number of secure integer domination in G. This paper is devoted to initiating the study of secure integer domination for forbidden graphs.

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Key Words and Phrases: Domination, Secure domination, Integer domination, forbidden graphs.

1. Introduction

In this article, a graph G that is simple and undirected, with a vertex set $V(G)$ and an edge set $E(G)$. The order of the graph, meaning the total number of vertices, is represented by $n = |V(G)|$. For each vertex $v \in V(G)$, the open neighbourhood of v, written as $N(v)$, consists of all vertices adjacent to v, $N(v) = \{u \in V(G) \mid uv \in E(G)\}$. The closed neighbourhood of v, denoted by $N[v]$, includes all neighbours of v along with v itself, $N[v] = N(v) \cup v$. The degree of a vertex v, written as $d(v)$, refers to the number of vertices connected directly to v. The minimum degree across all vertices in the graph is denoted by $\delta(G)$ and is defined as, $\delta(G) = min\{d(v) \mid v \in V(G)\}$. A set $S \subseteq V$ of vertices of a graph $G = (V, E)$ is said to be a dominating set when all the vertices $v \in V(G)$ belong either to S or are neighbours of a S element. The domination number of G, $\gamma(G)$ is the smallest possible cardinality of a dominating set of G. For all the basic terminology of graph theory we refer to [19].

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In graph theory, domination problems often become more tractable or exhibit specific structural properties when certain subgraphs are forbidden. A forbidden graph characterization is a method used to define families of graphs by specifying substructures that are not allowed to exist within any graph in that family. These forbid- den substructures can be general subgraphs, induced subgraphs, isomorphic subgraphs (topological minors), or graph minors. The set of these forbidden structures is also known as an obstruction set.

Determining the dominating structure of graphs is a challenging problem, which is one of the main motivations for studying graphs through the lens of forbidden subgraphs. The study of dominating subgraphs began in 1962 with the work of Wolk [18], who characterized P_4, C_4-free graphs. This line of research was further developed by M. B. Cozzens and L. L. Kelleher [10] who, in 1990, published a paper titled "Dominating Cliques in Graphs." A dominating clique is defined as a dominating set that induces a complete subgraph. Recent studies focus on analysing graphs with forbidden subgraphs, particularly those where the dominating set has size three.

In the $\{P_5, C_5\}$- free graphs, G. Basco and Z. Tuza [3] examined dominating cliques and considered dominating properties and induced subgraphs. They demonstrated that a graph G has every induced subgraph containing a dominating set if and only if G is $\{P_5, C_5\}$-free. This outcome helped to decrease the issue of determining domination graph numbers to the study of (P_t, C_t)-free graphs.

A later paper by J. Liu and H. Zhou [13] produced an interesting result of its own, Graphs with Forbidden Structure in Graphs with Some Forbidden Structure, they studied graphs which are $\{C_3, P_6, C_6\}$-free. They proved that when G is $\{C_3, P_6, C_6\}$-free then any connected induced subgraph of G is dominated by complete bipartite graph. Their efforts also led to the characterization of graphs which were free of (P_6, C_6). In G. Basco et al. [4] defined induced path dominated graphs in 2006. They focused on graphs whereby all the induced subgraphs that are connected contain a dominating induced path with not more than 4 vertices. Besides, E. Camby et al. [8] gave an alternative definition of P_k-free graphs. They proved that a graph G has no (P_k, C_k)-free subgraphs, i.e. G is (P_k, C_k)-free if and only if every connected induced subgraph of G contains a connected dominating set that is either P_k-free or a cycle C_k.

The secure domination number, and in particular the secure integer domination number, has gained interest due to its relevance in network security and fault tolerant systems. Studying how these parameters behave in H-free graphs (for various forbidden families H) may lead to new characterizations and bounds, as well as algorithmic approaches for efficiently computing these values.

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2. Secure Integer Dominating Function

The concept of secure integer domination was first introduced by Gowtham Priya et al. in [20]. This innovative extension of domination theory incorporates both security considerations and integer weight assignments to vertices, thereby enriching the structural analysis of graphs. Unlike classical domination, where each vertex is either dominated or not, secure integer domination assigns positive integers as dominating strengths, ensuring that every non-dominating vertex can be securely reassigned or "defended" by an adjacent dominant vertex without violating the domination condition. This framework not only enhances granularity in domination models but also captures stability and resilience within graph structures, making it highly relevant for applications in network security, resource allocation, and fault tolerant systems.

Definition 1: [20] A function $g: V(G) \rightarrow \{0, 1, 2, \ldots, k\}$ is called secure integer dominating function (SIDF) of G if it satisfies the following:

(1) $\forall y \in V(G)$, $\Sigma_{y \in V(G)} g(N[y]) \geq k$.
(2) $\forall y \in V_0$ $\exists z \in N(y) - V_0$ such that g_{yz} is an integer dominating function on G.

The secure integer dominating function has a weight equal to the value of $w(y) = \Sigma_{y \in V(G)} g(y)$. The minimum weight of secure integer dominating function is denoted by $\gamma_k^s(G)$ is called secure integer domination number (SIDN).

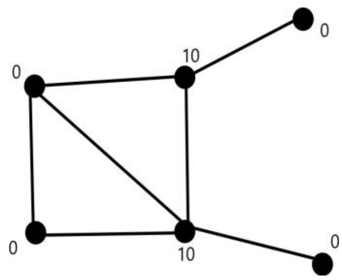

Figure 1: Graph with $\gamma_k(G) = 10$, when k=10.

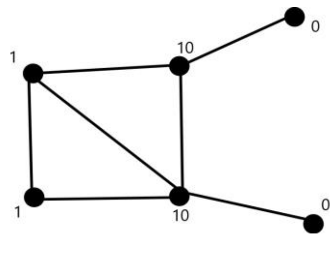

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Figure 2: Graph with $\gamma_k^s(G)$ = *22, when k=10.*

3. Results

Theorem 1. If G is {P_4, C_4} - free graph, then $\gamma_k(G) = k$.

Proof. Let G be a simple graph, which is $\{P_4, C_4\}$ – free. When G is $\{P_4, C_4\} - free$ maximum distance between any vertices will be two. That implies that G is complete graph. Let $v \in G$, for every other vertex $u \in G$, there is an edge between u and v. We can say that a vertex $v \in G$ is a dominating vertex. By the definition of SIDF,

1) If $v = k$, then $v_{n-1} = 0$.
2) If $v \leq k$, then $v_{n-1} = \{0, 1, 2, ..., k\}$.

Since we know that, cardinality of $V(G) = k$.

Theorem 2 . If G is {P_4, C_4} - free graph, then $\gamma_k^s(G) = k + (n - 2)$.

Proof. Let G be a connected $\{P_4, C_4\}$-free graph with n vertices and clique number $\omega(G)$. Since G is trivially perfect, it contains a universal vertex u adjacent to all other vertices. Consider a SIDF g: $V(G) \to \{0, 1, 2, . . . , k\}$. If G is complete ($\omega(G) = n$), assigning $g(u) = k$ and zero to all other vertices satisfies both the integer domination and security conditions, giving $\gamma_k^s(G) = k = k + n - \omega(G)$. If G is not complete, then $\omega(G) < n$, assigning $g(u) = k + 1$ and zero elsewhere ensures that all vertices are dominated and that any zero assigned vertex can be reinforced from u, satisfying the SIDF conditions. In this case, $\gamma_k^s(G) = k + 1 = k = (n- \omega(G)$, so the secure integer domination number is bounded in terms of k and the clique number by $k \neq \gamma_k^s(G) \neq k + (n - \omega(G)$. The lower bound is attained when G is complete, and the upper bound provides a tight estimate for noncomplete connected trivially perfect graphs.

Theorem 3. If G is {P_5, C_5} - free graph, then $\gamma_k^s(G) = (k + 1) max\{\omega(G), 3\}$.

Proof. Let G be a connected $\{P_5, C_5\}$-free graph. By the structural result of Bacsó and Tuza, such a graph admits a dominating set D that is either a clique of size at most $\omega(G)$ or an induced P_3. Assign $g(v) = k + 1$ for each $v \in D$ and $g(x) = 0$ otherwise. Since every vertex of G is dominated by D, each closed neighbourhood has weight at least k+ 1, satisfying the integer domination requirement. Moreover, for any vertex y with $g(y) = 0$, there exists a neighbour $z \in D$ from which one unit can be transferred after this transfer, the weight of each closed neighbourhood decreases by at most one, leaving every closed neighbourhood sum at least k, thereby satisfying the security condition. Hence this assignment is a secure integer dominating function of weight (k+1) | D |, showing that $\gamma_k^s(G) \neq (k + 1) \mid D \mid$. Since $\mid D \mid \leq max\{\omega(G), 3\}$, it follows that $\gamma_k^s(G) \leq max\{\omega(G),3\}$.

Theorem 4. If G is {P_6, C_6, C_3}-free graph, then $\gamma_k^s(G) = (k + 1)(a + b)$.

Proof. Let G be a connected $\{P_6, C_6, C_3\}$-free graph. By the characterization of P_6 -free graphs, every connected induced subgraph of a P_6-free graph contains either a dominating

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induced C_6 or a dominating (not necessarily induced) complete bipartite subgraph $K_{a,b}$, since G forbids C_6 the former possibility is excluded and hence G admits a dominating complete bipartite subgraph $K_{a,\,b}$ with vertex set D of size $|D| = a + b$. Define $g : V(G) \rightarrow \{0, 1, . . . , k\}$ by $g(v) = k + 1$ for $v \in D$ and $g(v) = 0$ otherwise. Every vertex has a neighbour in D, so each closed neighbourhood initially receives at least $k + 1$ and the integer domination condition $\sum_{x\,\epsilon N[y]}(g(x)) \geq k$ holds for all y. For any zero vertex y there is a defender $z \in D$ adjacent to y, and moving one unit from z to y leaves z with k and reduces every closed-neighbourhood sum by at most one, so the security condition is preserved. Thus, $\gamma_k^s(G)= (k + 1)\,(a + b)$.

Theorem 5. If G is $\{P_6, C_6, Claw\}$- free graph, then $\gamma_k^s(G) \leq k\,\omega(G)$.

Proof. Let G be a connected $\{P_6, C_6, Claw\}$- free graph. By the structural properties G contains a dominating clique of size at most $\omega(G)$. Assign $g(v) = k$ to each vertex in D and $g(x) = 0$ to all vertices outside D. Then for every vertex $y \in V(G)$, its closed neighbourhood $\sum_{x\,\epsilon N[y]}(g(x)$ is at least k, satisfying the integer domination requirement. Moreover, for any zero vertex $y \in D$, transferring one unit from a neighbour in D leaves all closed neighbourhood sums still at least k, satisfying the security condition. Therefore, this defines a secure integer dominating function with weight at most $k\,|D| \leq k\,\omega(G)$, giving $\gamma_k^s(G) \leq k\,\omega(G)$, while the integer domination condition ensures $\gamma_k^s(G) \leq k\,\omega(G)$. Hence, for any connected $\{P_6, C_6, claw\}$-free graph, the secure integer domination number satisfies, $\gamma_k^s(G) \leq k\,\omega(G)$.

Theorem 6. *If G is $\{P_5, K_4^-\}$- free graph, then* $\gamma_k^s(G) \leq k\,|V(G)| \leq k\,\omega(G)$.

Proof. Let C be a dominating cycle in G and assume every vertex off C has at least two neighbours on C. Define $g : V(G) \rightarrow \{0, 1, . . . , k\}$ by $g(v) = k$ for $v \in V(C)$ and $g(x) = 0$ otherwise. For any vertex y, its closed neighbourhood meets C, so $\sum_{x\,\epsilon N[y]}(g(x)) \geq k$, satisfying integer domination. Now take any zero vertex y (so $y \notin C$), by hypothesis y has two distinct neighbours $z_1, z_2 \in C$. Move one unit from z_1 to y; then $g'(z_1) = k - 1$ and $g'(y) = 1$, and all other values unchanged. Any closed neighborhood that previously contained at least two cycle vertices still contains at least one cycle vertex with value k, and any closed neighborhood that contained only one of z_1 and z_2 now contains the other with value k (or contains y with value 1 plus other k-contributions). Hence every closed neighborhood sum after the move is at least k, so the security condition holds. Therefore g is an SIDF of weight $k\,|V(C)|$, giving the claimed bound; since $|V(C)| \leq \omega(G)$ the last inequality follows.

4. Conclusion

In this study, we investigated the secure integer domination number in graphs forbidding specific induced subgraphs such as P_4, C_4, P_5, C_5 and claw-free structures, and established that these constraints significantly influence domination behavior by reducing complexity and enhancing domination stability. Our results show that the absence of long induced paths, short cycles, and claw configurations leads to tighter bounds and more efficient secure dominating sets, demonstrating improved resilience and control in such graph classes. This study not only advances the theoretical framework of secure integer domination under forbidden subgraph conditions but also provides a strong foundation for future research in optimizing domination parameters in structurally restricted networks.

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