International Journal of Applied Mathematics

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

r-PRIMITIVE NEAR-RINGS

Akin. J. S[1]   Pramila Inpa Rose. A[2]

[1]Research Scholar (Reg.no:21213112091007), Department of Mathematics,

Nesamony Memorial Christian College, Marthandam-629 165, Tamilnadu, India.

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamilnadu, India.

[2]Associate Professor and Head, Department of Mathematics,

Nesamony Memorial Christian College, Marthandam-629 165, Tamilnadu, India.

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamilnadu, India.

Email: [1]akinjsjohn@gmail.com [2]pramilapream@gmail.com

Abstract:

In this article, we are going to introduce the concept of r-primitive on the near-rings. For that first we give the definition of semi-type on the N-groups of near-rings. Then we study the properties semi-type N-groups. Using semi-type N-groups we define the definition of r-primitive near-rings. Finally, we study the properties of r-primitive near-rings.

Keywords: N-group, ideal, simple, monogenic, faithful, completely reducible.

2020 Mathematics Subject Classification (AMS): 16Y30.

1. Introduction:

In 1905, Dickson introduced the concept near-field [3]. It was the first near-ring founded. The first one to use the term "near-ring" were Zassenhaus in 1936 [5]. N-groups is the analogue of the concept of a module in ring theory. In this article we are going to use right near-rings and left near modules (left N-modules) as N-groups. Primitivity in near-rings are based on the concepts like faithful, monogenic and simple of N-groups of near-rings. In near-ring primitivity is called ν-primitive near-rings where $\nu \in \{0,1,2\}$ and mainly classified into three types as 0-primitive, 1-primitive and 2-primitive near-rings based on the types of N-groups of the near-rings. In this article we learn the properties of semi-type N-groups and r-primitive near-rings through examples, notes, remarks, propositions and theorems.

2. Preliminaries

Definition 2.1:[4]

A near-ring is a non-empty set N together with two binary operations " $+$ " and " $\cdot$ " such that

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

a) (N, +) is a group (not necessarily abelian).
b) (N, ·) is a semigroup.
c) $\forall\, n_1, n_2, n_3 \in$ N such that $(n_1+n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$ ("right distributive law").

Notation 2.2: [4]

The class of all near-rings will be denoted by $\mathcal{N}$.

Definition 2.3: [4]

- $N_0 = \{n \in N \mid n0 = 0\}$ is called the zero-symmetric part of N.
- $N_c = \{n \in N \mid n0 = n\} = \{n \in N \mid \forall\, n' \in N{:}\, nn' = n\}$ is called the constant part of N.

N_0 and N_c are itself near-rings.

Definition 2.4: [4]

N $\in \mathcal{N}$ is called zero symmetric (constant) if N = N_0 (N = N_c, respectively).

$\mathcal{N}_0$ ($\mathcal{N}_c$) stand for all classes of all zero symmetric (constant) near-rings.

Definition 2.5: [4]

Let (Γ, +) be a group with zero 0 and let N $\in \mathcal{N}$. Let $\mu : N \times \Gamma \to \Gamma\ \{(n,\gamma) \to n\gamma\}$. (Γ, μ) is called an N-group ("near-module over N") if $\forall\, \gamma \in \Gamma, \forall\, n, n' \in N$ such that $(n + n')\gamma = n\gamma + n'\gamma$ and $(nn')\gamma = n(n'\gamma)$. It is denoted by N^{Γ}.

Remark 2.6: [4]

If $\Gamma = N$ then we write it as N^{N}.

Notation 2.7: [4]

Let $N^{\mathcal{G}}$ be the class of all N-groups.

Definition 2.8: [4]

Let N $\in \mathcal{N}$. A normal subgroup I of $(N, +)$ is called ideal of N (I ⊴ N) if

- IN ⊆ I.
- $n(n' + i) - nn' \in$ I for all $n, n' \in$ N and $i \in$ I.

Notes 2.9:

- Normal subgroups R of (N, +) with RN ⊆ R are called right ideals of N ($R \trianglelefteq_r$ N).[4]
- Normal subgroups L of (N, +) with $n(n' + l) - nn' \in$ L for all $n, n' \in$ N & $l \in$ L are said to be left ideals (L $\trianglelefteq_l$ N).[4]
- If I is an ideal of N, then the factor group N/I becomes factor near-ring under the usual multiplication such that $(n_1 + I)(n_2 + I) = n_1n_2 + I$, $n_1, n_2 \in$ N. If Γ is an N/I-group (N-group) of near-ring N/I, then it is denoted by N/I^{Γ}.[2]

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Definition 2.10: [4]

Let $\mathrm{N} \in \mathcal{N}$ and $\mathrm{N}^{\Gamma} \in \mathrm{N}^{\mathcal{G}}$. A normal subgroup Δ of $(\Gamma, +)$ is called ideal of N^{Γ} ($\Delta \trianglelefteq_{\mathrm{N}} \Gamma$) if $\forall\, \gamma \in \Gamma\; \forall\, \delta \in \Delta\; \forall\, n \in \mathrm{N}$ $such\ that\ n(\gamma + \delta) - n\gamma \in \Delta$.

Remark 2.11: [4]

- $\{0\}$ and N are ideals of N.
- $\{0\}$ and Γ are ideals of N^{Γ}.

These ideals are called the trivial ideals.

Definition 2.12: [4]

- N (N^{Γ}) is simple if N (N^{Γ}) has no non-trivial ideals.
- N^{Γ} is called N-simple if N^{Γ} has no N-subgroup except Ω and Γ.

Definition 2.13: [4]

Let Δ_1, Δ_2 be subsets of $\mathrm{N}^{\Gamma} \in \mathrm{N}^{\mathcal{G}}$. Define $(\Delta_1 : \Delta_2) = \{\, n \in \mathrm{N} \mid n\Delta_2 \subseteq \Delta_1 \}$. $(0 : \Delta)$ is called the annihilator of Δ.

Definition 2.14: [4]

N^{Γ} is called faithful if $(0 : \Gamma) = \{0\}$.

Definition 2.15: [4]

N^{Γ} is monogenic if there exists $\gamma \in \Gamma$ such that $\mathrm{N}\gamma = \Gamma$.

i.e. we say that N^{Γ} is monogenic by γ and γ is called a generator of N^{Γ}.

Example 2.16:

Let near-ring N and it's N-group N^{Γ} be Klein's four group {0, a, b, c}. It's addition and multiplication are defined by following tables

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	b	0
c	a	a	c	a

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

{scheme 21, pg.no: 21 in [4]}

Then N^Γ is monogenic by $\gamma = b$.

Remark 2.17:

From definition of monogenic, if the N-group Γ is monogenic by γ then every element of N-group Γ is of form $n\gamma$ for any n belongs to near-ring N.

Note 2.18: [4]

Let N^Γ be monogenic by γ_0 then $L \trianglelefteq_l N \Rightarrow L\gamma_0 \trianglelefteq_N \Gamma$.

Definition 2.19: [4]

N^Γ is called strongly monogenic if N^Γ is monogenic and $\forall\, \gamma \in \Gamma$ such that $N\gamma = \{0\}$ or $N\gamma = \Gamma$.

Note 2.20:

Monogenic N-group N^Γ of near-ring N has more than one generator.

Example 2.21:

For the near-ring $N = Z_5$ under addition modulo 5 and multiplication is defined by

.	0	1	2	3	4
0	0	0	0	0	0
1	0	0	4	1	0
2	0	0	3	2	0
3	0	0	2	3	0
4	0	0	1	4	0

{scheme 6, pg.no:408 in [4]}

Let $\Gamma = Z_5$ be an N-group of near-ring $N = Z_5$. Then Γ has more than one monogenic generator.

Definition 2.22: [4]

Let $(I_k)_{k\in K}$ be ideals of N. Their sum $\sum_{k\in K} I_k$ is called an (internal) direct sum if each element of $\sum_{k\in K} I_k$ has a unique representation as a finite sum of elements of different I_k's.

In this case we write for the sum $\dot{\sum}_{k\in K} I_k$ (or $I_1 \dotplus I_2 \dotplus \ldots$) {In this article we denoted it by $\bigoplus_{k\in K} I_k$}.

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Definition 2.23: [4]

N (N^{Γ} with N $\in \mathscr{N}_0$) is called completely reducible if N (N^{Γ}) is the direct sum of simple ideals.

Remark 2.24: [4]

$N = N_0 \Leftrightarrow (\Delta \trianglelefteq_N \Gamma \Rightarrow \Delta \leq_N \Gamma)$ for all $\Gamma \in N^{\mathscr{G}}$.
Therefore, if N $\in \mathscr{N}_0$, then every ideal of N^{Γ} are N-subgroups of N^{Γ}.

Note 2.25: [4]

- For all $\Delta \leq_N \Gamma$, we have $(0{:}\,\Delta) \trianglelefteq N$.
- For all $\gamma \in \Gamma$, we have $(0{:}\,\gamma) \trianglelefteq_l N$.

Remark 2.26: [4]

Let Δ be a subset of $N^{\Gamma} \in N^{\mathscr{G}}$, then $(0 : \Delta) = \bigcap_{\delta\in\Delta}(0 : \delta)$.

Note 2.27:

Let N $\in \mathscr{N}_0$ and N^{Γ} be a completely reducible. If Δ is an ideal of N^{Γ} then Δ is simple or contains a simple ideal of N^{Γ}.

Remark 2.28: [2]

If the N-group N^{Γ} of near-ring N is completely reducible, then every ideal of N^{Γ} is a sum of simple ideals.

Definition 2.29: [4]

Let $N, N' \in \mathscr{N}$. If there exists a monomorphism $N \to N'$ we say that N is embeddable in N' and we write $N \hookrightarrow N'$.

Note 2.30: [4]

Each N-group can be embedded into some faithful N-group.

Definition 2.31: [4]

A monogenic N-group Γ with $\Gamma \neq \{0\}$ is said to be of

a) type 0 if N^{Γ} is simple.
b) type 1 if N^{Γ} is simple and strongly monogenic.
c) type 2 if Γ is N_0-simple.

Definition 2.32: [4]

A near-ring N is called ν-primitive where $\nu \in \{0,1,2\}$ on N^{Γ} if N^{Γ} is faithful and of type ν where $\nu \in \{0,1,2\}$. A near-ring N is ν-primitive where $\nu \in \{0,1,2\}$ if there exists $N^{\Gamma} \in N^{\mathscr{G}}$ such that N is ν-primitive where $\nu \in \{0,1,2\}$ on N^{Γ}.

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

3. Semi-type:

Definition 3.1:

An N^{Γ} of a near-ring N is of semi-type if N^{Γ} is monogenic and completely reducible.

Theorem 3.2:

Let $N \in \mathcal{N}_0$. If N^{Γ} is of semi-type, then $N\gamma$ is the direct sum of simple ideals, where γ is a monogenic generator.

Proof:

Let $N \in \mathcal{N}_0$.

Let Γ be an N-group of near-ring N.

Given Γ is of semi-type,

then Γ is monogenic and completely reducible.

Therefore, there exist $\gamma \in \Gamma$ such that $N\gamma = \Gamma$ and Γ is the direct sum of simple ideals.

$N\gamma = \bigoplus_{\alpha \in A} \Delta_{\alpha}$, where $\boldsymbol{\Delta_{\alpha}}$'s are simple ideals.

$N\gamma$ is the direct sum of simple ideals, where γ is a monogenic generator.

Theorem 3.3:

Let $N \in \mathcal{N}_0$. If N^{Γ} is of semi-type, then $N\gamma$ is completely reducible.

Proof:

Given $N \in \mathcal{N}_0$.

Let Γ be an N-group of near-ring N.

By theorem 3.2,

$N\gamma$ is the direct sum of simple ideals.

$\Rightarrow N\gamma$ is completely reducible.

Theorem 3.4:

Let $N \in \mathcal{N}_0$. If N^{Γ} is of semi-type, then $N\gamma$ is the sum of simple ideals.

Proof:

Given $N \in \mathcal{N}_0$

Let Γ be an N-group of semi-type.

By theorem 3.2,

Received: August 04 2025

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$N\gamma$ is the direct sum of simple ideals.

Therefore, $N\gamma$ is the sum of simple ideals.

Theorem 3.5:

Let N^{Γ} be of semi-type. If γ_0 is a monogenic generator of N^{Γ}. Then $L\gamma_0$ is simple ideal or contains simple ideals of N^{Γ}, for any left ideal L of N.

Proof:

Let N be a near-ring

Let Γ be a semi-type N-group.

Therefore, Γ is completely reducible and monogenic.

Let $\gamma_0 \in \Gamma$ be a monogenic generator of Γ.

Given L is a left ideal of N

By note 2.18,

$L\gamma_0 \trianglelefteq_N \Gamma$

Case (i)

L is a simple ideal.

To prove, $L\gamma_0$ is a simple ideal

Suppose L_1 is a non-trivial ideal of $L\gamma_0$

i.e. $L_1 \subset L\gamma_0$

$L_1 = \{l_1 | l_1 = l_1\gamma_0\}$

Also, Γ is completely reducible.

i.e. Γ is the direct sum of simple ideals.

$\Rightarrow \Gamma = \oplus_{\alpha\in A}\Delta_\alpha$, where Δ_α are simple ideals

Suppose L_1 is simple

$L_1 \subset L\gamma_0 \subset \Gamma$

$L_1 \subset \Gamma$

L_1 is one of the subgroups of Γ.

Since L_1 is non-trivial,

$$L_1 \neq \{0\} \text{ and } L_1 \neq L\gamma_0$$

Let $l_k \in L\gamma_0$ but $l_k \notin L_1$

$\Rightarrow l_k = l_k\gamma_0$, for $l_k \in L$

Received: August 04 2025

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

By definition of L_1,

$l_k \in L_1$

therefore, every element of $L\gamma_0$ is element of L_1

hence $L_1 = L\gamma_0$

therefore, $L\gamma_0$ has no non-trivial ideals.

Thus $L\gamma_0$ is a simple ideal of N-group Γ.

Case (ii)

If L is not simple

By note 2.18,

$L\gamma_0 \trianglelefteq_N \Gamma$

By note 2.27,

Therefore $L\gamma_0$ is simple or contains a simple ideal of N^Γ.

Theorem 3.6:

Let $N \in \mathscr{N}_0$. If N^Γ is a semi-type, then for all $\alpha \in A$, $\bigcap_{\delta\in\Delta_\alpha}(0{:}\,\delta)\,\gamma \trianglelefteq_N N^\Gamma$ where γ is a monogenic generator of N^Γ and $\boldsymbol{\Delta_\alpha} \leq_N N^\Gamma$.

Proof:

Let $N \in \mathscr{N}_0$ and Γ be an N-group of semi-type.

Therefore, Γ is monogenic and completely reducible

Given, Γ is monogenic by γ.

Since Γ is completely reducible,

Γ is the direct sum of simple ideals

i.e. $\Gamma = \bigoplus_{\alpha\in A} \boldsymbol{\Delta_\alpha}$, where $\boldsymbol{\Delta_\alpha}$'s are simple ideals

By remark 2.24,

Hence each $\boldsymbol{\Delta_\alpha}$'s are N-subgroups of Γ where $\alpha \in A$.

$\boldsymbol{\Delta_\alpha} \leq_N \Gamma$, for all $\alpha \in A$

By note 2.25,

$(0{:}\,\boldsymbol{\Delta_\alpha}) \trianglelefteq N$, for all $\alpha \in A$

by remark 2.26,

$\Rightarrow (0{:}\,\boldsymbol{\Delta_\alpha}) = \bigcap_{\delta\in\Delta_\alpha}(0{:}\,\delta)$, for all $\alpha \in A$

$\Rightarrow \bigcap_{\delta\in\Delta_\alpha}(0{:}\,\delta) \trianglelefteq N$, for all $\alpha \in A$ and where $\boldsymbol{\Delta_\alpha} \leq_N \Gamma$

Received: August 04 2025

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$\Rightarrow \bigcap_{\delta\in\Delta_\alpha}(0:\delta) \trianglelefteq_l$ N, for all $\alpha \in$ A and where $\Delta_\alpha \leq_N \Gamma$

Since Γ is monogenic by γ

By note 2.18,

$\bigcap_{\delta\in\Delta_\alpha}(0:\delta)\gamma \trianglelefteq_N \Gamma$, for all $\alpha \in$ A and where $\Delta_\alpha \leq_N \Gamma$

Theorem 3.7:

If N^Γ is monogenic by γ_0, then $(0:\gamma)\gamma_0 \trianglelefteq_N N^\Gamma$, for all $\gamma \in \Gamma$.

Proof:

Let Γ be a monogenic N-group by $\gamma_0 \in \Gamma$

By note 2.25,

$\Rightarrow (0:\gamma) \trianglelefteq_l$ N, for all $\gamma \in \Gamma$

By note 2.18,

$\Rightarrow (0:\gamma)\gamma_0 \trianglelefteq_N \Gamma$.

Theorem 3.8:

Let $N \in \mathscr{N}_0$. If N^N is completely reducible then $(0:\Delta_\alpha)$ and $N/(0:\Delta_\alpha)$ are completely reducible where Δ_α are simple ideals of N^N.

Proof:

Given $N \in \mathscr{N}_0$

Let N itself be of semi-type N-group.

Therefore, N-group N is completely reducible and monogenic

i.e. N^N is completely reducible and monogenic.

Since N^N is completely reducible,

N^N is the direct sum of simple ideals

$\Rightarrow N^N = \bigoplus_{\alpha\in A} \Delta_\alpha$, where Δ_α's are simple ideals

By remark 2.24,

$\Rightarrow \Delta_\alpha \leq_N N^N$, for all $\alpha \in$ A

By note 2.25,

$\Rightarrow (0:\Delta_\alpha) \trianglelefteq$ N, for all $\alpha \in$ A

By theorem 2.48 in [4]

Since N^N is completely reducible.

$(0:\Delta_\alpha)$ and $N/(0:\Delta_\alpha)$ are completely reducible.

Received: August 04 2025

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 3.9:

Let $N \in \mathcal{N}_0$. If N^{Γ} is monogenic and is the direct summand of simple ideals, then N^{Γ} is of semi-type.

Proof:

Given, $N \in \mathcal{N}_0$.

Let Γ be a monogenic N-group and direct summand of simple ideals.

By theorem 4.12 in [1]

Γ is completely reducible.

Since Γ is monogenic,

Γ is of semi-type.

Theorem 3.10:

If N^{Γ} is of semi-type, then $N\gamma$ is the sum of minimal ideals.

Proof:

Let N be a near-ring and Γ be its N-group.

Given Γ is of semi-type

$\Rightarrow \Gamma$ is monogenic and completely reducible

$\Rightarrow$ there exist $\gamma \in \Gamma$ such that $N\gamma = \Gamma$

Since Γ is completely reducible,

By theorem 2.48 in [4], Γ is the sum of minimal ideals.

Since $N\gamma = \Gamma$,

Hence, $N\gamma$ is the sum of minimal ideals.

Theorem 3.11:

Let N^{N} be of semi-type. If L is a left ideal of N then $L\gamma$ is simple or sum of simple ideals of N^{N} where γ is a monogenic generator of N^{N}.

Proof:

Let N be a near-ring.

Let $\Gamma = N$ be an N-group of near-ring N.

i.e. near-ring N itself is an N-group.

Given N-group $\Gamma = N$ is of semi-type

Therefore, $\Gamma = N$ is monogenic and completely reducible

Received: August 04 2025

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Let γ be a monogenic generator of $\Gamma = N$.

By note 2.18,

$L\gamma \trianglelefteq_N \Gamma$

$L\gamma \trianglelefteq_N N^N$

If $L\gamma$ is one of the simple ideals of $\Gamma = N$.

$\Rightarrow L\gamma$ is simple

If not,

By remark 2.28, $L\gamma$ is the sum of simple ideals of N^N.

Theorem 3.12:

Let N^Γ be of semi-type. If N is completely reducible then $L\gamma$ is the sum of simple ideals where L is ideal of N.

Proof:

Let N be a near-ring.

Let Γ be a N-group of N.

Given Γ is of semi-type

Therefore, Γ is monogenic and completely reducible

Let L be a left ideal of N.

If $\Gamma = N$

By theorem 3.11,

$L\gamma$ is sum of simple ideals of N^N.

If $\Gamma \neq N$

Since Γ is monogenic, Γ is monogenic by γ.

By theorem 3.5,

$L\gamma$ is simple ideal or contains simple ideals of N^Γ.

Suppose $L\gamma$ contains simple ideal of N^Γ.

To prove, $L\gamma$ is the sum of simple ideals of Γ.

Suppose not, $L\gamma$ is not a sum of simple ideals of Γ.

Let $\{\Delta_\alpha\}_{\alpha\in A}$be the collection of simple ideals of N^Γ.

Therefore $\Gamma = \bigoplus_{\alpha\in A}\Delta_\alpha$

By our assumption,

Received: August 04 2025

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Let $\{\Delta_\beta\}_{\beta<\alpha\in A}$ be the collection of simple ideals contained in Lγ

Since Lγ is not sum of simple ideal of Γ,

Lγ ≠ sum of Δ_β, where $\beta < \alpha \in A$

Then there is a subset Δ_γ of Lγ which is not an ideal such that Lγ = sum of Δ_β & Δ_γ.

For all $\delta_\gamma \in \Delta_\gamma$ and $n \in N$ and $\gamma' \in \Gamma$,

$$\Rightarrow n(\gamma' + \delta_\gamma) - n\gamma' \notin \Delta_\gamma$$

Since Γ is monogenic by γ,

Every element in Γ is of form nγ, for any $n \in N$.

Since $\gamma' \in \Gamma$,

$\gamma' = n_1\gamma$, for any $n_1 \in N$ and

Since $\delta_\gamma \in \Delta_\gamma \subset \Gamma$,

$\delta_\gamma = n'\gamma$, for any $n' \in N$

Consider

$$n(\gamma' + \delta_\gamma) - n\gamma' \notin \Delta_\gamma$$

$$n(n_1\gamma + n'\gamma) - n(n_1\gamma) \notin \Delta_\gamma$$

$$n((n_1 + n')\gamma) - (nn_1)\gamma \notin \Delta_\gamma$$

$$n(n_2\gamma) - n_3\gamma \notin \Delta_\gamma$$

$$(nn_2)\gamma - n_3\gamma \notin \Delta_\gamma$$

$$n_4\gamma + n_5\gamma \notin \Delta_\gamma$$

$$(n_4 + n_5)\gamma \notin \Delta_\gamma$$

$$n_6\gamma \notin \Delta_\gamma \subset \Gamma$$

$$n_6\gamma \notin \Gamma, \text{ for } n_6 \in N$$

Which is contradiction

Our assumption is wrong

Lγ is the sum of simple ideals.

Theorem 3.13:

Let I be an ideal of near-ring N. If N^Γ is of semi-type then N/I^Γ is also of semi-type.

Proof:

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Let I be an ideal of near-ring N.

Let Γ be an N-group.

First to prove, Γ is an N/I -group of the near-ring N/I.

i.e. to prove

for all n_1+ I, n_2+ I ∈ N/I and γ ∈ Γ,

i.$((n_1 + \mathrm{I}) + (n_2 + \mathrm{I}))\gamma = (n_1 + \mathrm{I})\gamma + (n_2 + \mathrm{I})\gamma$

ii.$((n_1 + \mathrm{I})(n_2 + \mathrm{I}))\gamma = (n_1 + \mathrm{I})((n_2 + \mathrm{I})\gamma)$

Consider

i$((n_1 + \mathrm{I}) + (n_2 + \mathrm{I}))\gamma = ((n_1 + i_1) + (n_2 + i_2))\gamma$, for $i_1, i_2 \in \mathrm{I}$

$= (n_3 + n_4)\gamma$, where $n_1 + i_1 = n_3$ and $n_2 + i_2 = n_4$

$= n_3\gamma + n_4\gamma$, where $n_1 + i_1 = n_3$ and $n_2 + i_2 = n_4$

$= (n_1 + i_1)\gamma + (n_2 + i_2)\gamma$

$= (n_1 + \mathrm{I})\gamma + (n_2 + \mathrm{I})\gamma$

$((n_1 + \mathrm{I}) + (n_2 + \mathrm{I}))\gamma = (n_1 + \mathrm{I})\gamma + (n_2 + \mathrm{I})\gamma$

ii$((n_1 + \mathrm{I})(n_2 + \mathrm{I}))\gamma = ((n_1 + i_1)(n_2 + i_2))\gamma$, for $i_1, i_2 \in \mathrm{I}$

$= (n_3 n_4)\gamma$, where $n_1 + i_1 = n_3$ and $n_2 + i_2 = n_4$

$= n_3\,(n_4\gamma)$, where $n_1 + i_1 = n_3$ and $n_2 + i_2 = n_4$

$= (n_1 + i_1)((n_2 + i_2)\gamma)$, for $i_1, i_2 \in \mathrm{I}$

$= (n_1 + \mathrm{I})((n_2 + \mathrm{I})\gamma)$

$((n_1 + \mathrm{I})(n_2 + \mathrm{I}))\gamma = (n_1 + \mathrm{I})((n_2 + \mathrm{I})\gamma)$

Γ is an N/I-group of near-ring N/I.

Since N^{Γ} is completely reducible, $\mathrm{N/I}^{\Gamma}$ is also completely reducible,

Since N^{Γ} is monogenic,

There exist γ ∈ Γ such that Nγ = Γ

To prove $\mathrm{N/I}^{\Gamma}$ is monogenic

i.e. to prove, there exist γ ∈ Γ such that N/I γ = Γ

suppose for all γ ∈ Γ such that N/I γ ≠ Γ

$\Rightarrow (n + \mathrm{I})\gamma \neq \Gamma$

$\Rightarrow (n + i)\gamma \notin \Gamma$, for $i \in \mathrm{I}$

$\Rightarrow n_1\gamma \notin \Gamma$, for $n_1 \in \mathrm{N}$

Since $n_1\gamma \in \mathrm{N}\gamma$

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$\Rightarrow N\gamma \neq \Gamma$

Which is a contradiction

Our assumption is wrong

Therefore, N/I^{Γ} is monogenic

Hence N/I^{Γ} is a semi-type N/I-group of a near-ring N/I.

4. r-Primitive Near-Rings:

Definition 4.1:

A near-ring N is said to be an r-primitive near-ring if it has (or is) N^{Γ} of semi-type and faithful.

Example 4.2:

The near-ring $N = Z_6$ under addition modulo 6 and multiplication is defined as

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

{scheme: 27, pg.no: 409 in [4]}

is an r-primitive near-ring for an N-group $\Gamma = Z_6$.

Theorem 4.3:

If N is an r-primitive near-ring then

a) For all $\gamma \in N^{\Gamma}$ and $0 \in N$, $0\gamma = 0$

b) For all non-zero $n \in N$ and $\gamma \in N^{\Gamma}$ is monogenic generator of N^{Γ} implies $n\gamma = \gamma_1$, for any $\gamma_1 \in N^{\Gamma}$.

c) For all non-zero $n \in N$ and $\gamma' \in N^{\Gamma}$ implies $n\gamma' = \gamma_1$, for any $\gamma_1 \in N^{\Gamma}$.

d) For all non-zero $n \in N$ and $\gamma' \in N^{\Gamma}$ implies $n\gamma' \neq 0$, for some $\gamma' \in N^{\Gamma}$.

Received: August 04 2025

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Proof:

Let N be an r-primitive near-ring.

Therefore, it has monogenic N-group Γ which is faithful and completely reducible.

a) For all $\gamma \in \Gamma$ and $0 \in N$

Since Γ is faithful, $(0{:}\,\Gamma) = \{0\}$

$\Rightarrow \{n \in N | n\Gamma = 0\} = \{0\}$

$\Rightarrow n\Gamma = 0$, for $n = 0 \in N$

$\Rightarrow n\gamma = 0$, for $n = 0 \in N$ and for all $\gamma \in \Gamma$.

Hence, $0\gamma = 0$, for all $\gamma \in \Gamma$.

b) Given, for all non-zero $n \in N$ and γ is monogenic generator of N-group Γ

Therefore $N\gamma = \Gamma$

Every element of Γ is of form $n\gamma$, for any $n \in N$.

For all $\gamma_1 \in \Gamma$,

$\gamma_1 = n\gamma$, for any $n \in N$

By (a)

$\Rightarrow \gamma_1 = n\gamma$, for any non-zero $n \in N$

$\Rightarrow n\gamma = \gamma_1$, for any non-zero $n \in N$

c) Given, for all non-zero $n \in N$ and $\gamma' \in \Gamma$

Since Γ is monogenic,

There exist $\gamma \in \Gamma$ such that $N\gamma = \Gamma$

i.e. every element of Γ is of form $n\gamma$, for any $n \in N$

given $\gamma' \in \Gamma$

$\Rightarrow \gamma' = n_1\gamma$, for any $n_1 \in N$

Now

$n\gamma' = n(n_1\gamma)$

$= (nn_1)\gamma$

$= n_2\gamma$

$= \gamma_1$, for any $\gamma_1 \in \Gamma$

$n\gamma' = \gamma_1$, for any $\gamma_1 \in \Gamma$

d) For all non-zero $n \in N$ and $\gamma' \in \Gamma$

To prove $n\gamma' \neq 0$, for some $\gamma' \in N^{\Gamma}$

Suppose $n\gamma' = 0$, for all $\gamma' \in N^{\Gamma}$

$\Rightarrow n\Gamma = 0$

Therefore, $n \in (0{:}\,\Gamma)$

$(0{:}\,\Gamma) \neq \{0\}$

Which is a contradiction

Our assumption is wrong

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$n\gamma' \neq 0$, for some $\gamma' \in N^{\Gamma}$.

Theorem 4.4:

Let N be a near-ring whose N^{Γ}'s annihilator is $\{0\}$ then the monogenic N^{Γ} is completely reducible if and only if N is an r-primitive near-ring on N^{Γ}.

Proof:

Let N be a near-ring and Γ be its N-group

Also given annihilator of Γ is $\{0\}$.

∴ clearly Γ is faithful.

Assume that, Γ is completely reducible and monogenic

Since Γ is faithful,

N is an r-primitive near-ring on Γ

Conversely,

Assume that, N is an r-primitive near-ring on Γ

Obviously, Γ is completely reducible and monogenic.

Remarks 4.5:

i. A completely reducible near-ring is r-primitive near-ring if it is faithful and monogenic as an N-group.
ii. A near-ring N whose N-group Γ where faithful, completely reducible and monogenic is r-primitive near-ring.
iii. A near-rings N where its monogenic N-group Γ where faithful and completely reducible are r-primitive near-rings.
iv. A near-ring in which its faithful N-group is completely reducible and monogenic. Then near- ring is an r-primitive near-ring.
v. A near-rings which has completely reducible N-group which is monogenic and faithful then near-ring N is an r-primitive near-ring.

Proposition 4.6:

If a near-ring N is an r-primitive near-ring then its

1) N^{Γ} is faithful.
2) N^{Γ} is monogenic.
3) N^{Γ} is completely reducible.

Proof:

Let N be an r-primitive near-ring.

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Let Γ be an N-group of N.

1) Since N is an r-primitive near-ring,
Γ is faithful

If Γ is not faithful

By note 2.30,

Γ is embedded into some faithful N-group Γ_1.

Therefore, Γ_1 is faithful

Hence N-group of near-ring N is faithful.

2) Since N is an r-primitive near-ring,
Γ is monogenic.

3) Since N is an r-primitive near-ring,
Γ is completely reducible.

Theorem 4.7:

If near-ring N is an r-primitive near-ring. Then

i. Every element of N^{Γ} is of form $n\gamma$, for any $n \in N$ and γ is a generator of N^{Γ}.
ii. $L\gamma$ is an ideal of N^{Γ}, where L is a left ideal of N.
iii. $L\gamma$ is faithful.

Proof:

Let N be an r-primitive near-ring

Let Γ be a N-group of N.

Γ is faithful and semi-type.

i. Since Γ is semi-type,
Γ is monogenic
Let γ be a generator of Γ.
i.e. $N\gamma = \Gamma$
therefore, every element of Γ is of form $n\gamma$, for any $n \in N$.

ii. Let L be a left ideal of N
Since Γ is of semi-type, Γ is monogenic by γ
By note 2.18,
Therefore, $L\gamma$ is an ideal of N^{Γ}.

iii. Since Γ is faithful
Therefore, $(0{:}\,\Gamma) = \{0\}$
Since Γ is monogenic by γ, $N\gamma = \Gamma$
Also L is a left ideal of near-ring N.
$\Rightarrow L\gamma \subseteq N\gamma = \Gamma$

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$\Rightarrow L\gamma \subseteq \Gamma$

$\Rightarrow (0: L\gamma) = \{0\}$

$L\gamma$ is faithful.

Proposition 4.8:

If N is an r-primitive near-ring on N^N then N is completely reducible.

Proof:

Let N be an r-primitive near-ring on N^N.

Therefore, the N-group $\Gamma = N$ is semi-type and faithful

Hence N is completely reducible.

Remarks 4.9:

1. If a near-ring N is 0-primitive near-ring on N^Γ then N is not r-primitive near-ring on N^Γ.

   Proof:

   Let N be a 0-primitive near-ring on N-group Γ.

   $\Rightarrow \Gamma$ is simple

   Therefore, Γ is not completely reducible

   Hence N is not r-primitive near-ring on Γ.

2. If N is an r-primitive near-ring on N-group Γ then N is not 0-primitive near-ring on N^Γ.

   Proof:

   Let N be an r-primitive near-ring on N-group Γ.

   $\Rightarrow \Gamma$ is completely reducible.

   Therefore, Γ is not simple.

   N is not 0-primitive near-ring on N-group Γ.

3. If N is a 1-primitive near-ring on N^Γ then N is not an r-primitive near-ring on N^Γ.
4. If N is an r-primitive near-ring on N^Γ then N is not 1-primitive near-ring on N^Γ.
5. Let $N \in \mathcal{N}_0$. If N is a 2-primitive near-ring on N^Γ then N is not an r-primitive near-ring on N^Γ.
6. Let $N \in \mathcal{N}_0$. If N is an r-primitive near-ring on N^Γ then N is not 2-primitive near-ring on N^Γ.

Definition 4.10:

An ideal I of near-ring N ($I \trianglelefteq N$) is called an r-primitive ideal of N if N/I is an r-primitive near-ring.

Proposition 4.11:

If N is an r-primitive near-ring on N^Γ and I is ideal of N then N/I is an r-primitive near-ring on N^Γ.

Proof:

Volume 38 No. 9s , 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Let N be r-primitive near-ring on N-group Γ.

Therefore, Γ is faithful and semi-type.

Let I be an ideal of N.

By theorem 3.13, $\mathrm{N/I}^{\Gamma}$ is of semi-type

To prove, Γ is a faithful $\mathrm{N/I}$-group

i.e. to prove $(0 : \Gamma) = \{0\}$

Suppose, $(0 : \Gamma) \neq \{0\}$

$\{n + \mathrm{I} \in \mathrm{N/I} \mid (n + \mathrm{I})\Gamma = 0\} \neq \mathrm{I}$

For some non-zero $n_1 + \mathrm{I} \in \mathrm{N/I}$,

$(n_1 + \mathrm{I})\Gamma = 0$

$(n_1 + \mathrm{I})\gamma = 0$, for all $\gamma \in \Gamma$

$(n_1 + i)\gamma = 0$, for $i \in \mathrm{I}$ and for all $\gamma \in \Gamma$

$n_2\gamma = 0$, for $n_2 = n_1 + i$ and $n_2 \neq 0$ and for all $\gamma \in \Gamma$

$n_2\Gamma = 0$ for $n_2 \neq 0$

Which is contradiction

Therefore, $(0 : \Gamma) = \mathrm{I}$

$\Rightarrow \mathrm{N/I}^{\Gamma}$ is faithful

$\Rightarrow \mathrm{N/I}^{\Gamma}$ is faithful and semi-type.

Hence $\mathrm{N/I}$ is an r-primitive near-ring on Γ.

Corollary 4.12:

In the above case I is an r-primitive ideal of N.

Reference:

[1] Beidleman, J. C., *On Near-rings and Near-ring modules*, Doctoral dissertation, Pennsylvania State University, 1964.

[2] Choudhary, S.C., *On Near-Rings and Near-Ring Modules*, Ph.D. thesis, Indian Institute of Technology Kanpur, 1972.

[3] Dickson, L. E., *Definition of a group and a field by independent postulates*, **Trans. Amer. Math. Soc. 6(1905)**, 198-204.

[4] Pilz, G., *Near-Rings: The Theory and its Applications*, North Holland, Amsterdam, 1983.

[5] Zassenhaus, H., *Über endliche Fastkörper*, **Abh. Math. Sem. Univ. Hamburg 11 (1935/36)**, 187-220.

Received: August 04 2025