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Almost (m, n)-semilattice

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Abstract

The concept of Almost (m, n)-semilattice is introduced and we prove some important results of Almost (m, n)-semilattice. In this paper, we prove that every Almost (m, n)-semilattice contains a semilattice. Moreover, the results and remarks on Almost (m, n)-semilattice are discussed with examples.

Keywords : Semilattice, Almost (m,n)-semilattice, Almost semilattice, (m,n)-semilattice..

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1. INTRODUCTION

Semigroups are generalizations of groups and rings. The concept of semilattice was introduced by F.Klein in1939.The concept of Almost Semilattice(ASL) was introduced be G.Nanaj Rao and T.G.Beyene [7] as a generalization of almost distributive lattice as a semilattice. Velrajan.M and Pramila Inpa Rose.A [8] introduced the concept of (m,n)-semilattice as a generalization of semilattice. A commutative semigroup S is called an (m,n)-semilattice if $x^n = x^m$, for all $x \in S$, where m and n are positive integers with $m < n$. For the background of semilattices, we refer the reader to [2, 3, 4, 5].

In this paper, we introduce the concept of Almost (m, n)-semilattice which is a generalization of semilattice and derive some properties of Almost (m, n)-semilattice. Also, we give few examples and theorems in an Almost (m, n)-semilattice. Moreover, we prove that every Almost (m, n)-semilattice contains a semilattice.

2. PRELIMINARIES

Definition 2.1. A partially ordered set or simply a poset $(P, \leq)$ is called a **lattice** if l.u.b.$\{x,y\}$ and g.l.b.$\{x,y\}$ exist for all $x, y \in P$.

Definition 2.2. A **semilattice** S is a nonempty set and ∘ is a binary operation on S satisfying:

1. $x\circ (y\circ z) = (x\circ y) \circ z$ (Associative law)
2. $x\circ y = y\circ x$ (Commutative law)
3. $x\circ x = x$, for all $x , y, z \in S$. (Idempotent law)

Definition 2.3. [7] An **Almost Semilattice** $(L, \circ)$ is a non-empty set and ∘ is a binary operation on L, satisfies the following conditions:

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(1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative law)

(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative law)

(3) $x \circ x = x$, for all $x, y, z \in L$. (Idempotent law).

Definition 2.4. [8] Let m and n be positive integers with $m < n$. A non empty set S together with a commutative and associative binary operation is called an **(m, n)-semilattice** if for each $a \in S$, $a^n = a^m$, where $a^n = aaa...a$ (n times).

3. MAIN RESULTS

Definition 3.1. Let m and n be positive integers with $m < n$. A non empty set N together with a binary operation $\circ$ is said to be an **Almost (m, n)-semilattice** if the following axioms holds:

(i) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost commutative law)
(ii) $x \circ (y \circ z) = (x \circ y) \circ z$. (Associative law)
(iii) $x^n = x^m$,
(i.e) $x^n = x \circ x \circ x ... \circ x$ (n times), ((m,n)-potent law)
for $x, y, z \in N$.

Example 3.2. The set N = {u, v, w} with the operation $\circ$ as defined below is an Almost (1,2)-semilattice.

$\circ$	u	v	w
u	u	u	u
v	u	v	w
w	u	v	w

Remark 3.3. In Example 3.2, the commutative law holds for the elements u, v and u, w but not for v, w.

Example 3.4. Let N ={1, 2, 3,...} be the set of all natural numbers. Define a binary operation $\circ$ on N by $x \circ y = x.y$ for all $x, y \in N$, where . is a usual multiplication.

N is not a semilattice, since $x \circ x = x.x \neq x$, for all $x\ (\neq 1) \in N$.

Similarly, N is not an Almost (m, n)-semilattice, since for $m < n$, $x^m \neq x^n$ for all $x\ (\neq 1) \in N$.

Example 3.5. Let N = {..., -3, -2, -1, 0, 1, 2, 3,...} be the set of all integers. Define a binary operation $\circ$ on N by $x \circ y = x.y$ for all $x, y \in N$, where . is a usual multiplication.

N is not a semilattice, since $x \circ x = x.x \neq x$, for all $x\ (\neq 0,1) \in N$.

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Similarly, N is not an Almost (m, n)-semilattice, since for $m < n$, $x^m \neq x^n$ for all $x\ (\neq 0,1) \in N$.

Example 3.6. Let $M = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$ be the group of all n^{th} roots of unity. Then, for any $m \in M$, $m^n = 1$, and hence, $m^{n+1} = m$.

Therefore, M is an Almost (1, n+1)-semilattice.

In general, if G is a finite abelian group of order n, then G is an Almost (1, n+1)-semilattice.

Example 3.7. Let G be an abelian group and $N = \{a \in G \mid a^{-1} = a\}$. Then for each $a \in N$, $a^2 = e$, hence $a^3 = a$.
Hence, N is an Almost (1,3)-semilattice.

Example 3.8. Let N = {a, b, c, d}. Define a binary operation ∘ on N as follows:

∘	a	b	c	d
a	a	a	a	a
b	a	b	c	b
c	a	c	c	c
d	a	c	c	d

Then, N is an Almost (1,2)-semilattice but is not a semilattice since $b \circ d \neq d \circ b$.

Remark 3.9. From commutativity, almost commutativity follows.

Remark 3.10. Every semilattice is an Almost (1,2)-semilattice and also an Almost (m, n)-semilattice, for any $1 \leq m < n$.

Remark 3.11. Every Almost (m, n)-semilattice is an Almost (m+k, n+k)-semilattice, for all $k \geq 1$. And every Almost (m, m+1)-semilattice is an Almost (m_1, n_1)-semilattice, where

$n_1 > m_1 \geq m$.

Remark 3.12. Every semilattice is an Almost (m, n)-semilattice.

Proof: Let N be a semilattice, then the associative law holds.

Also, by the commutativity of semilattice, almost commutative law follows.

And, by the idempotent law of semilattice,the axiom, $x^n = x^m$, $m < n$ where $x \in N$ holds.

But the converse does not hold.

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For example,

Let N = {a, b, c, d}. Define a binary operation ◦ on N as follows:

◦	a	b	c	d
a	a	a	a	a
b	a	b	c	b
c	a	c	a	c
d	a	d	c	d

Then, N is an Almost (2,3)-semilattice but is not a semilattice, since $c \circ c = a \neq c$.

Remark 3.13. Every semilattice is an (m,n)-semilattice.

But the converse is not true.

For example,

Let N = {a, b, c, d}. Define a binary operation ◦ on N as follows:

◦	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	c	d

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Then, N is an (2,3)-semilattice but is not a semilattice, since $c \circ c = a \neq c$.

Remark 3.14. Every (m, n)-semilattice is an Almost (m, n) semilattice.

Proof: Let N be an (m, n)-semilattice.

Then, the associative law and the axiom $x^n = x^m$, $m < n$ where $x \in N$ holds.

By the commutativity of (m, n)-semilattice, almost commutative law holds.

But the converse does not hold.

For example,

Let N = {a, b, c, d}. Define a binary operation ◦ on N as follows:

◦	a	b	c	d
a	a	a	a	a
b	a	c	a	c
c	a	a	a	c
d	a	c	a	d

Then, N is an Almost (3,4)-semilattice but not an (3,4)-semilattice, since $c \circ d \neq d \circ c$.

Remark 3.15. An Almost (m, n)-semilattice will be an (m, n)-semilattice if commutativity holds.

Remark 3.16. Every semilattice is an Almost semilattice.

By the commutativity of semilattice, almost commutative law holds.

But the converse does not hold.

For example,

Let N ={a, b, c}. Define a binary operation ◦ on N as follows:

◦	a	b	c
a	a	a	a

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b	a	b	c
c	a	b	c

Then, N is an Almost semilattice, but is not a semilattice, since $b \circ c = c \neq b = c \circ b$.

Remark 3.17. Every Almost semilattice is an Almost (m, n)-semilattice .

Proof: Let N be an Almost semilattice.

Then, the associative law and the almost commutative law holds.

Also, by the idempotent law of Almost semilattice, the axiom $x^n = x^m$, $m < n$ where $x \in N$ holds.

But the converse does not hold .

From the table of Remark 3.12, we get N is an Almost (2,3)-semilattice but is not a almost semilattice, since $c \circ c = a \neq c$.

Definition 3.18. An Almost (m, n)-semilattice N is called an **Almost (m, n)-semilattice with zero element** if there exists an element $\mathbf{0} \in N$ such that $\mathbf{0} \circ x = 0$, for all $x \in N$. And, the element **0** is called the zero element of N.

The product operation plays an important role as a means of constructing a new from the old. Now, we consider the product of Almost (m, n)-semilattices $(N_1, \circ_1)$ and $(N_2, \circ_2)$.

Theorem 3.19. If $(N_1, \circ_1)$ and $(N_2, \circ_2)$ are two Almost (m, n)-semilattices, then their product

$N_1 \times N_2$ is an Almost (m, n)-semilattice under the binary operation $\circ$ defined by

$(a_1, a_2) \circ (b_1, b_2) = (a_1 \circ_1 b_1, a_2 \circ_2 b_2)$, $\forall\, a_1, b_1 \in N_1$ and $a_2, b_2 \in N_2$.

Proof: Let $a_1, b_1, c_1 \in N_1$ and $a_2, b_2, c_2 \in N_2$.

$\Rightarrow (a_1, a_2) \in N_1 \times N_2$, $(b_1, b_2) \in N_1 \times N_2$ and $(c_1, c_2) \in N_1 \times N_2$.

$$\begin{aligned} \text{Now, } [(a_1, a_2) \circ (b_1, b_2)] \circ (c_1, c_2) &= [(a_1 \circ_1 b_1, a_2 \circ_2 b_2)] \circ (c_1, c_2) \\ &= ((a_1 \circ_1 b_1) \circ_1 c_1, (a_2 \circ_2 b_2) \circ_2 c_2) \\ &= ((b_1 \circ_1 a_1) \circ_1 c_1, (b_2 \circ_2 a_2) \circ_2 c_2) \\ &= ((b_1 \circ_1 a_1, b_2 \circ_2 a_2) \circ (c_1, c_2) \\ &= [(b_1, b_2) \circ (a_1, a_2)] \circ (c_1, c_2) . \end{aligned}$$

Thus, Almost commutative law holds.

$$\begin{aligned} \text{Also, } [(a_1, a_2) \circ (b_1, b_2)] \circ (c_1, c_2) &= [(a_1 \circ_1 b_1, a_2 \circ_2 b_2)] \circ (c_1, c_2) \\ &= ((a_1 \circ_1 b_1) \circ_1 c_1, (a_2 \circ_2 b_2) \circ_2 c_2) \\ &= (a_1 \circ_1 (b_1 \circ_1 c_1), a_2 \circ_2 (b_2 \circ_2 c_2)) \end{aligned}$$

$$= (a_1,a_2) \circ (b_1 \circ_1 c_1, b_2 \circ_2 c_2)$$

$$= (a_1, a_2)\circ[(b_1, b_2) \circ(c_1, c_2)]$$

Thus, Associative law holds.

For $(a_1, a_2) \in N_1 \times N_2$,

We have, $(a_1, a_2)^n = (a^{n1}, a^{n2})$

$$= (a^{m1}, a^{m2})$$

$$= (a_1, a_2)^m.$$

Thus, (m, n)-potent law holds.

Hence, $N_1 \times N_2$ is an Almost (m, n)-semilattice.

4. CONCLUSION

In this paper, we discussed the new concept and defined the conditions for an almost (m, n)-semilattice. Also, we discussed some examples, remarks and theorems based on almost (m, n)-semilattice.

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