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ANALYTICAL SOLUTION OF FRACTIONAL NEWTON'S LAW OF COOLING USING KUSHARE TRANSFORM

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ABSTRACT

In this paper, we highlight on application of the Kushare Integral Transform to solve Fractional Newton's Law of Cooling problems.

Keywords: Riemann–Liouville Derivative, KUSHARE Transform, Fractional Differential Equations, Mittag-Leffler function, Integral Transforms

1. INTRODUCTION

Since a long time ago, Fractional calculus is a mathematical concept that has become very popular among scientists and engineers. It has been used in various fields like physics, mechanics, signal processing, and electrical engineering since the 1990s. This paper introduces an exact solution for the fractional-order Newton's law of cooling using the Kushare Transform. By replacing the standard derivative with a Riemann-Liouville fractional derivative and applying the Kushare Transform, we obtain a mathematical solution for the cooling problem. Unlike the traditional exponential decay solution, this approach leads to a more general result involving the Mittag-Leffler function, highlighting the flexibility of fractional models. The study also demonstrates the effectiveness of the Kushare Transform in solving fractional differential equations. [1-7]

Classical Newton's law of cooling models the rate at which a body exchanges heat with its surroundings and is given by:

$$\frac{dT(t)}{dx} = -k(T(t) - T_a)$$

where is T(t) the temperature of the object, T_a is the surrounding temperature, and $k>0$ is the heat transfer coefficient.

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The fractional Newton's law of cooling replaces classical derivative with a Riemann–Liouville fractional derivative of order $\mu\epsilon(0,1]$:

$$D_t^{\mu}\mathrm{T}(t) = -k(T(t) - T_a)$$

We have to apply the Kushare Transform to above equation to get solution.

2. PRELIMINARIES

2.1 KUSHARE Transform: For a appropriate function f(t) ∈ A, the KUSHARE Transform is defined by:

$$K[f(t)] = S(v) = v\int_0^{\infty} f\,(t)e^{-tv^{\alpha}}dt, \quad \alpha \in \mathbb{R}\backslash\{0\}$$

This transform is linear and allows fractional exponents of the transform variable v.

It generalizes other known transforms:

- (α= 1) yields the Mahgoub Transform
- (α = 2) yields the Pourreza Transform
- (α = -1) yields the Elzaki Transform

3. FOR THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE:

If $0 < \mu \leq 1$, and f(t) is sufficiently smooth then

$$K\left[D_t^{\mu}f(t)\right] = v^{\alpha\mu}S(v) - \mathrm{v}^{\alpha(\mu-1)+1}f(0)$$

Proof:

Let $0 < \mu \leq 1$. The Riemann–Liouville fractional derivative of order μ is defined as:

$$D_t^{\mu}f(t) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^{\mu}}d\tau$$

$\mathrm{L}[D_t^{\mu}f(t)] = \mathrm{s}^{\mu}\mathrm{F(s)} - \mathrm{s}^{\mu-1}\mathrm{f(0)}$ ……. [From the Laplace transform of Riemann–Liouville derivative]

Where F(s)=L[f(t)].

Note that Kushare transform is related to the Laplace transform by the change of variable $\mathrm{s} = \mathrm{v}^{\alpha}$

So,

$$\mathrm{L}[D_t^{\mu}f(t)] = \mathrm{s}^{\alpha\mu}\mathrm{F(s)} - \mathrm{s}^{\alpha(\mu-1)}\mathrm{f(0)}$$

The Kushare Transform of f(t) is :

$\mathrm{K[f(t)]=S(v)=vF(v^{\alpha})}$

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$\Rightarrow F(v^{\alpha}) = \frac{S(v)}{v}$

We get ,

$$K[D_t^{\mu} f(t)] = v^{\mu\alpha}\frac{S(v)}{v} - v^{\alpha(\mu-1)}f(0) = v^{\alpha\mu-1}S(v) - v^{\alpha(\mu-1)}f(0)$$

Multiplying by v we get

$$K\left[D_t^{\mu} f(t)\right] = v^{\alpha\mu}S(v) - v^{\alpha(\mu-1)+1}f(0)$$

Hence the proof.

4. FRACTIONAL NEWTON'S LAW OF COOLING

The fractional version of Newton's Law of Cooling is given by:

$$D_t^{\mu}T(t) = -k(T(t) - T_a)$$

with the solution:

$$T(t) = T_a + (T_0 - T_a)E_{\mu}(-kt^{\mu})$$

$T(t)$ = temperature at time t

T_a = surrounding temperature

T_0 = initial temperature

E_{μ}=Mittag-Leffler function

K=fractional cooling constant

Proof:

We knoe that Kushare Transform of the Riemann–Liouville derivative is:

$$K[D_t^{\mu} f] = v^{\alpha\mu}S(v) - \sum_{k=0}^{n-1} v^{\alpha(\mu-k-1)+1}f^{k}(0)$$

Where $n - 1 < \mu \leq n$ and S(v)=K[f(t)]

Since $0 < \mu \leq 1$ and take n=1 we get

$$K\left[D_t^{\mu} f(t)\right] = v^{\alpha\mu}S(v) - v^{\alpha(\mu-1)+1}f(0)$$

Let , $\theta(t) = T(t) - T_a$ the fractional equation becomes:

$D_t^{\mu}\theta(t) = -k\theta(t)$, $\theta(0) = T_0 - T_a = \theta_0$

Apply Kushare Transform to above equation , we get

$$K[D_t^{\mu}\theta(t)] = K[-k\theta(t)]$$

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Since $K\left[D_t^{\mu}\theta(t)\right] = v^{\alpha\mu}S(v) - v^{\alpha(\mu-1)+1}\theta_0$ and $K[-k\theta(t)] = -kS(v)$

We get

$v^{\alpha\mu}S(v) - v^{\alpha(\mu-1)+1}\theta_0 = -kS(v)$

$$S(v) = \frac{v^{\alpha(\mu-1)+1}\theta_0}{v^{\alpha\mu} + k}$$

Using the known inverse transform correspondence (by analogy to Laplace and Elzaki methods), the inverse Kushare Transform yields:

$$\theta(t) = K^{-1}\{\frac{v^{\alpha(\mu-1)+1}\theta_0}{v^{\alpha\mu} + k}\}$$

$$\theta(t) = \theta_0 K^{-1}\{\frac{v^{\alpha(\mu-1)+1}}{v^{\alpha\mu} + k}\}$$

$$\theta(t) = \theta_0 E_\mu(-kt^\mu)$$

Thus the temperature function is:

$$T(t) = T_a + (T_0 - T_a)E_\mu(-kt^\mu)$$

Where $E_\mu(z)$ is the one parameter Mittag-Leffler function:

$$E_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)}$$

5. APPLICATION OF KUSHARE TRANSFORM

In this section using Kushare integral transform we solve problem based on fractional Newton's Law of Cooling .

Worked Example

The temperature of a body falls from 40°C to 36°C in 5 minutes when placed in a surrounding of constant temperature 16°C. Find the time taken for the temperature of the body to become 32°C.

Solution:

Given that

T_a = surrounding temperature = 16

T_0 = initial temperature =40

Temperature after 5 minutes: T(5)=36

We have to find the time t when T(t)=32

We know

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$D_t^{\mu}T(t) = -k(T(t) - T_a)$ with $T_0 = 40$

$$\theta(t) = T(t) - T_a$$

$\theta(0) = 24$ and $\theta(5) = 20$, we need to find t when $\theta(t) = 16$.

Using solution from Kushare Transform:

$$\theta(t) = \theta_0 E_{\mu}(-kt^{\mu})$$

$\theta(t) = 24E_{\mu}(-kt^{\mu})$ ----------- (a)

Now determine k by following way,

Put t=5 in above equation

$$\theta(5) = 20$$

So, $20 = 24E_{\mu}(-k5^{\mu})$

$$E_{\mu}(-k5^{\mu}) = \frac{20}{24} = \frac{5}{6}$$

So, $k = \frac{-1}{5\mu} {E_{\mu}}^{-1}(\frac{5}{6})$

This inverse is not analytic, but can be computed numerically for a chosen value of μ. Let's take a typical fractional order used in real heat models:

Assume: $\mu=0.9$

Using known values of the Mittag-Leffler function $E_{0.9}(-x)$ we find the value of x such that:

$$E_{0.9}(-x) = \frac{5}{6}$$

$$x = k5^{0.9}$$

From tables or numerical software (MATLAB, Python), we find:

$$E_{0.9}(-0.439) \approx 0.833 \approx \frac{5}{6}$$

So,

$$k5^{0.9} \approx 0.439$$

$$k \approx \frac{0.439}{5^{0.9}} \approx 0.1182$$

Now we have to find t when $\theta(t) = 16$

We solve equation (a)

$$16 = 24E_{0.9}(-0.1182t^{0.9})$$

$$E_{0.9}(-0.1182t^{0.9}) = \frac{2}{3}$$

Again using tables ,

$$E_{0.9}(-0.643) \approx 0.667 = \frac{2}{3}$$

So,

$$0.1182t^{0.9} \approx 0.643$$

$$t^{0.9} = \frac{0.643}{0.1182} \approx 5.44$$

So t $\approx 5.44^{1/0.9} \approx 6.46$ minutes.

Thus , The Temperature of the body to become 32°C in approximately 6.46 minutes.

6. CONCLUSION

This paper demonstrates the successful application of the Kushare Integral Transform to solve Fractional Newton's Law of Cooling problems.. Using this method, we obtain accurate and straightforward solutions for cooling-based problems. The approach simplifies the solving process while maintaining precision, proving the effectiveness of the Kushare Transform in thermal analysis.

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