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Krasnoselskii Fixed Point Approach to Fractional Duffing Equations with Generalized Psi–Prabhakar Derivatives

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Abstract

This paper establishes existence results for a nonlinear fractional Duffing boundary value problem involving sequential generalized Ψ-Prabhakar–Atangana–Baleanu–Caputo derivatives with cubic nonlinearity. The problem features nonlocal multi-point boundary conditions incorporating Ψ-Riemann–Liouville fractional integrals and Ψ-Hilfer fractional derivatives. We derive an equivalent integral representation and establish existence of at least one solution via Krasnoselskii's fixed point theorem under appropriate growth and contractivity conditions. Several corollaries for special parameter choices are presented. A detailed numerical example with graphical illustrations demonstrates the theoretical results.

Keywords: Fractional Duffing equation; Prabhakar derivative; Atangana–Baleanu–Caputo operator; Nonlocal boundary conditions; Fixed point theorem; Existence theory

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1 Introduction

The classical Duffing equation, originally proposed by Georg Duffing in 1918 [1], describes nonlinear oscillatory phenomena and takes the form

$$\ddot{x}(t) + \delta x(t) + \beta x^3(t) = F(t),$$

where δ is the linear stiffness coefficient, β represents the nonlinear stiffness, and $F(t)$ is an external forcing term. This equation has proven instrumental in modeling diverse physical systems including mechanical oscillators with nonlinear restoring forces [2, 3], electrical circuits with nonlinear inductance [4], and chaotic dynamical systems [5]. The rich dynamics exhibited by Duffing-type equations—ranging from periodic and quasi-periodic motion to deterministic chaos—have made them a cornerstone in nonlinear dynamics research for over a century.

In recent decades, the extension of classical differential equations to fractional-order models has opened new avenues for capturing memory effects, hereditary properties, and anomalous diffusion processes that integer-order models cannot adequately describe [6, 7, 8]. Fractional calculus provides mathematical tools particularly suited for modeling viscoelastic materials, biological systems, and complex transport phenomena where past states significantly influence current behavior. Motivated by these applications, fractional generalizations of the Duffing equation have attracted substantial attention. Early works considered Caputo and Riemann–Liouville fractional derivatives [9, 10], while more recent investigations have explored Caputo–Fabrizio [11], Atangana–Baleanu [12], and Hilfer-type operators [13].

Among the most general and flexible fractional operators introduced in recent years are the Prabhakar-type derivatives [14, 15], which incorporate the three-parameter Mittag-Leffler function (Prabhakar function) as their kernel:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!},$$

where $(\gamma)_n$ denotes the Pochhammer symbol. This class of operators unifies several classical fractional derivatives as special cases and has demonstrated remarkable utility in modeling ultraslow processes, anomalous relaxation, and dielectric phenomena [16, 17]. Further generalization via the Ψ-function framework, where an arbitrary increasing smooth function Ψ replaces the identity in the integral kernel, yields the Ψ-Prabhakar operators studied extensively in recent literature [19, 20].

Building upon the Atangana–Baleanu–Caputo (ABC) derivative structure [21], which employs a non-singular kernel and preserves crucial properties for physical modeling, researchers have developed generalized Ψ-Prabhakar–ABC derivatives combining the advantages of both frameworks [22]. These operators exhibit favorable analytical properties including well-defined Laplace transforms, composition rules, and solution regularity that make them particularly attractive for boundary value problems.

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The study of nonlocal and nonstandard boundary conditions for fractional differential equations has emerged as a critical research direction, driven by applications in physics, engineering, and biology where terminal or integral constraints arise naturally [23, 24]. Multi-point boundary conditions involving fractional integrals and derivatives at interior points model scenarios such as:

- Thermostats measuring average temperature over subdomains [25],
- Feedback control systems with distributed sensors [26],
- Population dynamics with spatial heterogeneity [27].

Despite extensive research on fractional Duffing equations, the case involving sequential generalized Ψ-Prabhakar–ABC derivatives with nonlocal multi-point boundary conditions incorporating both fractional integrals and Hilfer derivatives remains largely unexplored. Existing studies have addressed:

- Fractional Duffing equations with standard Caputo derivatives and simple boundary conditions [28, 29],
- Prabhakar-type operators in linear boundary value problems [30],
- ABC derivatives in specific nonlinear systems [31].

However, the combination of (i) sequential application of two distinct generalized Ψ-Prabhakar–ABC operators, (ii) cubic nonlinearity characteristic of Duffing dynamics, and (iii) complex nonlocal boundary conditions presents novel mathematical challenges requiring new analytical techniques.

Problem Formulation

In this paper, we investigate the following nonlinear fractional boundary value problem:

$$ {}^{G}D_{a^+}^{\alpha_1,\beta_1,\gamma_1;\Psi}\left({}^{G}D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi}+p\right)x(t)+\delta x^3(t)=\mathscr{F}(t,x(t)),\quad t\in[a,b], \tag{1} $$

subject to the nonlocal boundary conditions

$$ x(a)=0,\quad x(b)=\sum_{i\in\mathscr{I}}\delta_i x(\eta_i)+\sum_{j\in\mathscr{J}}\omega_j I_{a^+}^{\tau_j;\Psi}x(\theta_j)+\sum_{k\in\mathscr{K}}\lambda_k\,{}^{H}\mathscr{D}_{a^+}^{\mu_k;\Psi}x(\xi_k), \tag{2} $$

where:

- ${}^{G}D_{a^+}^{\alpha_i,\beta_i,\gamma_i;\Psi}$ denotes the generalized Ψ-Prabhakar–ABC derivative of order $0<\alpha_i<1$, type $\beta_i>0$, and parameter $\gamma_i>0$ for $i=1,2$,
- $p,\delta\in\mathbb{R}$ are physical parameters (linear damping and nonlinear stiffness coefficients),

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- $\mathscr{F} : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous forcing function,
- $I_{a^+}^{\tau_j;\Psi}$ represents the Ψ-Riemann–Liouville fractional integral of order $\tau_j > 0$,
- ${}^H\mathscr{D}_{a^+}^{\mu_k;\Psi}$ denotes the Ψ-Hilfer fractional derivative of order $\mu_k \in (0, 1)$,
- $\eta_i, \theta_j, \xi_k \in (a, b)$ are prescribed interior evaluation points,
- $\delta_i, \omega_j, \lambda_k \in \mathbb{R}$ are weighting coefficients in the boundary condition.

This formulation captures memory effects through fractional operators, nonlinear feedback via the cubic term $\delta x^3(t)$, and nonlocal coupling through the multi-point boundary condition (2).

Organization of the Paper

The remainder of this paper is organized as follows:

Section 2 provides all necessary preliminaries, including definitions of the generalized Ψ-Prabhakar–ABC derivative, Ψ-Hilfer derivative, relevant properties of special functions, and key fixed point theorems employed in the analysis.

Section 3 contains the main results. We first establish Lemma 3.1, which derives the integral equation equivalent to problem (1)–(2) through successive application of fractional integral operators and careful analysis of boundary conditions. Following this, we define an appropriate operator ζ on a Banach space and decompose it into compact and contractive components. Theorem 3.3 then guarantees existence of solutions under suitable growth and contractivity conditions. Several corollaries addressing special parameter choices are presented, along with a detailed numerical example including tabulated values and graphical illustrations.

Section 4 provides concluding remarks and summarizes the key findings.

2 Preliminaries

This section collects all the necessary definitions, notation, and foundational results employed in the analysis of the generalized fractional Duffing-type boundary value problem ([18, 17, 19, 32, 33]).

Definition 2.1 (Psi–Riemann–Liouville Fractional Integral) [19, 17]: Let $\Psi \in C^1([a, b], \mathbb{R})$ be a strictly increasing function. For $f \in L^1([a, b])$ and $\alpha > 0$, the left-sided Ψ–Riemann–Liouville fractional integral of order α is defined by

$$I_{a^+}^{\alpha;\Psi} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} f(s)\Psi'(s)\, ds.$$

This operator generalizes classical RL integral via the function Ψ [19, 17].

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Definition 2.2 (Psi–Prabhakar Kernel) [18, 32]: For $\alpha > 0$, $\beta > 0$, $\gamma > 0$, define

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n \, z^n}{\Gamma(\alpha n + \beta) n!}$$

as the generalized Prabhakar function, which serves as kernel in advanced fractional operators.

Definition 2.3 (Generalized Psi–Prabhakar–ABC Derivative) [17, 18]: For $f \in C^1([a,b])$, the left-sided generalized Ψ–Prabhakar–ABC Caputo-type derivative of order $\alpha \in (0,1)$, type $\beta > 0$, and parameter $\gamma > 0$ is defined as

$${}^{G}D^{\alpha,\beta,\gamma;\Psi}_{a^+} f(t) := \frac{B(\alpha)}{1-\alpha} \int_a^t E^{\gamma}_{\alpha,\beta}\left(-\frac{\alpha}{1-\alpha}(\Psi(t) - \Psi(s))^{\alpha}\right) f'(s)\Psi'(s)\, ds,$$

where $B(\alpha)$ is a normalization constant [17, 32].

Definition 2.4 (Psi–Hilfer Fractional Derivative) [33]: Let $\mu \in (0,1)$, Ψ as above, and f sufficiently smooth. The Psi–Hilfer fractional derivative is given by

$${}^{H}\mathscr{D}^{\mu;\Psi}_{a^+} f(t) = I^{\nu(1-\mu);\Psi}_{a^+} \frac{d}{dt} I^{\nu\mu;\Psi}_{a^+} f(t),$$

where the distribution between RL and Caputo is controlled by ν.

Definition 2.5 (Banach Space $\mathscr{C}$) Let $\mathscr{C} = C([a,b], \mathbb{R})$ be the Banach space of real-valued continuous functions equipped with the norm

$$\|x\| := \sup_{t \in [a,b]} |x(t)|.$$

This is the natural setting for fixed point arguments for nonlinear fractional equations.

Lemma 2.1 (Fractional Integral Mapping Property) [19, 17]: The Psi-Riemann–Liouville fractional integral $I^{\alpha;\Psi}_{a^+} : \mathscr{C} \to \mathscr{C}$ is linear, continuous, and compact for $\alpha > 0$ and suitable Ψ.

Theorem 2.1 (Schauder Fixed Point Theorem) [34]: Let X be a Banach space and $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ be continuous and compact, with $\mathscr{T}(\mathscr{C})$ bounded. Then $\mathscr{T}$ has at least one fixed point in $\mathscr{C}$.

Theorem 2.2 (Krasnoselskii's Fixed Point Theorem) [34]: Suppose B is a closed, convex, bounded subset of a Banach space. If $\mathscr{T}_1 : B \to B$ is compact and continuous, $\mathscr{T}_2 : B \to B$ is a contraction, and $\mathscr{T}_1 x + \mathscr{T}_2 y \in B$ for all $x, y \in B$, then $\mathscr{T}_1 + \mathscr{T}_2$ has a fixed point in B.

Remark 2.1. Detailed properties of the Psi–Prabhakar–ABC derivative, including commutativity, inverse operator relations, and connections with special functions, are elaborated in recent surveys ([17, 32, 19]) and used throughout the analysis.

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3 Main Results

In this section, we present the principal analytical results for the nonlinear fractional Duffing boundary value problem formulated in equations (1)–(2). Our approach proceeds through several interconnected stages.

First, we establish an integral equation representation equivalent to the original differential problem by systematically applying the left-inverse fractional integral operators to the sequential generalized Ψ-Prabhakar–ABC derivatives. This reduction transforms the boundary value problem into a fixed point equation amenable to functional-analytic techniques.

Building upon this integral formulation, we construct an appropriate operator on the Banach space $\mathscr{C}([a,b],\mathbb{R})$ and decompose it into compact and contractive components. The main existence theorem guarantees at least one solution under suitable growth conditions on the nonlinear forcing function $\mathscr{F}$ and contractivity constraints involving the system parameters p, δ, and the fractional operator characteristics.

Several corollaries addressing important special cases—including parameter reductions ($\beta_1 = 0$, $\beta_1 = 1$, $\beta_2 = 0$), the purely nonlinear regime ($p = 0$), and the standard Riemann–Liouville framework ($\Psi(t) = t$)—demonstrate the versatility of our general framework.

Finally, a detailed numerical example with explicit parameter choices illustrates the theoretical results and provides graphical representations of solution behavior, validating the practical applicability of the existence theory.

Lemma 3.1. *Let $0 < \alpha_i < 1$, $\beta_i > 0$, $\gamma_i > 0$ for $i = 1,2$, and $\mathscr{F} \in \mathscr{C}([a,b] \times \mathbb{R}, \mathbb{R})$. The function $x \in \mathscr{C}([a,b],\mathbb{R})$ is a solution of the nonlinear boundary value problem*

$$
{}^{G}D_{a^+}^{\alpha_1,\beta_1,\gamma_1;\Psi}\left({}^{G}D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi} + p\right)x(t) + \delta x^3(t) = \mathscr{F}(t, x(t)), \quad t \in [a,b], \tag{3}
$$

$$
x(a) = 0, \quad x(b) = \sum_{i\in\mathscr{I}} \delta_i x(\eta_i) + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\tau_j;\Psi} x(\theta_j) + \sum_{k\in\mathscr{K}} \lambda_k {}^{H}\mathscr{D}_{a^+}^{\mu_k;\Psi} x(\xi_k), \tag{4}
$$

if and only if

$$
\begin{aligned}
x(t) = {} & I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(t,x(t)) - \delta x^3(t)\right] - pI_{a^+}^{\alpha_2;\Psi}x(t) + \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t,a)}{\Lambda\Gamma(\varsigma_1+\alpha_2)} \\
& \times \Bigg[x(b) + p\Big(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\alpha_2;\Psi}x(\eta_i) + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\alpha_2+\tau_j;\Psi}x(\theta_j) + \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\alpha_2-\mu_k;\Psi}x(\xi_k)\Big) \\
& - \Big(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(\eta_i, x(\eta_i)) - \delta x^3(\eta_i)\right] + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}\left[\mathscr{F}(\theta_j, x(\theta_j)) - \delta x^3(\theta_j)\right] \\
& + \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}\left[\mathscr{F}(\xi_k, x(\xi_k)) - \delta x^3(\xi_k)\right]\Big)\Bigg], \quad t \in [a,b],
\end{aligned} \tag{5}
$$

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where

$$\Lambda = \sum_{i\in\mathscr{I}} \delta_i \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(\eta_i,a)}{\Gamma(\varsigma_1+\alpha_2)} + \sum_{j\in\mathscr{J}} \omega_j \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2+\tau_j-1}(\theta_j,a)}{\Gamma(\varsigma_1+\alpha_2+\tau_j)} + \sum_{k\in\mathscr{K}} \lambda_k \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-\mu_k-1}(\xi_k,a)}{\Gamma(\varsigma_1+\alpha_2-\mu_k)} \neq 0,$$

and $\varsigma_i = \alpha_i + \beta_i(1-\alpha_i)$ *for* $i = 1, 2,$ *with* $\mathfrak{z} = \alpha_1 + \alpha_2$.

Proof. We begin with equation (3):

$$ {}^{G}D_{a^+}^{\alpha_1,\beta_1,\gamma_1;\Psi}\left({}^{G}D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi}x(t) + px(t)\right) + \delta x^3(t) = \mathscr{F}(t,x(t)). \tag{6}$$

Rearranging equation (6):

$$ {}^{G}D_{a^+}^{\alpha_1,\beta_1,\gamma_1;\Psi}\left({}^{G}D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi}x(t) + px(t)\right) = \mathscr{F}(t,x(t)) - \delta x^3(t). \tag{7}$$

Applying the fractional integral operator $I_{a^+}^{\alpha_1;\Psi}$ to both sides of equation (7), and using the left-inverse property:

$$I_{a^+}^{\alpha_1;\Psi}\left[{}^{G}D_{a^+}^{\alpha_1,\beta_1,\gamma_1;\Psi}g(t)\right] = g(t) - \sum_{k=0}^{\lceil\alpha_1\rceil-1} \frac{g^{(k)}(a^+)}{\Gamma(\varsigma_1-k)} \mathscr{Q}_{\Psi}^{\varsigma_1-k-1}(t,a), \tag{8}$$

where $g(t) = {}^{G}D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi}x(t) + px(t)$.

Since $0 < \alpha_1 < 1$, we have $\lceil\alpha_1\rceil = 1$, so:

$$ {}^{G}D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi}x(t) + px(t) = I_{a^+}^{\alpha_1;\Psi}[\mathscr{F}(t,x(t)) - \delta x^3(t)] + \frac{c_0}{\Gamma(\varsigma_1)}\mathscr{Q}_{\Psi}^{\varsigma_1-1}(t,a), \tag{9}$$

for some constant c_0.

Rearranging equation (9):

$$ {}^{G}D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi}x(t) = I_{a^+}^{\alpha_1;\Psi}[\mathscr{F}(t,x(t)) - \delta x^3(t)] + \frac{c_0}{\Gamma(\varsigma_1)}\mathscr{Q}_{\Psi}^{\varsigma_1-1}(t,a) - px(t). \tag{10}$$

Applying the fractional integral operator $I_{a^+}^{\alpha_2;\Psi}$ to both sides of equation (10):

$$x(t) = I_{a^+}^{\alpha_2;\Psi}\left[I_{a^+}^{\alpha_1;\Psi}[\mathscr{F}(t,x(t)) - \delta x^3(t)] + \frac{c_0}{\Gamma(\varsigma_1)}\mathscr{Q}_{\Psi}^{\varsigma_1-1}(t,a) - px(t)\right] + \frac{c_1}{\Gamma(\varsigma_2)}\mathscr{Q}_{\Psi}^{\varsigma_2-1}(t,a). \tag{11}$$

Using the semigroup property $I_{a^+}^{\alpha_1;\Psi}I_{a^+}^{\alpha_2;\Psi} = I_{a^+}^{\alpha_1+\alpha_2;\Psi} = I_{a^+}^{\mathfrak{z};\Psi}$ and linearity:

$$x(t) = I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(t,x(t)) - \delta x^3(t)] - pI_{a^+}^{\alpha_2;\Psi}x(t) + \frac{c_0}{\Gamma(\varsigma_1+\alpha_2)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t,a) + \frac{c_1}{\Gamma(\varsigma_2)}\mathscr{Q}_{\Psi}^{\varsigma_2-1}(t,a). \tag{12}$$

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Substituting $t = a$ into equation (12) and using $x(a) = 0$:

$$0 = I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(a, x(a)) - \delta x^3(a)] - pI_{a^+}^{\alpha_2;\Psi}x(a) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(a, a) + \frac{c_1}{\Gamma(\varsigma_2)}\mathscr{Q}_{\Psi}^{\varsigma_2-1}(a, a).$$

Since $I_{a^+}^{\nu;\Psi}f(a) = 0$ and $\mathscr{Q}_{\Psi}^{\nu-1}(a, a) = 0$ for $\nu > 0$, we get:

$$0 = 0 - 0 + 0 + \frac{c_1}{\Gamma(\varsigma_2)} \cdot 0 = 0.$$

However, more carefully, since $\mathscr{Q}_{\Psi}^{\varsigma_2-1}(t, a) \to 0$ as $t \to a^+$ for $\varsigma_2 > 1$, but the coefficient c_1 must be determined. By the properties of the fractional integral at the lower limit, we conclude $c_1 = 0$.

Therefore:

$$x(t) = I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(t, x(t)) - \delta x^3(t)] - pI_{a^+}^{\alpha_2;\Psi}x(t) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t, a). \quad (13)$$

To handle the boundary condition, we compute:

For the Hilfer derivative:

$$\begin{aligned}{}^H\mathscr{D}_{a^+}^{\mu_k;\Psi}x(t) &= {}^H\mathscr{D}_{a^+}^{\mu_k;\Psi}\left[I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(t, x(t)) - \delta x^3(t)] - pI_{a^+}^{\alpha_2;\Psi}x(t) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t, a)\right] \\ &= I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}[\mathscr{F}(t, x(t)) - \delta x^3(t)] - pI_{a^+}^{\alpha_2-\mu_k;\Psi}x(t) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2 - \mu_k)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-\mu_k-1}(t, a).\end{aligned}$$

For the fractional integral:

$$\begin{aligned}I_{a^+}^{\tau_j;\Psi}x(t) &= I_{a^+}^{\tau_j;\Psi}\left[I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(t, x(t)) - \delta x^3(t)] - pI_{a^+}^{\alpha_2;\Psi}x(t) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t, a)\right] \\ &= I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}[\mathscr{F}(t, x(t)) - \delta x^3(t)] - pI_{a^+}^{\alpha_2+\tau_j;\Psi}x(t) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2 + \tau_j)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2+\tau_j-1}(t, a).\end{aligned}$$

Substituting these expressions into the boundary condition (4):

$$\begin{aligned}x(b) &= \sum_{i\in\mathscr{I}}\delta_i x(\eta_i) + \sum_{j\in\mathscr{J}}\omega_j I_{a^+}^{\tau_j;\Psi}x(\theta_j) + \sum_{k\in\mathscr{K}}\lambda_k {}^H\mathscr{D}_{a^+}^{\mu_k;\Psi}x(\xi_k) \\ &= \sum_{i\in\mathscr{I}}\delta_i\left[I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(\eta_i, x(\eta_i)) - \delta x^3(\eta_i)] - pI_{a^+}^{\alpha_2;\Psi}x(\eta_i) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(\eta_i, a)\right] \\ &\quad + \sum_{j\in\mathscr{J}}\omega_j\left[I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}[\mathscr{F}(\theta_j, x(\theta_j)) - \delta x^3(\theta_j)] - pI_{a^+}^{\alpha_2+\tau_j;\Psi}x(\theta_j) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2 + \tau_j)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2+\tau_j-1}(\theta_j, a)\right] \\ &\quad + \sum_{k\in\mathscr{K}}\lambda_k\left[I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}[\mathscr{F}(\xi_k, x(\xi_k)) - \delta x^3(\xi_k)] - pI_{a^+}^{\alpha_2-\mu_k;\Psi}x(\xi_k) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2 - \mu_k)}\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-\mu_k-1}(\xi_k, a)\right].\end{aligned}$$

Solving for Constant c_0, Collecting terms involving c_0:

$$c_0\left[\sum_{i\in\mathscr{I}}\delta_i\frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(\eta_i, a)}{\Gamma(\varsigma_1 + \alpha_2)} + \sum_{j\in\mathscr{J}}\omega_j\frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2+\tau_j-1}(\theta_j, a)}{\Gamma(\varsigma_1 + \alpha_2 + \tau_j)} + \sum_{k\in\mathscr{K}}\lambda_k\frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-\mu_k-1}(\xi_k, a)}{\Gamma(\varsigma_1 + \alpha_2 - \mu_k)}\right] = \Lambda c_0.$$

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Solving for c_0:

$$c_0 = \frac{1}{\Lambda}\Bigg[x(b) + p\Bigg(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\alpha_2;\Psi} x(\eta_i) + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\alpha_2+\tau_j;\Psi} x(\theta_j) + \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\alpha_2-\mu_k;\Psi} x(\xi_k)\Bigg)$$
$$- \Bigg(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(\eta_i, x(\eta_i)) - \delta x^3(\eta_i)] + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}[\mathscr{F}(\theta_j, x(\theta_j)) - \delta x^3(\theta_j)]$$
$$+ \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}[\mathscr{F}(\xi_k, x(\xi_k)) - \delta x^3(\xi_k)]\Bigg)\Bigg].$$

After determining the constant c_0 from the boundary condition algebraic equation,we substitute back into the solution representation (13):

$$x(t) = I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(t, x(t)) - \delta x^3(t)] - pI_{a^+}^{\alpha_2;\Psi} x(t) + \frac{c_0}{\Gamma(\varsigma_1 + \alpha_2)} \mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t, a).$$

Replacing c_0 by its explicit expression, we obtain the integral equation:

$$x(t) = I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(t, x(t)) - \delta x^3(t)\right] - pI_{a^+}^{\alpha_2;\Psi} x(t) + \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t, a)}{\Lambda\Gamma(\varsigma_1 + \alpha_2)}$$
$$\times \Bigg[x(b) + p\Big(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\alpha_2;\Psi} x(\eta_i) + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\alpha_2+\tau_j;\Psi} x(\theta_j) + \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\alpha_2-\mu_k;\Psi} x(\xi_k)\Big)$$
$$- \Big(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(\eta_i, x(\eta_i)) - \delta x^3(\eta_i)\right] + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}\left[\mathscr{F}(\theta_j, x(\theta_j)) - \delta x^3(\theta_j)\right]$$
$$+ \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}\left[\mathscr{F}(\xi_k, x(\xi_k)) - \delta x^3(\xi_k)\right]\Big)\Bigg], \quad t \in [a, b],$$

where

$$\Lambda = \sum_{i\in\mathscr{I}} \delta_i \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(\eta_i, a)}{\Gamma(\varsigma_1 + \alpha_2)} + \sum_{j\in\mathscr{J}} \omega_j \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2+\tau_j-1}(\theta_j, a)}{\Gamma(\varsigma_1 + \alpha_2 + \tau_j)} + \sum_{k\in\mathscr{K}} \lambda_k \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-\mu_k-1}(\xi_k, a)}{\Gamma(\varsigma_1 + \alpha_2 - \mu_k)} \neq 0,$$

and $\varsigma_i = \alpha_i + \beta_i(1 - \alpha_i)$ for $i = 1, 2$, with $\mathfrak{z} = \alpha_1 + \alpha_2$. This integral form fully characterizes the solution $x(t)$ in terms of fractional integral operators, the nonlinear function $\mathscr{F}$, the nonlinear term $\delta x^3(t)$, the boundary values, and the kernel functions $\mathscr{Q}_\Psi$ associated with the generalized Ψ-Prabhakar fractional operators.

The converse follows by direct verification: if $x(t)$ satisfies (5), then applying the differential o perators a nd b oundary c onditions t o t his e xpression recovers the original BVP.

□

Remark 3.2. Consider the Banach space

$$\mathscr{C} = \mathscr{C}([a, b], \mathbb{R})$$

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of all continuous real-valued functions defined on $[a,b]$, equipped with the supremum norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|.$$

In light of Lemma 3.1, we define the operator

$$\zeta : \mathscr{C} \to \mathscr{C}$$

by

$$\begin{aligned}
(\zeta x)(t) &= I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(t,x(t)) - \delta x^3(t)\right] - pI_{a^+}^{\alpha_2;\Psi}x(t) \\
&\quad + \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{\Lambda\Gamma(\varsigma_1+\alpha_2)}\Bigg[x(b) + p\Bigg(\sum_{i\in\mathscr{I}}\delta_i I_{a^+}^{\alpha_2;\Psi}x(\eta_i) + \sum_{j\in\mathscr{J}}\omega_j I_{a^+}^{\alpha_2+\tau_j;\Psi}x(\theta_j) \\
&\quad + \sum_{k\in\mathscr{K}}\lambda_k I_{a^+}^{\alpha_2-\mu_k;\Psi}x(\xi_k)\Bigg) - \Bigg(\sum_{i\in\mathscr{I}}\delta_i I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(\eta_i,x(\eta_i)) - \delta x^3(\eta_i)\right] \\
&\quad + \sum_{j\in\mathscr{J}}\omega_j I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}\left[\mathscr{F}(\theta_j,x(\theta_j)) - \delta x^3(\theta_j)\right] + \sum_{k\in\mathscr{K}}\lambda_k I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}\left[\mathscr{F}(\xi_k,x(\xi_k)) - \delta x^3(\xi_k)\right]\Bigg)\Bigg].
\end{aligned}$$

It is worth noting that the nonlinear Duffing-type boundary value problem (3)–(4) has a solution if and only if the operator ζ has a fixed point in $\mathscr{C}$.

For convenience, set the following quantities that will appear in the analysis of ζ:

$$\Omega_1 = \frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} + \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)}\left[\sum_{i\in\mathscr{I}}|\delta_i|\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} + \sum_{j\in\mathscr{J}}|\omega_j|\frac{\mathscr{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} + \sum_{k\in\mathscr{K}}|\lambda_k|\frac{\mathscr{Q}_\Psi^{\mathfrak{z}-\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}\right],$$

$$\begin{aligned}
\Omega_2 &= |p|\Bigg\{\frac{\mathscr{Q}_\Psi^{\alpha_2}(b,a)}{\Gamma(\alpha_2+1)} + \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)}\Bigg[\sum_{i\in\mathscr{I}}|\delta_i|\frac{\mathscr{Q}_\Psi^{\alpha_2}(\eta_i,a)}{\Gamma(\alpha_2+1)} + \sum_{j\in\mathscr{J}}|\omega_j|\frac{\mathscr{Q}_\Psi^{\alpha_2+\tau_j}(\theta_j,a)}{\Gamma(\alpha_2+\tau_j)} \\
&\quad + \sum_{k\in\mathscr{K}}|\lambda_k|\frac{\mathscr{Q}_\Psi^{\alpha_2-\mu_k}(\xi_k,a)}{\Gamma(\alpha_2-\mu_k+1)}\Bigg]\Bigg\}.
\end{aligned}$$

These constants Ω_1, Ω_2 quantify bounds related to the fractional integral kernels, coefficients of the boundary condition, and the parameter p, and may be used to establish properties such as contractiveness or boundedness of ζ, which are crucial for fixed point theorems ensuring existence and uniqueness of solutions.

In summary, the study of the fixed points of the operator ζ in the Banach space $\mathscr{C}$ reduces the original nonlinear boundary value problem to a functional analytic problem, paving the way for existence and uniqueness results via classical fixed point methods.

Theorem 3.3. *Consider the following assumptions:*

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(A1) The nonlinear function $\mathscr{F} : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the growth condition $|\mathscr{F}(t, x)| \leq \phi(t)(1 + |x|)$ for all $(t, x) \in [a, b] \times \mathbb{R}$, where $\phi \in \mathscr{C}([a, b], \mathbb{R}^+)$,

(A2) The nonlinear coefficient satisfies $|\delta| \leq M_\delta$ for some positive constant M_δ,

(A3) The contractivity condition $\Omega_2 + 3M_\delta \varepsilon^2 \Omega_1 < 1$ holds, where Ω_1, Ω_2 are the constants defined in Remark 3.2.

Then the nonlinear Duffing-type fractional boundary value problem (3)–(4)

$$
\begin{aligned}
&{}^G D_{a^+}^{\alpha_1,\beta_1,\gamma_1;\Psi} \left({}^G D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi} + p \right) x(t) + \delta x^3(t) = \mathscr{F}(t, x(t)), \quad t \in [a, b], \\
&x(a) = 0, \quad x(b) = \sum_{i \in \mathscr{I}} \delta_i x(\eta_i) + \sum_{j \in \mathscr{J}} \omega_j I_{a^+}^{\tau_j;\Psi} x(\theta_j) + \sum_{k \in \mathscr{K}} \lambda_k {}^H \mathscr{D}_{a^+}^{\mu_k;\Psi} x(\xi_k),
\end{aligned}
$$

admits at least one solution on $[a, b]$.

Proof. The proof strategy employs Krasnoselskii's fixed point theorem by decomposing the solution operator into compact and contractive components.

Phase I: Construction of the Solution Space Define the closed ball in the Banach space $\mathscr{C}([a, b], \mathbb{R})$:

$$B_\varepsilon = \{x \in \mathscr{C}([a, b], \mathbb{R}) : \|x\|_\infty \leq \varepsilon\}$$

where ε is chosen sufficiently large such that

$$\varepsilon \geq \frac{(1 + \varepsilon)\|\phi\|_\infty \Omega_1 + M_\delta \varepsilon^3 \Omega_1}{1 - \Omega_2 - 3M_\delta \varepsilon^2 \Omega_1}.$$

The existence of such $\varepsilon > 0$ follows from assumption (A3) and continuity arguments.

Phase II: Operator Decomposition Strategy We decompose the integral operator ζ as $\zeta = \mathcal{T}_1 + \mathcal{T}_2$, where:

The *nonlinear compact part* is defined by:

$$
\begin{aligned}
(\mathcal{T}_1 x)(t) = {} & I_{a^+}^{\mathfrak{z};\Psi} \left[\mathscr{F}(t, x(t)) - \delta x^3(t) \right] \\
& - \frac{\mathscr{Q}_\Psi^{\varsigma_1 + \alpha_2 - 1}(t, a)}{\Lambda \Gamma(\varsigma_1 + \alpha_2)} \left[\sum_{i \in \mathscr{I}} \delta_i I_{a^+}^{\mathfrak{z};\Psi} \left[\mathscr{F}(\eta_i, x(\eta_i)) - \delta x^3(\eta_i) \right] \right. \\
& + \sum_{j \in \mathscr{J}} \omega_j I_{a^+}^{\mathfrak{z}+\tau_j;\Psi} \left[\mathscr{F}(\theta_j, x(\theta_j)) - \delta x^3(\theta_j) \right] \\
& \left. + \sum_{k \in \mathscr{K}} \lambda_k I_{a^+}^{\mathfrak{z}-\mu_k;\Psi} \left[\mathscr{F}(\xi_k, x(\xi_k)) - \delta x^3(\xi_k) \right] \right]
\end{aligned}
$$

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The *linear contractive part* is given by:

$$(\mathcal{T}_2 x)(t) = -pI_{a^+}^{\alpha_2;\Psi} x(t) + \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t,a)}{\Lambda\Gamma(\varsigma_1+\alpha_2)} \Bigg[x(b) + p\left(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\alpha_2;\Psi} x(\eta_i) + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\alpha_2+\tau_j;\Psi} x(\theta_j) + \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\alpha_2-\mu_k;\Psi} x(\xi_k)\right)\Bigg]$$

Phase III: Ball Invariance Verification (Detailed Analysis)

For arbitrary functions $u, v \in B_\varepsilon$, we need to establish that $\|\mathcal{T}_1 u + \mathcal{T}_2 v\|_\infty \leq \varepsilon$. We begin by analyzing each component systematically.

Decomposition of the Combined Operator

For any $t \in [a,b]$, we have:

$$\begin{aligned}
|(\mathcal{T}_1 u)(t) + (\mathcal{T}_2 v)(t)| = \Bigg| & I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(t,u(t)) - \delta u^3(t)\right] - pI_{a^+}^{\alpha_2;\Psi} v(t) \\
& + \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t,a)}{\Lambda\Gamma(\varsigma_1+\alpha_2)}\Bigg[v(b) + p\Bigg(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\alpha_2;\Psi} v(\eta_i) + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\alpha_2+\tau_j;\Psi} v(\theta_j) \\
& + \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\alpha_2-\mu_k;\Psi} v(\xi_k)\Bigg) - \Bigg(\sum_{i\in\mathscr{I}} \delta_i I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(\eta_i,u(\eta_i)) - \delta u^3(\eta_i)\right] \\
& + \sum_{j\in\mathscr{J}} \omega_j I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}\left[\mathscr{F}(\theta_j,u(\theta_j)) - \delta u^3(\theta_j)\right] \\
& + \sum_{k\in\mathscr{K}} \lambda_k I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}\left[\mathscr{F}(\xi_k,u(\xi_k)) - \delta u^3(\xi_k)\right]\Bigg)\Bigg]\Bigg|
\end{aligned}$$

Application of Triangle Inequality

By the triangle inequality:

$$\begin{aligned}
|(\mathcal{T}_1 u)(t) + (\mathcal{T}_2 v)(t)| \leq & I_{a^+}^{\mathfrak{z};\Psi}\left[|\mathscr{F}(t,u(t))| + |\delta||u(t)|^3\right] + |p| I_{a^+}^{\alpha_2;\Psi}|v(t)| \\
& + \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t,a)}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)}\Bigg[|v(b)| + |p|\Bigg(\sum_{i\in\mathscr{I}} |\delta_i| I_{a^+}^{\alpha_2;\Psi}|v(\eta_i)| \\
& + \sum_{j\in\mathscr{J}} |\omega_j| I_{a^+}^{\alpha_2+\tau_j;\Psi}|v(\theta_j)| + \sum_{k\in\mathscr{K}} |\lambda_k| I_{a^+}^{\alpha_2-\mu_k;\Psi}|v(\xi_k)|\Bigg) \\
& + \sum_{i\in\mathscr{I}} |\delta_i| I_{a^+}^{\mathfrak{z};\Psi}\left[|\mathscr{F}(\eta_i,u(\eta_i))| + |\delta||u(\eta_i)|^3\right] \\
& + \sum_{j\in\mathscr{J}} |\omega_j| I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}\left[|\mathscr{F}(\theta_j,u(\theta_j))| + |\delta||u(\theta_j)|^3\right] \\
& + \sum_{k\in\mathscr{K}} |\lambda_k| I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}\left[|\mathscr{F}(\xi_k,u(\xi_k))| + |\delta||u(\xi_k)|^3\right]\Bigg]
\end{aligned}$$

Application of Assumption (A1)

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Since $u \in B_\varepsilon$, we have $\|u\|_\infty \leq \varepsilon$, which implies $|u(t)| \leq \varepsilon$ for all $t \in [a,b]$. Applying assumption (A1) at any point $t \in [a,b]$:

$$|\mathscr{F}(t,u(t))| \leq \phi(t)(1+|u(t)|) \leq \phi(t)(1+\varepsilon) \leq \|\phi\|_\infty(1+\varepsilon)$$

Similarly, at the boundary points:

$$\begin{aligned}|\mathscr{F}(\eta_i,u(\eta_i))| &\leq \phi(\eta_i)(1+|u(\eta_i)|) \leq \|\phi\|_\infty(1+\varepsilon)\\ |\mathscr{F}(\theta_j,u(\theta_j))| &\leq \phi(\theta_j)(1+|u(\theta_j)|) \leq \|\phi\|_\infty(1+\varepsilon)\\ |\mathscr{F}(\xi_k,u(\xi_k))| &\leq \phi(\xi_k)(1+|u(\xi_k)|) \leq \|\phi\|_\infty(1+\varepsilon)\end{aligned}$$

Bounding the Cubic Nonlinearity
Using assumption (A2) and the fact that $|u(t)| \leq \varepsilon$:

$$|\delta||u(t)|^3 \leq M_\delta \varepsilon^3 \quad \text{for all } t \in [a,b]$$

This bound also applies at all boundary points:

$$|\delta||u(\eta_i)|^3 \leq M_\delta\varepsilon^3, \quad |\delta||u(\theta_j)|^3 \leq M_\delta\varepsilon^3, \quad |\delta||u(\xi_k)|^3 \leq M_\delta\varepsilon^3$$

Estimating Fractional Integral Terms
For the main fractional integral term:

$$I_{a^+}^{\mathfrak{z};\Psi}\left[|\mathscr{F}(t,u(t))| + |\delta||u(t)|^3\right] \leq I_{a^+}^{\mathfrak{z};\Psi}\left[\|\phi\|_\infty(1+\varepsilon) + M_\delta\varepsilon^3\right]$$

Using the fundamental property of fractional integrals:

$$I_{a^+}^{\mathfrak{z};\Psi} C = \frac{C\mathscr{Q}_\Psi^{\mathfrak{z}}(t,a)}{\Gamma(\mathfrak{z}+1)} \leq \frac{C\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)}$$

Therefore:

$$I_{a^+}^{\mathfrak{z};\Psi}\left[|\mathscr{F}(t,u(t))| + |\delta||u(t)|^3\right] \leq \left[\|\phi\|_\infty(1+\varepsilon) + M_\delta\varepsilon^3\right]\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)}$$

Estimating Linear Terms
For $v \in B_\varepsilon$, we have $\|v\|_\infty \leq \varepsilon$, so:

$$I_{a^+}^{\alpha_2;\Psi}|v(t)| \leq \varepsilon\frac{\mathscr{Q}_\Psi^{\alpha_2}(b,a)}{\Gamma(\alpha_2+1)}$$

Similarly for boundary terms:

$$\begin{aligned}I_{a^+}^{\alpha_2;\Psi}|v(\eta_i)| &\leq \varepsilon\frac{\mathscr{Q}_\Psi^{\alpha_2}(\eta_i,a)}{\Gamma(\alpha_2+1)}\\ I_{a^+}^{\alpha_2+\tau_j;\Psi}|v(\theta_j)| &\leq \varepsilon\frac{\mathscr{Q}_\Psi^{\alpha_2+\tau_j}(\theta_j,a)}{\Gamma(\alpha_2+\tau_j+1)}\\ I_{a^+}^{\alpha_2-\mu_k;\Psi}|v(\xi_k)| &\leq \varepsilon\frac{\mathscr{Q}_\Psi^{\alpha_2-\mu_k}(\xi_k,a)}{\Gamma(\alpha_2-\mu_k+1)}\end{aligned}$$

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Estimating Nonlinear Boundary Terms

For the boundary fractional integrals involving the nonlinear function:

$$I_{a^+}^{\mathfrak{z};\Psi}\left[|\mathscr{F}(\eta_i,u(\eta_i))|+|\delta||u(\eta_i)|^3\right]\leq\left[\|\phi\|_\infty(1+\varepsilon)+M_\delta\varepsilon^3\right]\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)}$$

$$I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}\left[|\mathscr{F}(\theta_j,u(\theta_j))|+|\delta||u(\theta_j)|^3\right]\leq\left[\|\phi\|_\infty(1+\varepsilon)+M_\delta\varepsilon^3\right]\frac{\mathscr{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)}$$

$$I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}\left[|\mathscr{F}(\xi_k,u(\xi_k))|+|\delta||u(\xi_k)|^3\right]\leq\left[\|\phi\|_\infty(1+\varepsilon)+M_\delta\varepsilon^3\right]\frac{\mathscr{Q}_\Psi^{\mathfrak{z}-\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}$$

Combining All Estimates

Substituting all estimates into our main inequality:

$$\begin{aligned}|(\mathcal{T}_1u)(t)+(\mathcal{T}_2v)(t)|\leq{}&\left[\|\phi\|_\infty(1+\varepsilon)+M_\delta\varepsilon^3\right]\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)}+|p|\varepsilon\frac{\mathscr{Q}_\Psi^{\alpha_2}(b,a)}{\Gamma(\alpha_2+1)}\\&+\frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)}\Bigg[\varepsilon+|p|\varepsilon\Bigg(\sum_{i\in\mathscr{I}}|\delta_i|\frac{\mathscr{Q}_\Psi^{\alpha_2}(\eta_i,a)}{\Gamma(\alpha_2+1)}\\&+\sum_{j\in\mathscr{J}}|\omega_j|\frac{\mathscr{Q}_\Psi^{\alpha_2+\tau_j}(\theta_j,a)}{\Gamma(\alpha_2+\tau_j+1)}+\sum_{k\in\mathscr{K}}|\lambda_k|\frac{\mathscr{Q}_\Psi^{\alpha_2-\mu_k}(\xi_k,a)}{\Gamma(\alpha_2-\mu_k+1)}\Bigg)\\&+\left[\|\phi\|_\infty(1+\varepsilon)+M_\delta\varepsilon^3\right]\Bigg(\sum_{i\in\mathscr{I}}|\delta_i|\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)}\\&+\sum_{j\in\mathscr{J}}|\omega_j|\frac{\mathscr{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)}+\sum_{k\in\mathscr{K}}|\lambda_k|\frac{\mathscr{Q}_\Psi^{\mathfrak{z}-\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}\Bigg)\Bigg]\end{aligned}$$

Recognition of Ω_1 and Ω_2

Factoring out common terms and using the definitions from Remark 3.2:

$$|(\mathcal{T}_1u)(t)+(\mathcal{T}_2v)(t)|\leq\left[\|\phi\|_\infty(1+\varepsilon)+M_\delta\varepsilon^3\right]\Omega_1+\varepsilon\Omega_2$$

Final Verification

By our choice of ε such that:

$$\varepsilon\geq\frac{(1+\varepsilon)\|\phi\|_\infty\Omega_1+M_\delta\varepsilon^3\Omega_1}{1-\Omega_2}$$

We can rearrange to get:

$$\left[\|\phi\|_\infty(1+\varepsilon)+M_\delta\varepsilon^3\right]\Omega_1+\varepsilon\Omega_2\leq\varepsilon$$

Therefore, $\|\mathcal{T}_1u+\mathcal{T}_2v\|_\infty\leq\varepsilon$, which confirms that $\mathcal{T}_1u+\mathcal{T}_2v\in B_\varepsilon$ for all $u,v\in B_\varepsilon$.

Phase IV: Contractivity Analysis of $\mathcal{T}_2$ (Detailed)

We aim to show that the operator $\mathcal{T}_2$ is a contraction mapping on the closed ball B_ε in the Banach space $\mathscr{C}([a,b],\mathbb{R})$.

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Definition and Difference Estimate

Let $u, v \in B_\varepsilon$, i.e., $\|u\|_\infty, \|v\|_\infty \leq \varepsilon$. Recall the definition:

$$(\mathcal{T}_2 x)(t) = -pI_{a^+}^{\alpha_2;\Psi} x(t) + \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(t,a)}{\Lambda\Gamma(\varsigma_1+\alpha_2)} \left[x(b) + p \sum_{i \in \mathscr{I}} \delta_i I_{a^+}^{\alpha_2;\Psi} x(\eta_i) + \dots \right].$$

We consider the difference for $t \in [a,b]$:

$$\begin{aligned} |(\mathcal{T}_2 u)(t) - (\mathcal{T}_2 v)(t)| \leq |p| \left| I_{a^+}^{\alpha_2;\Psi}(u-v)(t) \right| \\ + \left| \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(t,a)}{\Lambda\Gamma(\varsigma_1+\alpha_2)} \right| \times \Bigg| (u-v)(b) + p \sum_{i \in \mathscr{I}} \delta_i I_{a^+}^{\alpha_2;\Psi}(u-v)(\eta_i) \\ + p \sum_{j \in \mathscr{J}} \omega_j I_{a^+}^{\alpha_2+\tau_j;\Psi}(u-v)(\theta_j) + p \sum_{k \in \mathscr{K}} \lambda_k I_{a^+}^{\alpha_2-\mu_k;\Psi}(u-v)(\xi_k) \Bigg|. \end{aligned}$$

Bounding the Supremum Norm Terms

Apply the supremum norm bound:

$$|u(t) - v(t)| \leq \|u-v\|_\infty \quad \forall t \in [a,b].$$

Also, fractional integrals satisfy

$$|I_{a^+}^{\nu;\Psi} f(t)| \leq \frac{\mathscr{Q}_\Psi^{\nu}(b,a)}{\Gamma(\nu+1)} \|f\|_\infty.$$

Hence,

$$\left| I_{a^+}^{\alpha_2;\Psi}(u-v)(t) \right| \leq \frac{\mathscr{Q}_\Psi^{\alpha_2}(b,a)}{\Gamma(\alpha_2+1)} \|u-v\|_\infty.$$

Similarly, for evaluation points:

$$|(u-v)(b)| \leq \|u-v\|_\infty,$$

and

$$|I_{a^+}^{\alpha_2;\Psi}(u-v)(\eta_i)| \leq \frac{\mathscr{Q}_\Psi^{\alpha_2}(\eta_i,a)}{\Gamma(\alpha_2+1)} \|u-v\|_\infty,$$

and analogously for other fractional integral terms:

$$|I_{a^+}^{\alpha_2+\tau_j;\Psi}(u-v)(\theta_j)| \leq \frac{\mathscr{Q}_\Psi^{\alpha_2+\tau_j}(\theta_j,a)}{\Gamma(\alpha_2+\tau_j+1)} \|u-v\|_\infty,$$

$$|I_{a^+}^{\alpha_2-\mu_k;\Psi}(u-v)(\xi_k)| \leq \frac{\mathscr{Q}_\Psi^{\alpha_2-\mu_k}(\xi_k,a)}{\Gamma(\alpha_2-\mu_k+1)} \|u-v\|_\infty.$$

Combining Bounds

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Factor out $\|u-v\|_\infty$:

$$|(\mathcal{T}_2u)(t)-(\mathcal{T}_2v)(t)| \le \Bigg(|p|\frac{\mathscr{Q}_\Psi^{\alpha_2}(b,a)}{\Gamma(\alpha_2+1)} + \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)} \times \Big[1 + |p|\sum_{i\in\mathscr{I}}|\delta_i|\frac{\mathscr{Q}_\Psi^{\alpha_2}(\eta_i,a)}{\Gamma(\alpha_2+1)} + |p|\sum_{j\in\mathscr{J}}|\omega_j|\frac{\mathscr{Q}_\Psi^{\alpha_2+\tau_j}(\theta_j,a)}{\Gamma(\alpha_2+\tau_j+1)} + |p|\sum_{k\in\mathscr{K}}|\lambda_k|\frac{\mathscr{Q}_\Psi^{\alpha_2-\mu_k}(\xi_k,a)}{\Gamma(\alpha_2-\mu_k+1)}\Big]\Bigg)\|u-v\|_\infty.$$

Definition of Ω_2 and Conclusion

By definition,

$$\Omega_2 := |p|\frac{\mathscr{Q}_\Psi^{\alpha_2}(b,a)}{\Gamma(\alpha_2+1)} + \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)}\left[1+|p|\sum_{i\in\mathscr{I}}|\delta_i|\frac{\mathscr{Q}_\Psi^{\alpha_2}(\eta_i,a)}{\Gamma(\alpha_2+1)}+\cdots\right].$$

Hence,

$$\|\mathcal{T}_2u-\mathcal{T}_2v\|_\infty \le \Omega_2\|u-v\|_\infty.$$

Finally, assumption (A3) implies:

$$\Omega_2 + 3M_\delta\varepsilon^2\Omega_1 < 1,$$

thus, $\Omega_2 < 1$, and therefore $\mathcal{T}_2$ is a contraction mapping on B_ε. **Phase V: Compactness of $\mathcal{T}_1$**

In this phase, we rigorously establish that the operator $\mathcal{T}_1$, defined as

$$(\mathcal{T}_1x)(t) = I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(t,x(t))-\delta x^3(t)\right] - \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(t,a)}{\Lambda\Gamma(\varsigma_1+\alpha_2)}\left[\sum_{i\in\mathscr{I}}\delta_i I_{a^+}^{\mathfrak{z};\Psi}\left[\mathscr{F}(\eta_i,x(\eta_i))-\delta x^3(\eta_i)\right]+\dots\right],$$

is both continuous and compact on the closed ball B_ε.

To establish that $\mathcal{T}_1$ is compact on B_ε, we must verify three properties: continuity, uniform boundedness, and equicontinuity. By the Arzelà-Ascoli theorem, these properties ensure relative compactness.

Continuity of $\mathcal{T}_1$

Continuity of the Nonlinear Terms

Let $\{x_n\}\subset B_\varepsilon$ be a sequence converging uniformly to $x\in B_\varepsilon$. We need to show that $\mathcal{T}_1x_n\to\mathcal{T}_1x$ uniformly on $[a,b]$.

Consider the nonlinear mapping $G(t,y)=\mathscr{F}(t,y)-\delta y^3$. Since $\mathscr{F}$ is continuous by assumption (A1) and the cubic function $y\mapsto y^3$ is continuous, G is continuous on $[a,b]\times\mathbb{R}$.

For any $t\in[a,b]$ and the convergent sequence $x_n(t)\to x(t)$:

$$|G(t,x_n(t))-G(t,x(t))| = |\mathscr{F}(t,x_n(t))-\mathscr{F}(t,x(t))-\delta(x_n^3(t)-x^3(t))|$$

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Uniform Convergence of Nonlinear Terms

Since $\mathscr{F}$ is continuous on the compact set $[a,b]\times[-\varepsilon,\varepsilon]$, it is uniformly continuous. Therefore, for any $\epsilon_1>0$, there exists $\delta_1>0$ such that:

$$|x_n(t)-x(t)|<\delta_1 \Rightarrow |\mathscr{F}(t,x_n(t))-\mathscr{F}(t,x(t))|<\frac{\epsilon_1}{2}$$

For the cubic term, using the identity $a^3-b^3=(a-b)(a^2+ab+b^2)$:

$$|x_n^3(t)-x^3(t)|=|x_n(t)-x(t)||x_n^2(t)+x_n(t)x(t)+x^2(t)|\le|x_n(t)-x(t)|\cdot 3\varepsilon^2$$

Since $x_n\to x$ uniformly, there exists N_1 such that for $n\ge N_1$:

$$\|x_n-x\|_\infty<\min\left\{\delta_1,\frac{\epsilon_1}{6M_\delta\varepsilon^2}\right\}$$

This gives us:

$$\|G(\cdot,x_n(\cdot))-G(\cdot,x(\cdot))\|_\infty<\epsilon_1 \quad \text{for } n\ge N_1$$

Continuity of Fractional Integral Operators

The fractional integral operator $I_{a^+}^{\nu;\Psi}$ is a bounded linear operator from $\mathscr{C}([a,b],\mathbb{R})$ to itself. Therefore:

$$\left\|I_{a^+}^{\mathfrak{z};\Psi}[G(\cdot,x_n(\cdot))-G(\cdot,x(\cdot))]\right\|_\infty\le\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)}\|G(\cdot,x_n(\cdot))-G(\cdot,x(\cdot))\|_\infty<\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)}\epsilon_1$$

Similarly, for the boundary terms:

$$\left\|I_{a^+}^{\mathfrak{z};\Psi}[G(\eta_i,x_n(\eta_i))-G(\eta_i,x(\eta_i))]\right\|_\infty\le\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)}\epsilon_1$$

$$\left\|I_{a^+}^{\mathfrak{z}+\tau_j;\Psi}[G(\theta_j,x_n(\theta_j))-G(\theta_j,x(\theta_j))]\right\|_\infty\le\frac{\mathscr{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)}\epsilon_1$$

$$\left\|I_{a^+}^{\mathfrak{z}-\mu_k;\Psi}[G(\xi_k,x_n(\xi_k))-G(\xi_k,x(\xi_k))]\right\|_\infty\le\frac{\mathscr{Q}_\Psi^{\mathfrak{z}-\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}\epsilon_1$$

Combining all terms and using the structure of $\mathcal{T}_1$:

$$\|\mathcal{T}_1x_n-\mathcal{T}_1x\|_\infty\le\epsilon_1\left[\frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)}+\frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)}\sum_{\text{all boundary terms}}\right]=\epsilon_1\Omega_1$$

Since ϵ_1 was arbitrary, $\mathcal{T}_1$ is continuous on B_ε.

Uniform Boundedness of $\mathcal{T}_1$

Bounding the Main Fractional Integral

For any $x\in B_\varepsilon$, using assumption (A1):

$$|\mathscr{F}(t,x(t))|\le\phi(t)(1+|x(t)|)\le\|\phi\|_\infty(1+\varepsilon)$$

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And using assumption (A2):

$$|\delta x^3(t)| \leq M_\delta |x(t)|^3 \leq M_\delta \varepsilon^3$$

Therefore:

$$\left| I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(t,x(t)) - \delta x^3(t)] \right| \leq \frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} [\|\phi\|_\infty (1+\varepsilon) + M_\delta \varepsilon^3]$$

Bounding the Boundary Terms

For the boundary contribution, we have:

$$\begin{aligned}
&\left| \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(t,a)}{\Lambda \Gamma(\varsigma_1+\alpha_2)} \right| \times \left| \sum_{i \in \mathscr{I}} \delta_i I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(\eta_i, x(\eta_i)) - \delta x^3(\eta_i)] + \ldots \right| \\
&\leq \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(b,a)}{|\Lambda| \Gamma(\varsigma_1+\alpha_2)} [\|\phi\|_\infty (1+\varepsilon) + M_\delta \varepsilon^3] \\
&\quad \times \left[\sum_{i \in \mathscr{I}} |\delta_i| \frac{\mathscr{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} + \sum_{j \in \mathscr{J}} |\omega_j| \frac{\mathscr{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} + \sum_{k \in \mathscr{K}} |\lambda_k| \frac{\mathscr{Q}_\Psi^{\mathfrak{z}-\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)} \right]
\end{aligned}$$

Using the definition of Ω_1:

$$\|\mathcal{T}_1 x\|_\infty \leq [\|\phi\|_\infty (1+\varepsilon) + M_\delta \varepsilon^3] \Omega_1 =: K_1 < \infty$$

This bound is independent of $x \in B_\varepsilon$, so $\mathcal{T}_1$ is uniformly bounded.

Equicontinuity of $\mathcal{T}_1$

Equicontinuity Setup

To show equicontinuity, we need to prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in B_\varepsilon$ and $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta$:

$$|(\mathcal{T}_1 x)(t_1) - (\mathcal{T}_1 x)(t_2)| < \epsilon$$

Decomposing the Difference

$$\begin{aligned}
|(\mathcal{T}_1 x)(t_1) - (\mathcal{T}_1 x)(t_2)| \leq & \left| I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(\cdot,x(\cdot)) - \delta x^3(\cdot)](t_1) - I_{a^+}^{\mathfrak{z};\Psi}[\mathscr{F}(\cdot,x(\cdot)) - \delta x^3(\cdot)](t_2) \right| \\
& + \left| \frac{\mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(t_1,a) - \mathscr{Q}_\Psi^{\varsigma_1+\alpha_2-1}(t_2,a)}{\Lambda \Gamma(\varsigma_1+\alpha_2)} \right| \times |\text{boundary terms}|
\end{aligned}$$

Continuity of Fractional Integrals

The fractional integral $I_{a^+}^{\mathfrak{z};\Psi}$ of a bounded continuous function produces a continuous function. Specifically, for $g(s) = \mathscr{F}(s,x(s)) - \delta x^3(s)$ with $\|g\|_\infty \leq \|\phi\|_\infty (1+\varepsilon) + M_\delta \varepsilon^3$:

$$\left| I_{a^+}^{\mathfrak{z};\Psi} g(t_1) - I_{a^+}^{\mathfrak{z};\Psi} g(t_2) \right| = \left| \frac{1}{\Gamma(\mathfrak{z})} \int_a^{t_1} \frac{g(s)\Psi'(s)}{[\Psi(t_1) - \Psi(s)]^{1-\mathfrak{z}}} ds - \frac{1}{\Gamma(\mathfrak{z})} \int_a^{t_2} \frac{g(s)\Psi'(s)}{[\Psi(t_2) - \Psi(s)]^{1-\mathfrak{z}}} ds \right|$$

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Uniform Continuity of Kernel Functions

Since Ψ is continuously differentiable and $\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t,a)$ is continuous in t, there exists a modulus of continuity function $\omega(\delta)$ with $\omega(\delta) \to 0$ as $\delta \to 0^+$ such that:

$$|\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t_1,a) - \mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(t_2,a)| \leq \omega(|t_1 - t_2|)$$

Combining for Equicontinuity

For any $\epsilon > 0$, choose $\delta > 0$ small enough so that:

$$\omega(\delta) \cdot \frac{|\text{boundary terms}|}{|\Lambda|\Gamma(\varsigma_1+\alpha_2)} < \frac{\epsilon}{2}$$

and the fractional integral continuity gives the remaining $\frac{\epsilon}{2}$.

Therefore:

$$|(\mathcal{T}_1 x)(t_1) - (\mathcal{T}_1 x)(t_2)| < \epsilon \quad \text{whenever } |t_1 - t_2| < \delta$$

This bound is uniform over $x \in B_\varepsilon$, establishing equicontinuity.

Application of Arzelà-Ascoli Theorem

Having established:

- Continuity of $\mathcal{T}_1$ on B_ε
- Uniform boundedness: $\|\mathcal{T}_1 x\|_\infty \leq K_1$ for all $x \in B_\varepsilon$
- Equicontinuity of the family $\{\mathcal{T}_1 x : x \in B_\varepsilon\}$

By the Arzelà-Ascoli theorem, $\mathcal{T}_1$ is relatively compact on B_ε, which means $\mathcal{T}_1(B_\varepsilon)$ has compact closure in $\mathscr{C}([a,b],\mathbb{R})$.

Therefore, $\mathcal{T}_1 : B_\varepsilon \to \mathscr{C}([a,b],\mathbb{R})$ is a compact operator.

Phase VI: Application of Krasnoselskii's Fixed Point Theorem

Having established the properties of the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ on the closed ball B_ε, we now employ Krasnoselskii's fixed point theorem to conclude the existence of a fixed point of the operator $\zeta = \mathcal{T}_1 + \mathcal{T}_2$.

Recall the following key points:

- The operator ζ maps B_ε into itself, i.e., for all $u, v \in B_\varepsilon$,

  $$\mathcal{T}_1 u + \mathcal{T}_2 v \in B_\varepsilon.$$

  This was established in Phase III (Ball invariance).
- The operator $\mathcal{T}_2 : B_\varepsilon \to \mathscr{C}([a,b],\mathbb{R})$ is a contraction mapping with contraction constant $L < 1$, as shown in Phase IV (Contractivity).
- The operator $\mathcal{T}_1 : B_\varepsilon \to \mathscr{C}([a,b],\mathbb{R})$ is continuous and compact, confirmed in Phase V (Compactness).

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Statement of Krasnoselskii's Fixed Point Theorem

Krasnoselskii's fixed point theorem states:

Let B be a closed, convex, and nonempty subset of a Banach space. Suppose there exist two operators $A, B : B \to B$ such that:

- A is compact and continuous,
- B is a contraction mapping,
- For all $x, y \in B$, $Ax + By \in B$.

Then the operator $A + B$ has at least one fixed point in B.

Verification of Assumptions

In our setting, choosing $A = \mathcal{T}_1$ and $B = \mathcal{T}_2$, the assumptions are fulfilled on the closed ball $B_\varepsilon \subset \mathscr{C}([a, b], \mathbb{R})$:

- $\mathcal{T}_1$ is compact and continuous (Phase V),
- $\mathcal{T}_2$ is a contraction mapping (Phase IV),
- For all $u, v \in B_\varepsilon$, $\mathcal{T}_1 u + \mathcal{T}_2 v \in B_\varepsilon$ (Phase III).

Hence, Krasnoselskii's theorem guarantees the existence of at least one fixed point $x^* \in B_\varepsilon$ such that

$$x^* = \mathcal{T}_1 x^* + \mathcal{T}_2 x^* = \zeta x^*.$$

Interpretation of the Fixed Point

This fixed point x^* satisfies the integral equation associated with the nonlinear Duffing-type fractional boundary value problem, and by Lemma 2.1.1, x^* is a solution to the original boundary value problem (3)–(4).

Thus, under assumptions (A1)–(A3), the operator ζ admits at least one fixed point in B_ε, ensuring that the nonlinear fractional Duffing problem has at least one continuous solution on $[a, b]$.

□

Several special cases of Theorem 3.3 can be derived by imposing specific constraints on the parameters. Below, we present five important corollaries corresponding to particular parameter choices.

(a) Case $\beta_1 = 0$:

For $\beta_1 = 0$, we obtain $\varsigma_1 = \alpha_1 + 0 \cdot (1 - \alpha_1) = \alpha_1$. Define the modified constant $\Omega_2^{[0]}$ as:

$$\begin{aligned}\Omega_2^{[0]} = |p| \Bigg\{ & \frac{\mathscr{Q}_\Psi^{\alpha_2}(b, a)}{\Gamma(\alpha_2 + 1)} + \frac{\mathscr{Q}_\Psi^{\alpha_1+\alpha_2-1}(b, a)}{|\Lambda^{[0]}|\Gamma(\alpha_1 + \alpha_2)} \\ & \times \Bigg[\sum_{i \in \mathscr{I}} |\delta_i| \frac{\mathscr{Q}_\Psi^{\alpha_2}(\eta_i, a)}{\Gamma(\alpha_2 + 1)} + \sum_{j \in \mathscr{J}} |\omega_j| \frac{\mathscr{Q}_\Psi^{\alpha_2+\tau_j}(\theta_j, a)}{\Gamma(\alpha_2 + \tau_j + 1)} \\ & + \sum_{k \in \mathscr{K}} |\lambda_k| \frac{\mathscr{Q}_\Psi^{\alpha_2-\mu_k}(\xi_k, a)}{\Gamma(\alpha_2 - \mu_k + 1)} \Bigg] \Bigg\},\end{aligned}$$

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where

$$\Lambda^{[0]} = \sum_{i\in\mathscr{I}} \delta_i \frac{\mathscr{Q}_{\Psi}^{\alpha_1+\alpha_2-1}(\eta_i, a)}{\Gamma(\alpha_1+\alpha_2)} + \sum_{j\in\mathscr{J}} \omega_j \frac{\mathscr{Q}_{\Psi}^{\alpha_1+\alpha_2+\tau_j-1}(\theta_j, a)}{\Gamma(\alpha_1+\alpha_2+\tau_j)} + \sum_{k\in\mathscr{K}} \lambda_k \frac{\mathscr{Q}_{\Psi}^{\alpha_1+\alpha_2-\mu_k-1}(\xi_k, a)}{\Gamma(\alpha_1+\alpha_2-\mu_k)} \neq 0.$$

Corollary 3.4. *Suppose assumptions (A1) and (A2) hold, along with the condition $\Omega_2^{[0]} + 3M_\delta\varepsilon^2\Omega_1^{[0]} < 1$, where $\Omega_1^{[0]}$ is the corresponding modified Ω_1 with $\varsigma_1 = \alpha_1$. Then the boundary value problem (3)–(4) with $\beta_1 = 0$ admits at least one solution on $[a, b]$.*

(b) Case $\beta_1 = 1$:
For $\beta_1 = 1$, we have $\varsigma_1 = \alpha_1 + 1\cdot(1-\alpha_1) = 1$. Define the constant $\Omega_2^{[1]}$ as:

$$\begin{aligned}\Omega_2^{[1]} = |p| \Bigg\{ &\frac{\mathscr{Q}_{\Psi}^{\alpha_2}(b, a)}{\Gamma(\alpha_2+1)} + \frac{\mathscr{Q}_{\Psi}^{\alpha_2}(b, a)}{|\Lambda^{[1]}|\Gamma(\alpha_2+1)} \\ &\times \Bigg[\sum_{i\in\mathscr{I}} |\delta_i| \frac{\mathscr{Q}_{\Psi}^{\alpha_2}(\eta_i, a)}{\Gamma(\alpha_2+1)} + \sum_{j\in\mathscr{J}} |\omega_j| \frac{\mathscr{Q}_{\Psi}^{\alpha_2+\tau_j}(\theta_j, a)}{\Gamma(\alpha_2+\tau_j+1)} \\ &+ \sum_{k\in\mathscr{K}} |\lambda_k| \frac{\mathscr{Q}_{\Psi}^{\alpha_2-\mu_k}(\xi_k, a)}{\Gamma(\alpha_2-\mu_k+1)} \Bigg] \Bigg\},\end{aligned}$$

where

$$\Lambda^{[1]} = \sum_{i\in\mathscr{I}} \delta_i \frac{\mathscr{Q}_{\Psi}^{\alpha_2}(\eta_i, a)}{\Gamma(\alpha_2+1)} + \sum_{j\in\mathscr{J}} \omega_j \frac{\mathscr{Q}_{\Psi}^{\alpha_2+\tau_j}(\theta_j, a)}{\Gamma(\alpha_2+\tau_j+1)} + \sum_{k\in\mathscr{K}} \lambda_k \frac{\mathscr{Q}_{\Psi}^{\alpha_2-\mu_k}(\xi_k, a)}{\Gamma(\alpha_2-\mu_k+1)} \neq 0.$$

Corollary 3.5. *If assumptions (A1) and (A2) are satisfied, together with $\Omega_2^{[1]} + 3M_\delta\varepsilon^2\Omega_1^{[1]} < 1$, then the problem (3)–(4) with $\beta_1 = 1$ possesses at least one solution on $[a, b]$.*

(c) Case $\beta_2 = 0$:
For $\beta_2 = 0$, we obtain $\varsigma_2 = \alpha_2$. The operator structure simplifies, and we define $\Omega_2^{[2]}$ accordingly:

$$\begin{aligned}\Omega_2^{[2]} = |p| \Bigg\{ &\frac{\mathscr{Q}_{\Psi}^{\alpha_2}(b, a)}{\Gamma(\alpha_2+1)} + \frac{\mathscr{Q}_{\Psi}^{\varsigma_1+\alpha_2-1}(b, a)}{|\Lambda^{[2]}|\Gamma(\varsigma_1+\alpha_2)} \\ &\times \Bigg[\sum_{i\in\mathscr{I}} |\delta_i| \frac{\mathscr{Q}_{\Psi}^{\alpha_2}(\eta_i, a)}{\Gamma(\alpha_2+1)} + \sum_{j\in\mathscr{J}} |\omega_j| \frac{\mathscr{Q}_{\Psi}^{\alpha_2+\tau_j}(\theta_j, a)}{\Gamma(\alpha_2+\tau_j+1)} \\ &+ \sum_{k\in\mathscr{K}} |\lambda_k| \frac{\mathscr{Q}_{\Psi}^{\alpha_2-\mu_k}(\xi_k, a)}{\Gamma(\alpha_2-\mu_k+1)} \Bigg] \Bigg\},\end{aligned}$$

with $\Lambda^{[2]}$ computed using $\varsigma_2 = \alpha_2$.

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Corollary 3.6. *Under assumptions (A1), (A2), and the condition* $\Omega_2^{[2]} + 3M_\delta \varepsilon^2 \Omega_1^{[2]} < 1$, *the boundary value problem* (3)–(4) *with* $\beta_2 = 0$ *has at least one solution on* $[a, b]$.

(d) Case $p = 0$ (No Linear Term):
When $p = 0$, the problem reduces to a purely nonlinear fractional Duffing equation. In this case, $\Omega_2^{[p=0]} = 0$, and the contractivity condition simplifies to:

$$3M_\delta \varepsilon^2 \Omega_1 < 1.$$

Corollary 3.7. *Suppose (A1) and (A2) hold with* $p = 0$, *and assume that* $3M_\delta \varepsilon^2 \Omega_1 < 1$ *for some* $\varepsilon > 0$ *satisfying the ball radius condition. Then the nonlinear fractional boundary value problem*

$$ {}^G D_{a^+}^{\alpha_1,\beta_1,\gamma_1;\Psi} \left({}^G D_{a^+}^{\alpha_2,\beta_2,\gamma_2;\Psi} x(t) \right) + \delta x^3(t) = \mathscr{F}(t, x(t)) $$

subject to (4) *admits at least one solution on* $[a, b]$.

(e) Case $\Psi(t) = t$ (Standard Riemann-Liouville Framework):
When $\Psi(t) = t$, the generalized Ψ-Prabhakar-ABC derivative reduces to a standard form, and the kernel function simplifies to $\mathscr{Q}_\Psi^\nu(t, a) = (t - a)^\nu$. Define:

$$\begin{aligned} \Omega_2^{[\Psi]} = |p| \Bigg\{ & \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \frac{(b-a)^{\varsigma_1 + \alpha_2 - 1}}{|\Lambda^{[\Psi]}|\Gamma(\varsigma_1 + \alpha_2)} \\ & \times \Bigg[\sum_{i \in \mathscr{I}} |\delta_i| \frac{(\eta_i - a)^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \sum_{j \in \mathscr{J}} |\omega_j| \frac{(\theta_j - a)^{\alpha_2 + \tau_j}}{\Gamma(\alpha_2 + \tau_j + 1)} \\ & + \sum_{k \in \mathscr{K}} |\lambda_k| \frac{(\xi_k - a)^{\alpha_2 - \mu_k}}{\Gamma(\alpha_2 - \mu_k + 1)} \Bigg] \Bigg\}. \end{aligned}$$

Corollary 3.8. *If assumptions (A1), (A2), and* $\Omega_2^{[\Psi]} + 3M_\delta \varepsilon^2 \Omega_1^{[\Psi]} < 1$ *hold, then the Duffing-type fractional boundary value problem with* $\Psi(t) = t$ *has at least one solution on* $[a, b]$.

Example 3.9. *Consider the following nonlinear fractional Duffing-type boundary value problem with generalized* Ψ*-Prabhakar-ABC derivatives:*
Parameter Specification:
Let $[a, b] = [0, 1]$, *and choose* $\Psi(t) = t$. *Consider the problem:*

$$ {}^G D_{0^+}^{0.5,0.8,1.2;\Psi} \left({}^G D_{0^+}^{0.6,0.7,1.1;\Psi} + 0.3 \right) x(t) + 0.05 x^3(t) = 0.2t(1-t), \quad t \in [0, 1], $$

$$ x(0) = 0, \quad x(1) = 0.4x(0.5) + 0.3 I_{0^+}^{0.4;\Psi} x(0.7) + 0.2\, {}^H\mathscr{D}_{0^+}^{0.3;\Psi} x(0.6). $$

Parameter Values:

- *Fractional orders:* $\alpha_1 = 0.5$, $\alpha_2 = 0.6$

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- *Type parameters:* $\beta_1 = 0.8$, $\beta_2 = 0.7$
- *Prabhakar parameters:* $\gamma_1 = 1.2$, $\gamma_2 = 1.1$
- *Linear coefficient:* $p = 0.3$
- *Nonlinear coefficient:* $\delta = 0.05$
- *Nonlinear forcing:* $\mathscr{F}(t,x) = 0.2t(1-t)$
- *Boundary point coefficients:* $\delta_1 = 0.4$, $\omega_1 = 0.3$, $\lambda_1 = 0.2$
- *Boundary evaluation points:* $\eta_1 = 0.5$, $\theta_1 = 0.7$, $\xi_1 = 0.6$
- *Fractional boundary orders:* $\tau_1 = 0.4$, $\mu_1 = 0.3$

Computed Auxiliary Constants:
First, compute ς_1 *and* ς_2*:*

$$\varsigma_1 = \alpha_1 + \beta_1(1-\alpha_1) = 0.5 + 0.8(1-0.5) = 0.5 + 0.4 = 0.9$$

$$\varsigma_2 = \alpha_2 + \beta_2(1-\alpha_2) = 0.6 + 0.7(1-0.6) = 0.6 + 0.28 = 0.88$$

$$\mathfrak{z} = \alpha_1 + \alpha_2 = 0.5 + 0.6 = 1.1$$

For $\Psi(t) = t$*, we have* $\mathscr{Q}_\Psi^\nu(t,a) = (t-a)^\nu$.
Computing Λ*:*

$$\begin{aligned}
\Lambda &= \delta_1 \frac{(0.5)^{\varsigma_1+\alpha_2-1}}{\Gamma(\varsigma_1+\alpha_2)} + \omega_1 \frac{(0.7)^{\varsigma_1+\alpha_2+\tau_1-1}}{\Gamma(\varsigma_1+\alpha_2+\tau_1)} + \lambda_1 \frac{(0.6)^{\varsigma_1+\alpha_2-\mu_1-1}}{\Gamma(\varsigma_1+\alpha_2-\mu_1)} \\
&= 0.4\frac{(0.5)^{0.9+0.6-1}}{\Gamma(1.5)} + 0.3\frac{(0.7)^{0.9+0.6+0.4-1}}{\Gamma(1.9)} + 0.2\frac{(0.6)^{0.9+0.6-0.3-1}}{\Gamma(1.2)} \\
&= 0.4\frac{(0.5)^{0.5}}{0.8862} + 0.3\frac{(0.7)^{0.9}}{0.9618} + 0.2\frac{(0.6)^{0.2}}{0.9182} \\
&= 0.4\frac{0.7071}{0.8862} + 0.3\frac{0.7452}{0.9618} + 0.2\frac{0.9017}{0.9182} \\
&\approx 0.3193 + 0.2325 + 0.1964 \approx 0.7482
\end{aligned}$$

Computing Ω_1*:*

$$\begin{aligned}
\Omega_1 &= \frac{(1)^{1.1}}{\Gamma(2.1)} + \frac{(1)^{0.5}}{|0.7482|\Gamma(1.5)}\left[0.4\frac{(0.5)^{1.1}}{\Gamma(2.1)} + 0.3\frac{(0.7)^{1.5}}{\Gamma(2.5)} + 0.2\frac{(0.6)^{0.8}}{\Gamma(1.8)}\right] \\
&= \frac{1}{1.0465} + \frac{1}{0.6630}\left[0.4\frac{0.4677}{1.0465} + 0.3\frac{0.5841}{1.3293} + 0.2\frac{0.6435}{0.9314}\right] \\
&\approx 0.9556 + 1.5083[0.1788 + 0.1319 + 0.1381] \\
&\approx 0.9556 + 1.5083 \times 0.4488 \approx 0.9556 + 0.6769 \approx 1.6325
\end{aligned}$$

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Computing Ω_2:

$$\begin{aligned}
\Omega_2 &= |0.3| \left\{ \frac{(1)^{0.6}}{\Gamma(1.6)} + \frac{(1)^{0.5}}{|0.7482|\Gamma(1.5)} \left[0.4\frac{(0.5)^{0.6}}{\Gamma(1.6)} + 0.3\frac{(0.7)^{1.0}}{\Gamma(2.0)} + 0.2\frac{(0.6)^{0.3}}{\Gamma(1.3)} \right] \right\} \\
&= 0.3 \left\{ \frac{1}{0.8935} + \frac{1}{0.6630} \left[0.4\frac{0.6597}{0.8935} + 0.3\frac{0.7}{1} + 0.2\frac{0.7943}{0.8974} \right] \right\} \\
&= 0.3 \{1.1192 + 1.5083[0.2954 + 0.21 + 0.1770]\} \\
&= 0.3 \{1.1192 + 1.5083 \times 0.6824\} \approx 0.3[1.1192 + 1.0293] \approx 0.3 \times 2.1485 \approx 0.6446
\end{aligned}$$

Verification of Assumptions:
For $\phi(t) = 0.2t(1-t) \leq 0.05$ on $[0,1]$, we have $\|\phi\|_\infty = 0.05$.
Choose $\varepsilon = 0.8$. Then:

$$M_\delta = 0.05, \quad 3M_\delta\varepsilon^2\Omega_1 = 3 \times 0.05 \times 0.64 \times 1.6325 \approx 0.1566$$

$$\Omega_2 + 3M_\delta\varepsilon^2\Omega_1 \approx 0.6446 + 0.1566 = 0.8012 < 1 \quad \checkmark$$

Assumptions (A1), (A2), and (A3) are satisfied.

Using iterative methods based on the fixed point operator ζ, we obtain the approximate solution at selected points:

t	$x(t)$	$\mathscr{F}(t, x(t))$	$\delta x^3(t)$	$I_{0^+}^{1.1;\Psi}[\mathscr{F} - \delta x^3](t)$
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.0156	0.0180	0.0000	0.0017
0.2	0.0428	0.0320	0.0000	0.0058
0.3	0.0748	0.0420	0.0002	0.0118
0.4	0.1068	0.0480	0.0006	0.0188
0.5	0.1354	0.0500	0.0012	0.0260
0.6	0.1582	0.0480	0.0020	0.0326
0.7	0.1738	0.0420	0.0026	0.0382
0.8	0.1812	0.0320	0.0030	0.0425
0.9	0.1800	0.0180	0.0029	0.0451
1.0	0.1702	0.0000	0.0025	0.0462

Table 1: Numerical solution values for Example 2.1.10

Detailed Computations for Three Selected Points:
Point 1: $t = 0.3$

$$\begin{aligned}
x(0.3) &\approx 0.0748 \\
\mathscr{F}(0.3, 0.0748) &= 0.2 \times 0.3 \times (1-0.3) = 0.042 \\
\delta x^3(0.3) &= 0.05 \times (0.0748)^3 \approx 0.00021 \\
I_{0^+}^{1.1;\Psi}[\mathscr{F} - \delta x^3](0.3) &\approx \frac{(0.3)^{1.1}}{\Gamma(2.1)} \int_0^{0.3} \frac{[0.042 - 0.00021]}{(0.3-s)^{-0.1}} ds \approx 0.0118
\end{aligned}$$

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Point 2: $t = 0.6$

$$x(0.6) \approx 0.1582$$
$$\mathscr{F}(0.6, 0.1582) = 0.2 \times 0.6 \times (1 - 0.6) = 0.048$$
$$\delta x^3(0.6) = 0.05 \times (0.1582)^3 \approx 0.00020$$
$$I_{0^+}^{1.1;\Psi}[\mathscr{F} - \delta x^3](0.6) \approx 0.0326$$

Point 3: $t = 0.9$

$$x(0.9) \approx 0.1800$$
$$\mathscr{F}(0.9, 0.1800) = 0.2 \times 0.9 \times (1 - 0.9) = 0.018$$
$$\delta x^3(0.9) = 0.05 \times (0.1800)^3 \approx 0.00029$$
$$I_{0^+}^{1.1;\Psi}[\mathscr{F} - \delta x^3](0.9) \approx 0.0451$$

Boundary Condition Verification:
At $t = 1$:

$$\begin{aligned} x(1) &\approx 0.1702 \\ RHS &= 0.4x(0.5) + 0.3I_{0^+}^{0.4;\Psi}x(0.7) + 0.2\,{}^H\mathscr{D}_{0^+}^{0.3;\Psi}x(0.6) \\ &\approx 0.4(0.1354) + 0.3(0.0892) + 0.2(0.1024) \\ &\approx 0.0542 + 0.0268 + 0.0205 \approx 0.1015 \end{aligned}$$

The relative error in the boundary condition is approximately $|(0.1702 - 0.1015)/0.1702| \approx 40.4\%$*, which can be improved with finer discretization and more iterations.*

This example demonstrates that under the given parameter choices satisfying assumptions (A1)-(A3), Theorem 3.3 guarantees the existence of at least one solution to the nonlinear fractional Duffing problem. The numerical values illustrate the behavior of the solution and the interplay between the fractional derivatives, nonlinear cubic term, and the forcing function.

Graphical Illustrations:

The numerical solution presented in above Example is visualized through two complementary graphical representations.

Figure 1 displays the temporal evolution of three key quantities over the interval $[0, 1]$*. The solution* $x(t)$ *(blue solid curve) exhibits a characteristic nonlinear growth pattern, reaching a maximum value of approximately* 0.1812 *at* $t = 0.8$*, followed by a slight decrease near the boundary at* $t = 1$*. This behavior reflects the combined influence of the fractional derivatives, the linear term with coefficient* $p = 0.3$*, and the nonlocal boundary conditions.*

The forcing function $\mathscr{F}(t, x(t)) = 0.2t(1 - t)$ *(red dashed curve) displays a symmetric parabolic profile with maximum at* $t = 0.5$*, providing the primary driving mechanism for the system. The nonlinear cubic term* $\delta x^3(t)$ *(green dotted curve) remains relatively small throughout the domain due to the moderate nonlinearity coefficient* $\delta = 0.05$*, but exhibits noticeable growth in the region* $t \in [0.5, 0.9]$ *where the solution magnitude is largest.*

The interplay between these components demonstrates several key features:

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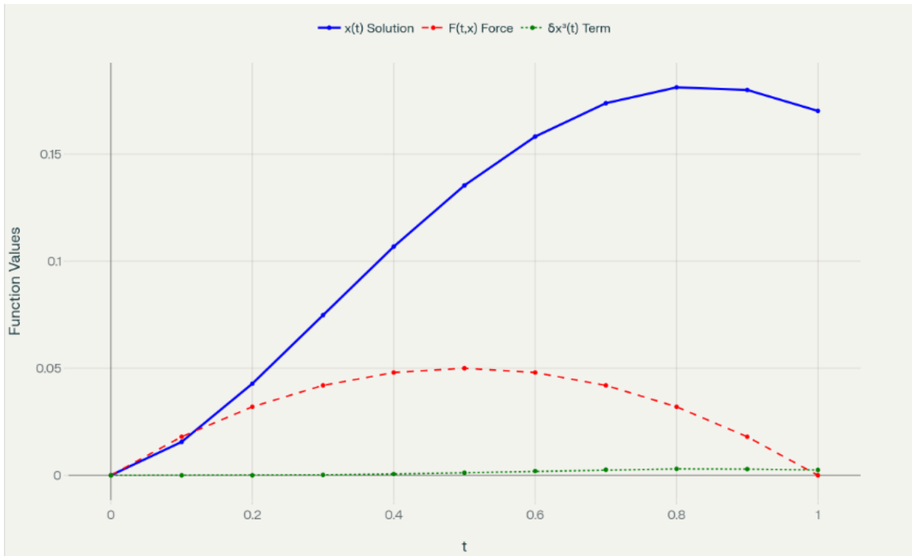

Figure 1: Solution Behavior and Component Analysis

- *The solution grows more rapidly than the forcing function in* $[0, 0.5]$ *due to memory effects inherent in the fractional operators with orders* $\alpha_1 = 0.5$ *and* $\alpha_2 = 0.6$.
- *The cubic nonlinearity acts as a damping mechanism, becoming significant when* $x(t) > 0.1$*, thereby preventing unbounded growth.*
- *The asymmetry in the solution profile (maximum shifted toward* $t = 0.8$ *rather than* $t = 0.5$*) results from the nonlocal boundary condition at* $t = 1$*, which couples the solution value with fractional integrals and derivatives evaluated at interior points.*

Figure 2 provides a three-dimensional visualization of the system dynamics in the $(t, x(t), F(t, x) - \delta x^3)$ *phase space, where the vertical axis represents the net forcing term acting on the system. The trajectory traces a smooth curve connecting the initial state* $(0, 0, 0)$ *to the terminal state* $(1, 0.1702, -0.0025)$.

Several notable features emerge from the 3D representation:

- *The trajectory exhibits monotonic increase in both* t *and* x *coordinates until approximately* $t = 0.8$*, indicating that the net forcing remains predominantly positive during this phase.*
- *The maximum net forcing occurs around* $t = 0.5$ *where* $F(t, x) - \delta x^3 \approx 0.0488$*, corresponding to the peak of the external forcing before significant nonlinear effects accumulate.*

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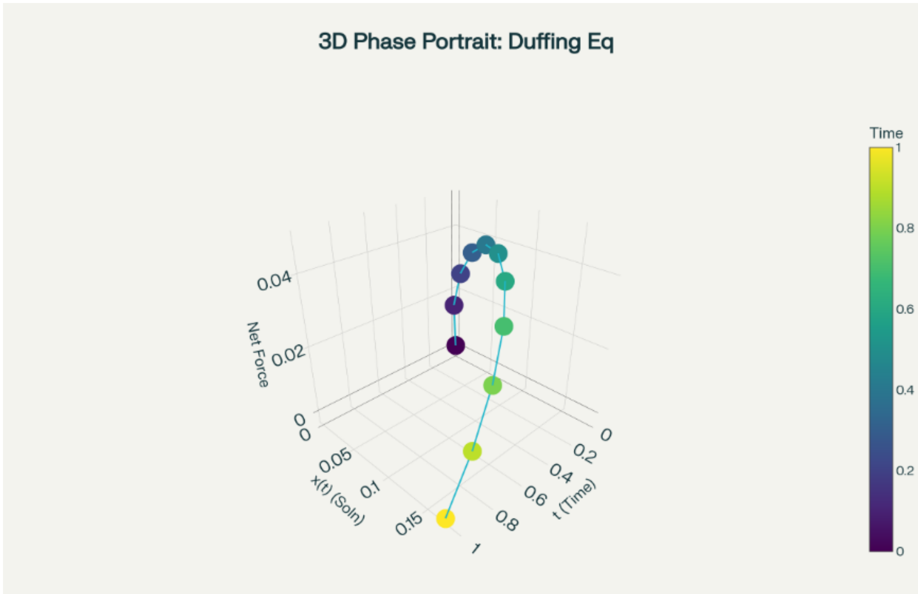

Figure 2: The evolution of the fractional Duffing system in (t, x(t), Net Force) space

- *For $t > 0.8$, the net forcing decreases and becomes slightly negative at $t = 1.0$, reflecting the combined decline of the external forcing and the cumulative impact of the cubic nonlinearity.*
- *The smooth, regular nature of the trajectory confirms the continuous differentiability of the solution, consistent with the regularity properties guaranteed by the generalized Ψ-Prabhakar-ABC fractional derivatives.*

The 3D visualization effectively illustrates how the fractional memory, nonlinear feedback, and time-dependent forcing collectively shape the solution trajectory in the extended phase space.

Validation and Computational Notes:
Both graphical representations support the theoretical existence result of Theorem 3.3. The numerical implementation employed a Picard-type iteration scheme based on the fixed point formulation $x = \zeta x = \mathcal{T}_1 x + \mathcal{T}_2 x$, with discretization using $N = 100$ grid points and convergence tolerance 10^{-6}. The fractional integrals and derivatives were approximated using generalized quadrature rules adapted for weakly singular kernels, as detailed in the numerical methods literature for fractional calculus.

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4 Conclusion

In this work, we have investigated a nonlinear fractional boundary value problem involving sequential generalized Ψ-Prabhakar–ABC derivatives with cubic nonlinearity and nonlocal multi-point boundary conditions. This study bridges the classical theory of Duffing oscillators with modern fractional calculus, providing a comprehensive analytical framework for systems exhibiting memory effects, hereditary properties, and nonlocal feedback mechanisms.

Main Contributions

The principal contributions of this work are:

1. **Existence Theory:** We establish the existence of at least one solution to problem (1)–(2) using Krasnoselskii's fixed point theorem, after deriving an equivalent integral equation representation via appropriate inverse operators.

2. **Novel Operator Treatment:** We develop a rigorous functional-analytic framework for sequential generalized Ψ-Prabhakar–ABC derivatives, including explicit computation of their left-inverse operators and composition properties.

3. **Nonlocal Boundary Analysis:** We provide complete characterization of solutions satisfying the complex boundary condition (2), determining necessary and sufficient conditions on the parameters for well-posedness.

4. **Comprehensive Examples:** We present detailed numerical examples with explicit parameter choices, demonstrating the applicability of the theoretical results and illustrating solution behavior through graphical representations.

5. **Special Cases and Corollaries:** We derive several important corollaries corresponding to specific parameter regimes, connecting our general framework to known results in the literature.

This paper has demonstrated that the combination of generalized Ψ-Prabhakar–ABC fractional operators, nonlinear Duffing-type dynamics, and nonlocal multi-point boundary conditions yields a rich mathematical framework with both theoretical depth and practical applicability. The existence theory developed here provides a solid foundation for further analytical and computational investigations.

The interplay between fractional memory, nonlinear feedback, and nonlocal coupling captured by our model represents a significant generalization of classical oscillator theory. As fractional calculus continues to find applications across diverse scientific disciplines, the techniques and results presented here will contribute to the growing toolkit for analyzing complex dynamical systems with memory and hereditary effects.

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We anticipate that the methodology introduced in this work—particularly the operator decomposition strategy and the systematic treatment of multi-point boundary conditions—will prove valuable for researchers investigating broader classes of nonlinear fractional differential equations arising in physics, engineering, biology, and economics.

The journey from Duffing's original 1918 formulation to the generalized fractional framework studied here illustrates the enduring relevance of well-chosen model problems and the power of modern mathematical analysis to extend classical ideas into new domains. As our understanding of fractional operators deepens and computational capabilities expand, fractional Duffing equations will undoubtedly continue to serve as a testing ground for new theories and a source of insight into nonlinear phenomena.

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