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Area of Ovals with Constant Width

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Abstract

We investigate ovals of constant width obtained from a simple closed parametric function in this paper. We measure the area enclosed within the parametric curve and compare it relatively to the area of a correspondingly particular circle. We show that the area of the oval approaches the area of the corresponding circle with increasing width, regardless of the number of vertices. We prove that the area of the oval from a vertex to the next one equals the total area divided by the number of counted vertices, and the area from a vertex to its opposite is equal to half of the total area. Bounds for the area of the oval at all possible values of the number of vertices that satisfy the condition of convexity are derived. A correlation between a range of constant widths and a predetermined maximum number of possible vertices for the resulting ovals is established. Theoretical findings are supported by numerical data and accompanying figures.

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1 Introduction

Ovals with constant width are important in mathematics and engineering due to the interesting property that the oval width, which denotes the measured distance between parallel line segments on the curve of the oval, is constant regardless of the orientation. This unique interesting property is considered in numerous applications, such as generating distinctive designs for items like coins and rollers.

Since this kind of ovals maintain a constant width while they are different in form, away from the regular circle, then the area of this kind of ovals is a practical and significant topic in geometry. It is not only crucial for mathematics, but it is also very beneficial to understand the relationship between geometry and physical applications.

In this work, we study the area of smooth closed convex curves in $\mathbb{R}^2$, called ovals. Moving around ovals shows that every point on an oval has a corresponding point, also on the oval, with the property that the two points have parallel tangent lines. Such points are opposite to each other and have the same normal line. In general, the orthogonal distance from a tangent segment at a point to the tangent segment at its opposite point is a function of the parameter of the oval. If such a distance is constant, then we say that the oval is of constant width [1]. This is equivalent to saying that normal segments connecting opposite points always have the same length.

Al-Banawi [2] did a study of some geometrical concepts of ovals in the plane in terms of the orthogonal distance measured from the tangent segment on a any point of the oval to the point of origin, which defines the usual support function [3]. The support function is beneficial in building a formula that parameterizes the oval, which helps in constructing other formulas regarding the focal curve relative to the oval and its length [2].

Al-Banawi and Al-Btoush [4] exploited the idea of approximating the support function by the usual units of the Fourier series representation, sines and cosines, to study optimization of curvature along an oval with constant width with a new formula for the area included within such closed convex curves. The development of the smooth support

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function by the Fourier series was also used by Resnikoff [5] to study other properties regarding the geometry of ovals having constant width in the plane.

Al-rabtah and Al-Banawi [6] explored planar ovals of constant width and compared them with specific circles. After counting them, they utilized the vertices on the oval to measure the curve's length between the opposite points. Barbier's theorem was again reproved in their work. Additionally, they introduced a formula that measures the distance to any point on the oval from the origin, and demonstrated that the distance function's extreme values are found at the vertices and their opposite points. Their study compared these ovals with particular circles, establishing that the variations between the specific circles and the ovals' distances apart from the point of origin are considered small and lie within a specific range. They further proved that all numerous types of ovals are geometrically and analytically bounded between two well-defined specific circles that their centers lie at the origin, with being a general form for the curve established by Fillmore [7].

The width of an oval in $\mathbb{R}^2$ was modeled by Al-Banawi and Jaradat [8] in a form of an ordinary differential equation that is second-order, homogeneous, linear with constant coefficients. This model demonstrated that coincidence of normal lines occur and two opposite points lying on a curve with constant width corresponds to the same focal point.

A $\mathbb{C}^3$ curve with constant width was analyzed by Fu and Zhou [9] using the corresponding characteristic equation to a system with time delay. With regard to the impact of the derivative rank on a non-circular smooth curve with constant width, they gave the simulation findings. A conjecture put out in their work [9] sought to demonstrate that the unique curve with constant width, where its parametric equation remains a smooth curve with constant width regardless of the number of evaluated derivatives, is a circle.

Fillmore presented a parametric formula corresponding to a curve with constant width having three vertices in [7]. Considering that Fillmore's formulation was combined by the concept of shadow [10], our work extends this formulation to accommodate an odd number of ver-

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tices of more than three.

It should be noted that our method makes use of a precise parametric formula that allows us to jointly examine and characterize several ovals. Our approach facilitates the presenting of simulation results that show the theoretical analysis concerning numerous features of curves with constant width.

Since our work here is concentrated on the area within an oval of constant width, it is worth mentioning that Ara in [11] proved that, in the hyperbolic plane, the Reuleaux triangle has an area that is a smaller than that for any other convex curve of the same constant width. In [12], Gallego and Solanes studied relations between the area of a convex set and its perimeter and diameter. In [13], Kim and Chai obtained an upper bound for the evaluated area of convex curves having constant relative breadth in Minkowski plane. In [14], Santaló followed an end of a segment whose other end traced a fixed curve, and managed to derive some theorems regarding the area bounded by ovals.

The rest of the work in this article is as follows: In Section 2, we give a summary of the work done by Al-rabtah and Al-Banawi [6] regarding ovals with constant width. The summary describes two main branches, detecting vertices on ovals of constant width and measuring distance from the origin to points (in particular vertices) on ovals of constant width. In Section 3, we start our new work and show our new results regarding the area of ovals of constant width. We give formulas for the areas enclosed within an oval with constant width, the area inside an oval of constant width on a subdomain, and the area inside an oval of constant width from any vertex to its corresponding opposite point. Furthermore, we study the difference and the relative difference between special areas. In Section 4, we give a closer look at the difference between the general area and an area of a particular circle, which allows us to introduce different cases supported by excellent inequalities and new relations between area and number of vertices. In Section 5, we support our work by numerical examples, accompanied by data and figures. Section 6 is the conclusion.

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2 Ovals with Constant Width

It was demonstrated in [2] that if G is a parametric rule for the oval and $\beta(\gamma)$ represents the support function, then we have the following at the variable γ:

$$G(\gamma) = (\beta(\gamma)\cos\gamma - \beta'(\gamma)\sin\gamma,\ \beta(\gamma)\sin\gamma + \beta'(\gamma)\cos\gamma)\,.$$

The work in [2] was based on a formula that was first proposed by Fillmore [7] and generalized from there. Our study is actually based on the following parametric rule,

$$\begin{aligned} G(\gamma) = (&a\cos\gamma + \cos\gamma\cos n\gamma + n\sin\gamma\sin n\gamma, \\ &a\sin\gamma + \sin\gamma\cos n\gamma - n\cos\gamma\sin n\gamma), \end{aligned} \tag{1}$$

where $\gamma \in [0, 2\pi]$, the real number a is positive, and the parameter n is an odd natural number, while the value $n = 3$ is considered in [7].

A parametrization form of ovals with constant width having a sufficient condition of convexity was stated by Al-rabtah and Al-Banawi [6]. Convexity is the property where the oval's curvature is positive throughout. In terms of geometry, this indicates that the oval completely rests on one side of the tangent at every given location on the oval.

Theorem 1. [6] *Let G be a parametric rule defined by the Formula (1). Then, if $a-(n^2-1)\cos n\gamma > 0$, then the resulting image of G represents an oval with constant width $2a$.*

It should be noted that, given the determination of a, Theorem 1 gives us the potential values of the odd integer n. In actuality, one can select n such that $n < \sqrt{a+1}$ given a specific value of a.

A vertex on an oval is a rectangular point where the curvature attains its relative extreme value in the geometry of plane curves [15]. We use the following definition in our investigation.

Definition 2. *[6]* Given a parametric rule G at γ, let $K(\gamma)$ represent its curvature. If the curvature of G at v is the maximum, then v is a vertex on the trace of the image of G.

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A main result in [6] regarding counting vertices along an oval with constant width is the next reformulated theorem.

Theorem 3. [6] *If G represents an oval with constant width as in the form (1), then the parametric function G has n number of vertices, specifically at*

$$\gamma = \frac{2s\pi}{n},\ s = 0, 1, \ldots, n-1.$$

While Al-rabtah and Al-Banawi [6] were interested in calculating the length of the curve between two opposite points on an oval of constant width, they also managed to prove that if one of the opposite points is a vertex, then the length in between is half of the length of the oval itself. They also managed to show that the distance traveled along the curve between any two consecutive vertices multiplied by the number of counted vertices is equal to the total curve length.

The rest of the work done by Al-rabtah and Al-Banawi [6] was about the distance to any specific point on the oval measured from the origin defined by Formula (1). There, they gave a formula for the distance with special formulas regarding particular circles. They also proved that the distance function gains its absolute maximum values at the vertices, and its absolute minimum values at the corresponding opposite points of vertices. And finally, they managed to find the intersection points between any curve defined by Formula (1) and the circles with radii $a+1$ and $a-1$, and concluded that all ovals obtained from Formula (1) are located within the two specific circles centering at the origin of the Cartesian plane; the first circle is of radius equals to $a+1$, while the second one is of radius $a-1$.

In [16], a study of ovals with constant width was carried out in polar coordinates. Along with several additional findings, the results from [6] were reproduced in polar coordinates.

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3 Area of an Oval

Now we state our new results regarding the area of an oval with constant width.

Theorem 4. *The area enclosed by the parametric curve* (1) *whose rectangular coordinates are,*

$$x = a\cos\gamma + \cos\gamma\cos n\gamma + n\sin\gamma\sin n\gamma, \text{ and}$$
$$y = a\sin\gamma + \sin\gamma\cos n\gamma - n\cos\gamma\sin n\gamma,$$

is given by

$$A(n) = \pi a^2 + \frac{1-n^2}{2}\pi. \tag{2}$$

In particular, if $n = 1$, then the parametric curve is a circle of radius a, with area $A(1) = \pi a^2$.

Proof. We use Green's Theorem since the curve is simple closed, where the area is given by

$$A = \frac{1}{2}\oint_c x\,dy - y\,dx.$$

Here, $x\,dy - y\,dx = a^2 + a(2-n^2)\cos n\gamma + (1-n^2)\cos^2 n\gamma$.
Thus,

$$\begin{aligned} A(n) &= \frac{1}{2}\int_0^{2\pi}\Big(a^2 + a(2-n^2)\cos n\gamma + (1-n^2)\cos^2 n\gamma\Big)d\gamma \\ &= \frac{1}{2}\int_0^{2\pi}\left(a^2 + a(2-n^2)\cos n\gamma + \frac{1-n^2}{2}(1+\cos 2n\gamma)\right)d\gamma \\ &= \frac{1}{2}\left[\left(a^2\gamma + \frac{a(2-n^2)}{n}\sin n\gamma + \frac{1-n^2}{2}\left(\gamma + \frac{\sin 2n\gamma}{2n}\right)\right)\right]_0^{2\pi} \end{aligned}$$

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$$= \frac{1}{2}\left(2\pi a^2 + 0 + \frac{1-n^2}{2}(2\pi + 0)\right) - \frac{1}{2}(0) = \pi a^2 + \frac{1-n^2}{2}\pi.$$

If $n = 1$, then $x = a\cos\gamma + 1$, $y = a\sin\gamma$, representing a circle of radius a, centered at the point $(1,0)$, with area $A(1) = \pi a^2 + 0 = \pi a^2$. □

Theorem 5. *The area of the oval enclosed by the parametric curve* (1) *whose rectangular coordinates are*

$$x = a\cos\gamma + \cos\gamma\cos n\gamma + n\sin\gamma\sin n\gamma,$$
$$y = a\sin\gamma + \sin\gamma\cos n\gamma - n\cos\gamma\sin n\gamma,$$

from a determined vertex point to the next traced one on the oval is

$$A(n) = \frac{1}{n}\left(\pi a^2 + \frac{1-n^2}{2}\pi\right),$$

where n *is the number of vertices. i.e.*

$$A(n) = \frac{1}{n}\left(\textit{Total Area of Oval}\right).$$

Proof. From the proof of Theorem 4, and since the n vertices of the oval are given by

$$\gamma = \frac{2s\pi}{n}, \ s = 0, 1, \ldots, n-1,$$

then we have,

$$A(n) = \frac{1}{2}\int_{2s\pi/n}^{2(s+1)\pi/n}\left(a^2 + a(2-n^2)\cos n\gamma + (1-n^2)\cos^2 n\gamma\right)d\gamma$$
$$= \frac{1}{2}\left[\left(a^2\gamma + \frac{a(2-n^2)}{n}\sin n\gamma + \frac{1-n^2}{2}\left(\gamma + \frac{\sin 2n\gamma}{2n}\right)\right)\right]_{2s\pi/n}^{2(s+1)\pi/n}$$
$$= \frac{1}{2}\left(\frac{2(s+1)\pi}{n}a^2 + 0 + \frac{1-n^2}{2}\left(\frac{2(s+1)\pi}{n} + 0\right)\right)$$

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$$-\frac{1}{2}\left(\frac{2s\pi}{n}a^2+0+\frac{1-n^2}{2}\left(\frac{2s\pi}{n}+0\right)\right)=\frac{1}{n}\left(\pi a^2+\frac{1-n^2}{2}\pi\right),$$

which is equal to $(1/n)$ times the total area of the oval, given by Formula (2) in Theorem 4. □

Theorem 6. *The area of the oval enclosed by the parametric curve* (1) *from γ to $\gamma+\pi$ is given by*

$$A(n)=\frac{1}{2}\left[\pi a^2-\frac{2a(2-n^2)}{n}\sin n\gamma+\frac{1-n^2}{2}\pi\right]. \tag{3}$$

Proof. From the proof of Theorem 4 we have,

$$\begin{aligned}
A(n)&=\frac{1}{2}\int_{\gamma}^{\gamma+\pi}\left(a^2+a(2-n^2)\cos n\gamma+(1-n^2)\cos^2 n\gamma\right)d\gamma\\
&=\frac{1}{2}\left[\left(a^2\gamma+\frac{a(2-n^2)}{n}\sin n\gamma+\frac{1-n^2}{2}\left(\gamma+\frac{\sin 2n\gamma}{2n}\right)\right)\right]_{\gamma}^{\gamma+\pi}\\
&=\frac{1}{2}\left[(\gamma+\pi)a^2+\frac{a(2-n^2)}{n}\sin n(\gamma+\pi)\right.\\
&\qquad\left.+\frac{1-n^2}{2}\left(\gamma+\pi+\frac{\sin 2n(\gamma+\pi)}{2n}\right)\right]\\
&\quad-\frac{1}{2}\left[\gamma a^2+\frac{a(2-n^2)}{n}\sin n\gamma+\frac{1-n^2}{2}\left(\gamma+\frac{\sin 2n\gamma}{2n}\right)\right]\\
&=\frac{1}{2}\left[(\gamma+\pi)a^2-\frac{a(2-n^2)}{n}\sin n\gamma+\frac{1-n^2}{2}\left(\gamma+\pi+\frac{\sin 2n\gamma}{2n}\right)\right]\\
&\quad-\frac{1}{2}\left[\gamma a^2+\frac{a(2-n^2)}{n}\sin n\gamma+\frac{1-n^2}{2}\left(\gamma+\frac{\sin 2n\gamma}{2n}\right)\right]\\
&=\frac{1}{2}\left[\pi a^2-\frac{2a(2-n^2)}{n}\sin n\gamma+\frac{1-n^2}{2}\pi\right].
\end{aligned}$$

□

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Corollary 7. *The area of the oval enclosed by the parametric curve (1) from a vertex point at γ to its corresponding opposite point at $\gamma + \pi$ is given by*

$$A(n) = \frac{1}{2}\left[\pi a^2 + \frac{1-n^2}{2}\pi\right], \tag{4}$$

which is equal to half of the total area of the oval.

Proof. From Formula (3), and since $\sin n\gamma = 0$ at the vertices where $\gamma = (2s\pi)/n$, $s = 0, 1, 2, \ldots, n-1$, then

$$\begin{aligned} A(n) &= \frac{1}{2}\left[\pi a^2 - \frac{2a(2-n^2)}{n}\sin n\gamma + \frac{1-n^2}{2}\pi\right] \\ &= \frac{1}{2}\left[\pi a^2 - \frac{2a(2-n^2)}{n}(0) + \frac{1-n^2}{2}\pi\right] = \frac{1}{2}\left[\pi a^2 + \frac{1-n^2}{2}\pi\right]. \quad \square \end{aligned}$$

For the parameter n satisfying the condition of convexity: $n^2 - 1 < a$, we define the difference $D(n) = A(1) - A(n)$. Then, $D(n) = \pi(n^2-1)/2$. Now, $0 \le D(n) < a\pi/2$, and so $D(n)$ has a lower bound 0, namely at $n = 1$, and an upper bound $a\pi/2$, which is obtained at the largest possible value of the parameter n that satisfies the condition of convexity.

Corollary 8. *For each parameter a, the difference between the two areas $D(n) = A(1) - A(n)$ increases as $n \longrightarrow \sqrt{a+1}$, and the two areas are equal if $n = 1$, that is, $D(1) = 0$.*

Corollary 9. *For the difference $D(n) = A(1) - A(n)$, the relative difference is*

$$\frac{D(n)}{A(1)} < \frac{a\pi/2}{\pi a^2} = \frac{1}{2a}.$$

As $a \longrightarrow \infty$, the relative difference approaches zero.

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4 A Closer Look at D(n) and A(n)

While $D(n) = \frac{\pi}{2}(n^2 - 1)$ with condition of convexity $n^2 - 1 < a$, one can consider the following three main cases:

Case 1: If $a + 1$ is a perfect square, and $\sqrt{a+1}$ is an odd positive integer, then a suitable choice for the largest possible value of the parameter n that satisfies the condition of convexity $n < \sqrt{a+1}$ is $\sqrt{a+1} - 2$. Hence,

$$\begin{aligned} D &= \frac{\pi}{2}(n^2 - 1) = \frac{\pi}{2}(a + 1 - 4\sqrt{a+1} + 4 - 1) \\ &= \frac{a\pi}{2} + 2\pi\left(1 - \sqrt{a+1}\right). \end{aligned}$$

Therefore, the area $A(n)$ of the oval, for all possible values of the parameter n, satisfies

$$A_1 \equiv \pi a^2 - \frac{a + 4 - 4\sqrt{a+1}}{2}\pi \leq A(n) \leq \pi a^2. \tag{5}$$

Case 2: If $a + 1$ is a perfect square, and $\sqrt{a+1}$ is an even positive integer, then a suitable choice for the largest possible value of the parameter n that satisfies the condition of convexity $n < \sqrt{a+1}$ is $\sqrt{a+1} - 1$. Hence,

$$\begin{aligned} D &= \frac{\pi}{2}(n^2 - 1) = \frac{\pi}{2}(a + 1 - 2\sqrt{a+1}) \\ &= \frac{a\pi}{2} + \frac{\pi}{2}\left(1 - 2\sqrt{a+1}\right). \end{aligned}$$

Therefore, the area $A(n)$ of the oval, for all possible values of the parameter n, satisfies

$$A_2 \equiv \pi a^2 - \frac{a + 1 - 2\sqrt{a+1}}{2}\pi \leq A(n) \leq \pi a^2. \tag{6}$$

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Case 3: If $a+1$ is not a perfect square, then a suitable choice for the largest possible value of the parameter n that satisfies the condition of convexity $n < \sqrt{a+1}$ is $\lfloor \sqrt{a+1} \rfloor$ being odd, and here we have three cases:

- If a is a perfect square and $\sqrt{a}$ is an odd positive integer, then the largest suitable choice for n is $\sqrt{a}$. Hence,

$$D = \frac{\pi}{2}(n^2-1) = \frac{\pi}{2}((\sqrt{a})^2-1) = \frac{(a-1)}{2}\pi.$$

- If a is not a perfect square and $\lfloor \sqrt{a+1} \rfloor$ is an even positive integer, then the largest suitable choice for n is $\lfloor \sqrt{a+1} \rfloor - 1$. Hence,

$$\begin{aligned} D &= \frac{\pi}{2}(n^2-1) = \frac{\pi}{2}\left((\lfloor \sqrt{a+1} \rfloor - 1)^2 - 1\right) < \frac{\pi}{2}((\sqrt{a})^2-1) \\ &= \frac{(a-1)}{2}\pi. \end{aligned}$$

- If a is not a perfect square and $\lfloor \sqrt{a+1} \rfloor$ is an odd positive integer, then the largest suitable choice for n is $\lfloor \sqrt{a+1} \rfloor - 2$. Hence,

$$\begin{aligned} D &= \frac{\pi}{2}(n^2-1) = \frac{\pi}{2}\left((\lfloor \sqrt{a+1} \rfloor - 2)^2 - 1\right) \\ &< \frac{\pi}{2}\left((\lfloor \sqrt{a+1} \rfloor - 1)^2 - 1\right) < \frac{\pi}{2}((\sqrt{a})^2-1) = \frac{(a-1)}{2}\pi. \end{aligned}$$

Therefore, in the case when $a+1$ is not a perfect square, the area $A(n)$ of the oval, for all possible values of the parameter n, satisfies

$$A_3 \equiv \pi a^2 - \frac{a-1}{2}\pi \leq A(n) \leq \pi a^2. \tag{7}$$

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Corollary 10. *Upon finding the area of the oval generated by the parametric function (1) at the largest possible parameter value n that satisfies the condition of convexity $n < \sqrt{a+1}$, the minimum areas in the above three main cases satisfy the inequality*

$$A_3 < A_2 < A_1 < \pi a^2 \equiv A(1),$$

where $A(1)$ is the largest oval's area obtained at $n = 1$, which represents a circle of radius a.

Remark 1. Consider finding the area of the oval generated by the parametric function (1). Let $a \in [n^2, (n+2)^2 - 1]$, where n is a determined positive odd integer. Then n is the largest possible parameter value in (1) that satisfies the condition of convexity $n < \sqrt{a+1}$.

Corollary 11. *Let $a \in [n^2, (n+2)^2 - 1]$, where n is a determined positive odd integer. The minimum oval's area A_3 occurs at $a = n^2$, and therefore $a + 1$ is not a perfect square. While the minimum oval's area A_2 occurs at $a = (n+1)^2 - 1$, and therefore $a+1$ is a perfect square, and $\sqrt{a+1}$ is an even positive integer. And finally, the minimum oval's area A_1 occurs at $a = (n+2)^2 - 1$, and therefore $a+1$ is a perfect square, and $\sqrt{a+1}$ is an odd positive integer.*

Corollary 12. *Let $a \in [n^2, (n+2)^2 - 1]$, where n is a determined positive odd integer. Then the area $A(n)$ of the oval generated by the parametric function (1), at the largest possible value of n that satisfies the condition of convexity, for any value of a in the prescribed interval, is determined by, either,*

$$A_3 \equiv \pi a^2 - \frac{a-1}{2}\pi \leq A(n) \leq \pi a^2 - \frac{a+1-2\sqrt{a+1}}{2}\pi \equiv A_2,$$

or,

$$A_2 \equiv \pi a^2 - \frac{a+1-2\sqrt{a+1}}{2}\pi \leq A(n) \leq \pi a^2 - \frac{a+4-4\sqrt{a+1}}{2}\pi \equiv A_1.$$

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5 Simulation Results

In order to demonstrate and support our findings and the theoretical analysis of this paper, we include a few simulation results in this section.

Example 1. Consider the oval with constant width when $a = 30$ and $n = 3$, where the area bounded by the parametric curve is divided into two and three subareas. We show that the result in the case of three subareas is consistent with Theorem 5, and the result in the case of two subareas is consistent with conclusion of Corollary 7.

In Figure 1, the area is divided into three subareas, the first one starts from the first vertex at $\gamma = 0$ to the second vertex at $\gamma = 2\pi/3$, the second subarea stars from the second vertex at $\gamma = 2\pi/3$ to the third vertex at $\gamma = 4\pi/3$, and the last subarea starts from the third vertex at $\gamma = 4\pi/3$ back to the first vertex. The area of each subarea is equal to $896\pi/3$, which is $1/3$ of the total area of the oval $A(n = 3) = 896$. This result is consistent with Theorem 5.

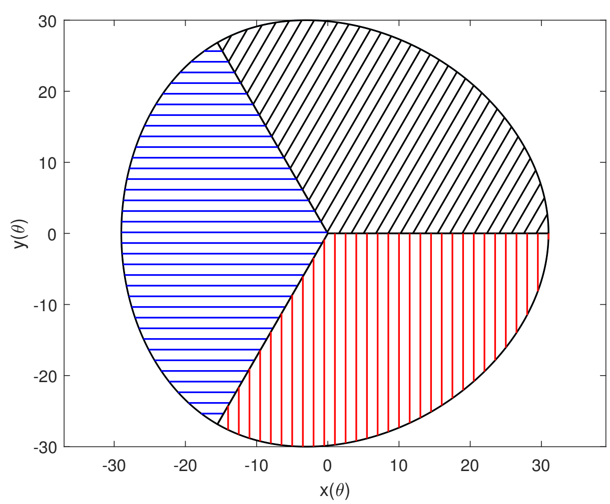

Figure 1: Three parts area of the Oval with $a = 30$ and $n = 3$. Areas from each vertex to the next one.

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In Figure 2, the area is divided into two subareas, obtained by splitting the oval by the line segment connecting the vertex at $\gamma = 2\pi/3$ with its opposite, the area of each subarea is equal to $896\pi/2$, which is $1/2$ of the total area of the oval $A(n = 3) = 896$. This result is consistent with Corollary 7.

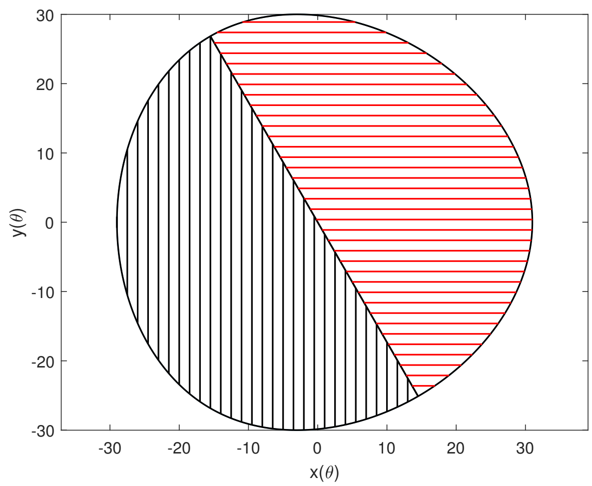

Figure 2: Two parts area of the Oval with $a = 30$ and $n = 3$. Area from a vertex to its opposite.

Example 2. Consider the relative difference for the area $A(n)$ of the ovals with different values of a, at the largest possible values of the parameter n, compared to the area of the circle $A(1)$. We show that the results satisfy the conclusions of Corollary 9.

Table 1 Shows the relative difference, $D(n)/A(1)$, for the area $A(n)$ of the ovals, at the largest possible values of the parameter n, compared to the area of the circle $A(1)$, obtained when $n = 1$ is substituted in Formula (2). The first column in the table shows different values of a, the second column shows the areas of the circles $A(1)$ at each value of a with parameter value $n = 1$, the third column shows the largest possible value of the parameter n for each value of a that satisfies the condition

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of convexity $n^2 - 1 < a$, while the fourth column shows the area $A(n)$ of the ovals at the corresponding values of a and n, and finally, the last column shows the relative differences in the areas of the ovals at the corresponding values a and n, compared to the area A(1) of the corresponding circle. All results in the last column satisfy the upper bound value for the relative differences in area given by Corollary 9. We also notice that the relative differences in the areas approaches zero as the values of a become large, i.e. the relative difference approaches zero as $a \longrightarrow \infty$, regardless of the largest possible values of the parameter n. Of course, if the values of the parameter n are smaller than the largest possible values determined by the condition of convexity, then the relative differences in areas become smaller than the results shown in the table.

Table 1: Relative differences for the area A(n) of the ovals at the largest possible values of n, with different values for a, compared to the area of the circle A(1).

a	Area of Circle A(1)	Largest possible n	Area of Oval A(n)	Rel. Difference in Area
24	$1.809\,557 \times 10^3$	3	$1.796\,991 \times 10^3$	6.9444×10^{-3}
60	$1.130\,973 \times 10^4$	7	$1.123\,434 \times 10^4$	6.6667×10^{-3}
190	$1.134\,115 \times 10^5$	13	$1.131\,476 \times 10^5$	2.3269×10^{-3}
500	$7.853\,982 \times 10^5$	21	$7.847\,070 \times 10^5$	8.8000×10^{-4}
1300	$5.309\,292 \times 10^6$	35	$5.307\,369 \times 10^6$	3.6213×10^{-4}
2000	$1.256\,637 \times 10^7$	43	$1.256\,347 \times 10^7$	2.3100×10^{-4}
2500	$1.963\,495 \times 10^7$	49	$1.963\,118 \times 10^7$	1.9200×10^{-4}
3900	$4.778\,362 \times 10^7$	61	$4.777\,778 \times 10^7$	1.2229×10^{-4}
35000	$3.848\,451 \times 10^9$	187	$3.848\,396 \times 10^9$	1.4273×10^{-5}
40000	$5.026\,548 \times 10^9$	199	$5.026\,486 \times 10^9$	1.2375×10^{-5}
50000	$7.853\,982 \times 10^9$	223	$7.853\,904 \times 10^9$	9.9456×10^{-6}
70000	$1.539\,380 \times 10^{10}$	263	$1.539\,370 \times 10^{10}$	7.0580×10^{-6}

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Example 3. We discuss the areas of the ovals, with different values of a where $a+1$ is a perfect square, at all possible values of the parameter n that satisfy the condition of convexity $n < \sqrt{a+1}$, and compare those areas with the main results discussed in Section 4.

Figure 3 shows the areas of the ovals when $a = 80$ where $a + 1 = 81$ is a perfect square and $\sqrt{a+1} = 9$ is an odd positive integer. We notice that the areas $A(n)$ of the ovals at all possible odd values of the parameter n in the interval $[1, 7]$ lie between the maximum area $A(1)$ and the minimum area A_1, where $A(1)$ is obtained at $n = 1$, which represents a circle of radius a, and the area $A(7)$ at the largest possible value $n = 7$ equals A_1, which agree with Inequality (5).

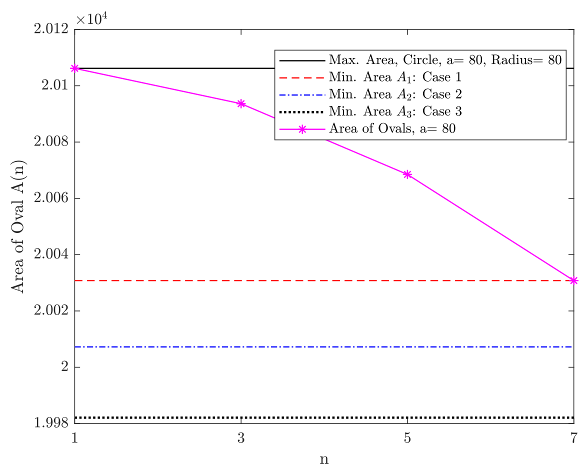

Figure 3: Areas of Ovals at n and $a = 80$, where $a + 1 = 81$ is a perfect square & $\sqrt{a+1} = 9$ is odd.

Figure 4 shows the areas of the ovals when $a = 63$ where $a + 1 = 64$ is a perfect square and $\sqrt{a+1} = 8$ is an even positive integer. We notice that the areas $A(n)$ of the ovals at all possible odd values of the parameter n in the interval $[1, 7]$ lie between the maximum area $A(1)$ and the minimum area A_2, where $A(1)$ is obtained when $n = 1$, which represents a circle of radius a, and the area $A(7)$ at the largest possible value $n = 7$ equals A_2, which agrees with Inequality (6).

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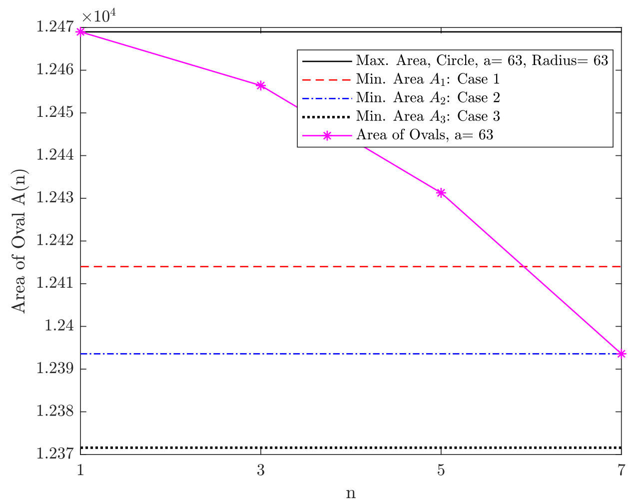

Figure 4: Areas of Ovals at n and $a = 63$, where $a + 1 = 64$ is a perfect square & $\sqrt{a + 1} = 8$ is even.

Example 4. We discuss the areas of the ovals, with different values of a where $a + 1$ is not a perfect square, at all possible values of the parameter n that satisfy the condition of convexity $n < \sqrt{a + 1}$, and compare these areas with the results discussed in Section 4.

Figure 5 shows the areas of the ovals when $a = 49$ where $a + 1 = 50$ is not a perfect square but a is a perfect square and $\sqrt{a} = 7$ is an odd positive integer. We notice that the areas $A(n)$ of the ovals at all possible odd values of the parameter n in the interval $[1, 7]$ lie between the maximum area $A(1)$ and the minimum area A_3, where $A(1)$ is obtained when $n = 1$, which represents a circle of radius a, and the area $A(7)$ at the largest possible value $n = 7$ equals A_3, which agrees with Inequality (7).

Figure 6 shows the areas of the ovals when $a = 52$ where $a + 1 = 53$ and a are not perfect squares, and $\lfloor \sqrt{a + 1} \rfloor = 7$ is an odd positive integer, thus, the largest possible value of n that satisfies the condition of convexity is $n = 7$. We notice that the areas $A(n)$ of the ovals at all pos-

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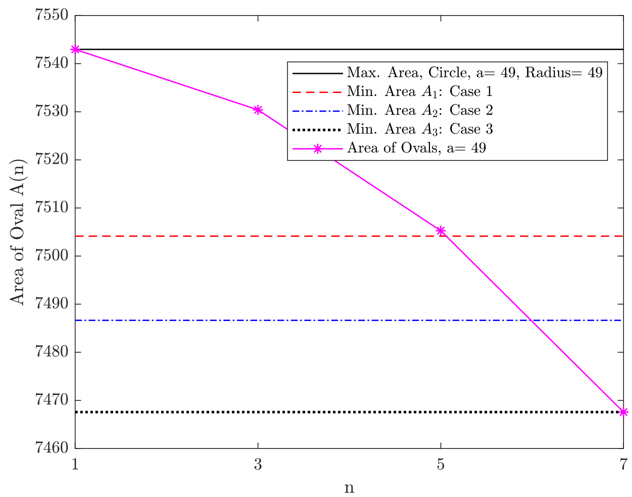

Figure 5: Area of Ovals at n compared to the special cases. $a = 49$, $a + 1 = 50$ is not a perfect square, but a is a perfect square & $\sqrt{a} = 7$ is odd.

sible odd values of the parameter n in the interval $[1, 7]$ lie between the maximum area $A(1)$ and the minimum area A_3, where $A(1)$ is obtained at $n = 1$, which represents a circle of radius a, and the area $A(7)$ at the largest possible value $n = 7$ is greater than A_3, which agrees with Inequality (7). Actually, the area $A(7)$ of the oval when $n = 7$ lies between A_3 and A_2.

Figure 7 shows the areas of the ovals when $a = 76$ where $a + 1 = 77$ and a are not perfect squares, and $\lfloor \sqrt{a + 1} \rfloor = 8$ is an even positive integer, thus, the largest possible value of n that satisfies the condition of convexity is $n = 7$. We notice that the areas $A(n)$ of the ovals at all possible odd values of the parameter n in the interval $[1, 7]$ lie between the maximum area $A(1)$ and the minimum area A_3, and the area $A(7)$ at the largest possible value $n = 7$ is greater than A_3, which agrees with Inequality (7). Actually, the area $A(7)$ of the oval when $n = 7$ lies between A_2 and A_1.

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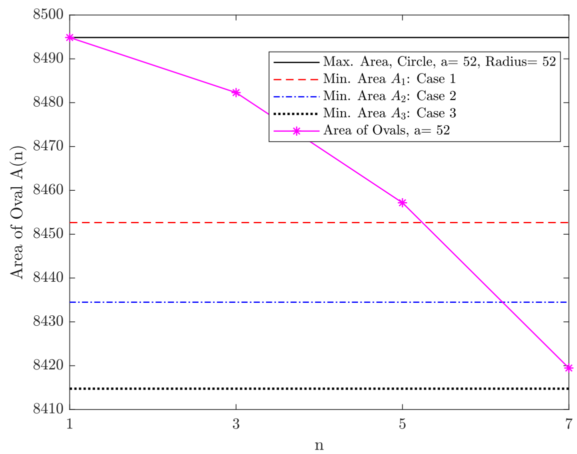

Figure 6: Area of Ovals at n compared to the special cases. $a = 52$, $a + 1 = 53$ is not a perfect square & $\lfloor \sqrt{a+1} \rfloor = 7$ is odd.

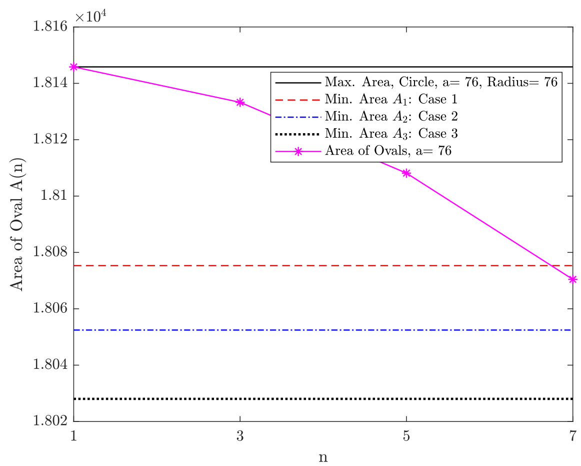

Figure 7: Area of Ovals at n compared to the special cases. $a = 76$, $a + 1 = 77$ is not a perfect square & $\lfloor \sqrt{a+1} \rfloor = 8$ is even.

Example 5. We discuss the areas of the ovals, with different values of a that belong to certain interval $[n^2, (n+2)^2 - 1]$, where n is the largest

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possible value that satisfies the condition of convexity $n < \sqrt{a+1}$, and show that the results agree with Corollary 12.

Figure 8 shows the areas of the ovals at $a \in [n^2, (n+2)^2 - 1] = [49, 80]$, with the largest possible value of the parameter n that satisfies the condition of convexity $n < \sqrt{a+1}$, that is, $n = 7$. We notice that the minimum value $A(7) = 7.4676 \times 10^3$ occurs at $a = n^2 = 49$, where $a + 1 = 50$ is not a perfect square, which agrees with the lower bound A_3 in Case 3. The maximum value $A(7) = 2.0031 \times 10^4$ occurs at $a = (n+2)^2 - 1 = 80$, where $a+1 = 81$ is a perfect square and $\sqrt{a+1} = 9$ is an odd positive integer, which agrees with the lower bound A_1 in Case 1. The area $A(7) = 1.2394 \times 10^4$ is obtained at $a = (n+1)^2 - 1 = 63$, where $a + 1 = 64$ is a perfect square and $\sqrt{a+1} = 8$ is an even positive integer, which agrees with the lower bound A_2 in Case 2. Now, the area at any value a in the interval $[n^2, (n+1)^2 - 1] = [49, 63]$ lies between A_3 and A_2. And finally, the area at any value a in the interval $[(n+1)^2 - 1, (n+2)^2 - 1] = [63, 80]$ lies between A_2 and A_1, which agrees with the results in Corollary 12.

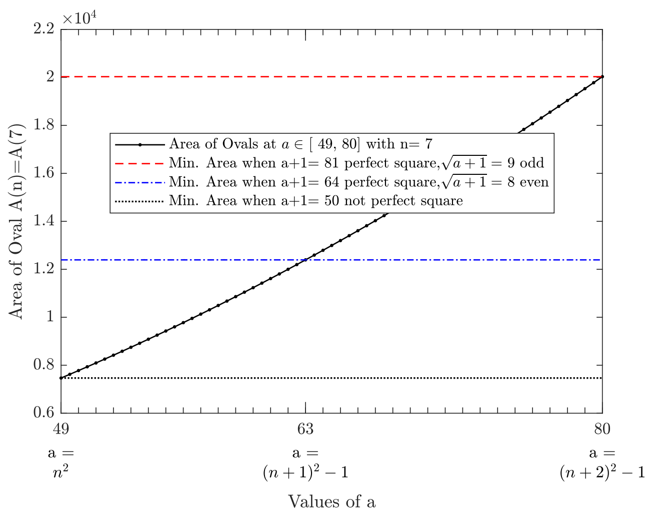

Figure 8: Areas of the Ovals at $a \in [n^2, (n+2)^2 - 1]$, $n = 7$, and the largest possible value of the parameter n for each a.

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6 Conclusions

The measurement of the area of a geometrical figure is a scheme of inducing more properties regarding it. In this work, we have studied intrinsic properties about the area of ovals with constant width. A useful methodology is to compare the results of a concept of geometry for a shape with those known for closer figures to it. We measured the area of the oval and compared it to the area of the particular circle that is related to it and showed that the relative area difference of the oval compared to that of the circle is less than the reciprocal of its width and approaching zero as the width tends to infinity, regardless of the number of vertices. The last results allow researchers to make a decision regarding a boundary circle of ovals with constant width by one of them that fits the circle more than the others. The area of the oval from one vertex to the next one is equal to the total area of the oval divided by the number of vertices, and the area from a vertex to its opposite is equal to half of the total area.

This work established important results in the subject of optimization where the concept of extreme values of the area within an oval had been widely discussed. In order to find the minimum and maximum values of the area of the oval, depending on the number of vertices, we found that we have three main cases to determine the largest number of vertices that satisfies the condition of convexity, area bounds are determined in each case. Thus, the frequency of cases gave the work conclusions regarding convexity and area bounds. Additionally, an important idea in this work was to have a closer look at the domain (interval) of plotting an oval with constant width. We established a correlation between an interval with endpoints depending on a predetermined odd positive integer greater than one and ovals of constant width related to that interval, and in addition to finding the areas and determining area bounds, we showed that the maximum number of vertices of all possible ovals is that the already predefined odd number.

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