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PROPERTIES OF WAVE PACKET TRANSFORM IN THE QPFT DOMAIN

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Abstract

In this paper we introduce a quadratic-phase Fourier wave packet transform, formulated through a combination of quadratic-phase integral operator and wave packet trans-form concepts. We explore various fundamental properties of this transform, including Parsevals relation and a recon-struction formula.

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1 Introduction

The Wave Packet Transform (WPT) was initially proposed by Posch in 1992 [9], formulated through the framework of the Weyl operator. Functionally analogous to the wavelet transform, WPT facilitates the representation of signals in a joint time-frequency domain by employing a linear operator applied to a structured family of functions termed wave packets. This approach enables enhanced localization and resolution of signal features across both time and frequency dimensions. In recent advancements, WPT has emerged as a powerful tool across diverse signal processing applications, including wireless communications, signal denoising, and image compression [6, 7]. Owing to its increased adaptability and improved analytical precision, WPT has demonstrated superior effectiveness over conventional wavelet transform methods in multiple domains [6, 7, 8, 9, 10, 24, 25].

The classical Fourier Transform (FT) serves as a fundamental analytical technique for stationary signals due to its global frequency resolution. However, its limitation in handling time-varying characteristics makes it unsuitable for non-stationary signal analysis, where wavelet-based methods offer superior adaptability [3, 4]. In recent developments, the Quadratic Phase Fourier Transform (QPFT) has garnered significant attention owing to its diverse utility in signal and image processing domains. Despite its advantages, QPFT exhibits limitations in accurately resolving components within the quadratic phase spectrum, particularly in applications demanding high localization precision.

The fractional Fourier transform (FRFT), a natural generalization of the classical Fourier transform, has been extensively developed

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in both theoretical frameworks and applied domains [1, 7, 26]. Furthermore, the linear canonical transform [5], which extends the FRFT, has emerged as a highly effective tool for analyzing non-stationary signals and performing advanced time-frequency signal processing.
Castro et al. [2] generalized the existing Fourier-related transforms by formulating the Quadratic Phase Fourier Transform (QPFT), thereby unifying and extending earlier developments. Subsequently, Prasad and Sharma [15] contributed significantly to the theoretical advancement of the QPFT by establishing a range of analytical results [15, 16, 17, 18, 19, 20, 21]. The QPFT for a function $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$ is formally defined as follows:

$$(\mathcal{Q}_{r,s}^{a,b,c} f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} \mathcal{K}_{r,s}^{a,b,c}(\omega, x) f(x) dx, \tag{1}$$

where $\mathcal{K}_{r,s}^{a,b,c}(\omega, x) = \sqrt{\frac{b}{2\pi i}}\, e^{i\Omega_{r,s}^{a,b,c}(\omega,x)}$ with $\Omega_{r,s}^{a,b,c}(\omega, x) = ax^2 + bx\omega + c\omega^2 + rx + s\omega$ which is known as the quadratic-phase function, for every $a, b \neq 0, c, r, s \in \mathbb{R}$.
Inversion of QPFT is given by

$$f(x) = \int_{\mathbb{R}} \mathcal{K}_{-s,-r}^{-c,-b,-a}(x, \omega)(\mathcal{Q}_{r,s}^{a,b,c} f)(\omega) d\omega, \tag{2}$$

where

$$\mathcal{K}_{-s,-r}^{-c,-b,-a}(x, \omega) = \sqrt{\frac{bi}{2\pi}}\, e^{-i\Omega_{s,r}^{c,b,a}(x,\omega)}.$$

If the functions $f, g \in L^2(\mathbb{R})$, then its Parseval's identity for QPFT domain is given as follows:

$$\int_{\mathbb{R}} (\mathcal{Q}_{r,s}^{a,b,c} f)(\omega)\overline{(\mathcal{Q}_{r,s}^{a,b,c} g)(\omega)} d\omega = \int_{\mathbb{R}} f(x)\overline{g(x)} dx. \tag{3}$$

And (3) known as Plancherel's identity when $f = g$ and is given by

$$\int_{\mathbb{R}} \left|(\mathcal{Q}_{r,s}^{a,b,c} f)(\omega)\right|^2 d\omega = \int_{\mathbb{R}} |f(x)|^2 dx. \tag{4}$$

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The manuscript is organized into four distinct sections to systematically present the study. Section 1 introduces the Quadratic Phase Fourier Transform (QPFT) and establishes its foundational aspects, including a derivation of its Parseval-type identity. Section 2 provides a focused overview of wave packet transform methodologies, encompassing the standard, fractional, and linear canonical variants. In Section 3, key analytical properties of the proposed Quadratic Phase Fourier Wave Packet Transform (QPFWPT) are rigorously derived and discussed. Finally, Section 4 offers concluding remarks and summarizes the principal findings of the work.

2 Preliminaries

Shi et al.[23] extended the framework of classical mother wavelets[3, 4] by introducing a novel class of fractional wavelets. Building on this foundation, Wei et al.[11] further advanced the theory by formulating the concept of the linear canonical wavelet transform. Subsequently, Prasad and Sharma [15] generalized the results of[11] and proposed the Quadratic-Phase Wavelet (QPW) representation for functions $f(x) \in L^2(\mathbb{R})$, defined as follows:

$$\Omega^{a}_{r}{}_{\beta,\alpha}(x) = \frac{1}{\sqrt{\alpha}} \ \left(\frac{x-\beta}{\alpha}\right) e^{-ia(x^2-\beta^2)-ir(x-\beta)}, \quad \beta \in \mathbb{R},\ \alpha > 0. \quad (5)$$

And the admissibility condition for the quadratic-phase wavelet transform is given by

$$0 < C_{\psi}^{\Omega_r^a} = \int_{\mathbb{R}} \frac{|\mathcal{Q}_{r,s}^{a,b,c}[e^{-ia(.)^2-ir(.)}\psi](\omega)|^2}{|\omega|} d\omega < \infty. \quad (6)$$

To facilitate the analysis of both stationary and non-stationary signals, several advanced signal processing frameworks have been developed, including the wave packet transform, the fractional wave packet transform, and the linear canonical wave packet transform, as established in the literature [5, 8, 9, 14, 24, 25].

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The wave packet transform (WPT) [9, 24, 25] can be viewed as

$$\Big(\mathbb{W}P_{\psi}f\Big)(\omega,\beta,\alpha)=\frac{1}{\sqrt{2\pi\alpha}}\int_{\mathbb{R}}e^{-i\omega x}\overline{\psi}\Big(\frac{x-\beta}{\alpha}\Big)f(x)dx, \tag{7}$$

where the function $\frac{1}{\sqrt{2\pi\alpha}}e^{-i\omega x}\psi\Big(\frac{x-\beta}{\alpha}\Big)$ is known as wavelet packets. Huang and Suter [6] defined the fractional wave packet transform(FrWPT) as

$$\Big(\mathbb{W}P_{\psi}^{\theta}f\Big)(\omega,\beta,\alpha)=\int_{\mathbb{R}}\mathcal{K}^{\theta}(\omega,x)\psi_{\beta,\alpha}(x)dx, \tag{8}$$

where $\mathcal{K}^{\theta}(\omega,x)$ is the kernel of fractional Fourier wavelet transform. Later on Prasad et al. [14] defined the continuous fractional wave packet transform(CFrWPT) as

$$\begin{aligned}\Big(C\mathbb{W}P_{\psi}^{\theta}f\Big)(\omega,\beta,\alpha) &= \frac{C^{\theta}}{\sqrt{\alpha}}\int_{\mathbb{R}}e^{\frac{i}{2}(x^2+\omega^2)\cot\theta-i\omega x\csc\theta}\\ &\times\ e^{\frac{i}{2}(x^2-\beta^2)\cot\theta}\overline{\psi}\Big(\frac{x-\beta}{\alpha}\Big)f(x)dx, \end{aligned}\tag{9}$$

where $C^{\theta}=\sqrt{\frac{1-i\cot\theta}{2\pi}}$. Moreover, Li and Wei [8] defined the linear canonical wave packet transform associated with parameters $A=(a,b,c,d)$ as

$$\Big(\mathbb{W}_{\psi}^{A}f\Big)(\omega,\beta,\alpha)=\frac{1}{\sqrt{i2\pi\alpha b}}\int_{\mathbb{R}}\ \Big(\frac{x-\beta}{\alpha}\Big)f(x)e^{i[\frac{a}{2b}x^2-\frac{1}{b}\omega x+\frac{d}{2b}\omega^2]}dx,\ b\neq 0. \tag{10}$$

Recently, Bhat et al.[12] established the quadratic-phase Fourier wave packet transform (QPFWPT) which is generalization of linear canonical wave packet transform and given as follows:

$$\begin{aligned}\Big(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}}f\Big)(\xi,\beta,\alpha) &= \int_{\mathbb{R}}\mathcal{K}^{\Omega_{r,s}^{a,b,c}}(\xi,x)\overline{\psi_{\beta,\alpha}^{\Omega_{r}^{a}}}(x)dx,\ \beta\in\mathbb{R},\ \alpha>0,\\ &= \sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}}e^{i(ax^2+bx\xi+c\xi^2+rx+s\xi)}\frac{1}{\sqrt{\alpha}}\overline{\ }\Big(\frac{x-\beta}{\alpha}\Big)\\ &\times\ \overline{f}(x)e^{ia(x^2-\beta^2)+ir(x-\beta)}dx\end{aligned} \tag{11}$$

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We have $f_\xi^{\Omega_{r,s}^{a,b,c}}(x) = \sqrt{\frac{b}{2\pi i}} e^{i\Omega_{r,s}^{a,b,c}(\xi,x)} f(x)$ then its QPFT is given by

$$\hat{f}_\xi^{\Omega_{r,s}^{a,b,c}}(\omega) = \sqrt{\frac{b}{2\pi i}} e^{-ic\omega\xi} \mathcal{Q}^{\Omega_{r,s}^{a,b,c}} \Big[e^{ia(.)^2+ir(.)} f\Big](\omega+\xi). \tag{12}$$

Using the Parseval's identity, the quadratic-phase Fourier wave packet transform (QPFWPT) is defined as

$$\Big(\mathbb{W}P_\psi^{\Omega_{r,s}^{a,b,c}} f\Big)(\xi,\beta,\alpha) = \Big\langle f_\xi^{\Omega_{r,s}^{a,b,c}}, \psi_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}} \Big\rangle = \Big\langle \hat{f}_\xi^{\Omega_{r,s}^{a,b,c}}, \hat{\psi}_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}} \Big\rangle. \tag{13}$$

Hence (13) rewrite as

$$\begin{aligned} \Big(\mathbb{W}P_\psi^{\Omega_{r,s}^{a,b,c}} f\Big)(\xi,\beta,\alpha) &= \sqrt{\frac{b\alpha}{2\pi i}} \int_{\mathbb{R}} e^{-i\Omega_{r,s}^{a,b,c}(\omega,\beta)+ic(\alpha\omega)^2+is(\alpha\omega)-ic\omega\xi} \\ &\quad \times \mathcal{Q}^{\Omega_{r,s}^{a,b,c}} \Big[e^{ia(.)^2+ir(.)} f\Big](\omega+\xi) \\ &\quad \times \overline{\mathcal{Q}_{r,s}^{a,b,c}[e^{-ia(.)^2-ir(.)}\psi](\alpha\omega)} d\omega. \end{aligned} \tag{14}$$

3 Properties of wave packet transform associated with quadratic-phase Fourier transform

Bhat et al.[12] investigated certain analytical properties of the quadratic-phase wave packet transform within the intersection space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, following an approach conceptually aligned with the earlier contributions of Prasad and Kundu[13]. However, given that most practical signals encountered in real-world applications predominantly belong to the Hilbert space $L^2(\mathbb{R})$, we have extended the theoretical framework of the Quadratic-Phase Wave Packet Transform (QFWPT) by deriving additional properties that are more suitable for application in signal processing contexts. Inspired by the methodology outlined in the work of Prasad and Ansari[22], this section is devoted to the formal development and presentation of several significant properties of the wave packet transform within the quadratic-phase Fourier transform (QPFT) domain.

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Theorem 1. *If $\psi_{\beta,\alpha}^{\Omega_r^a}$ be the QPW and $f \in L^2(\mathbb{R})$, then we have the following properties of the QPFWPT:*
(1) *Linearity: If $f(x) = mg(x) + nh(x)$, then*

$$\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha) = m\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} g\right)(\xi,\beta,\alpha) + n\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} h\right)(\xi,\beta,\alpha).$$

(2) *Scaling: If $D_m f(x) = \frac{1}{\sqrt{m}} f(\frac{x}{m})$, then*

$$(i) \quad \left(\mathbb{W}^{\Omega_{dr,s}^{a,b,c}} D_m f\right)(\xi,\beta,\alpha) = \left(\mathbb{W}^{\Omega_{rm,s}^{am^2,b,c}} f\right)\left(\xi, \frac{\beta}{m}, \frac{\alpha}{m}\right).$$

$$(ii) \quad \left(\mathbb{W}_{D_m\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha) = \left(\mathbb{W}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,m\alpha).$$

(3) *Parity: If $Pf(x) = f(-x)$, then*

$$\left(\mathbb{W}_{P\psi}^{\Omega_{r,s}^{a,b,c}} Pf\right)(\xi,\beta,\alpha) = \left(\mathbb{W}^{\Omega_{-r,s}^{a,b,c}} f\right)(\xi,-\beta,\alpha).$$

(4) *Translation:*
(*i*)*If $T_m\psi(x) = \psi(x-m)$, then*

$$\left(\mathbb{W}_{T_m\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha) = \left(\mathbb{W}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta+m\alpha,\alpha).$$

(*ii*)*If $\Im_{ar} f(x) = e^{-2airx} f(x-r)$, then*

$$\left(\mathbb{W}^{\Omega_{r,s}^{a,b,c}} \Im_{ar} f\right)(\xi,\beta,\alpha) = e^{-2ai\beta r}\left(\mathbb{W}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta-r,\alpha).$$

(5) *Anti-linearity: If $\psi(x) = m\varphi(x) + n\phi(x)$, then*

$$\left(\mathbb{W}_{m\varphi+n\phi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha) = \overline{m}\left(\mathbb{W}_{\varphi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha) + \overline{n}\left(\mathbb{W}_{\phi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha).$$

Proof. It is straightforward, so we left. □

Theorem 2. *If ψ_1 and ψ_2 are two wavelets and $\left(\mathbb{W}P_{\psi_1}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)$ and $\left(\mathbb{W}P_{\psi_2}^{\Omega_{r,s}^{a,b,c}} g\right)(\xi,\beta,\alpha)$ denote the QPFWPT of signals $f, g \in L^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \left(\mathbb{W}P_{\psi_1}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)\overline{\left(\mathbb{W}P_{\psi_2}^{\Omega_{r,s}^{a,b,c}} g\right)(\xi,\beta,\alpha)}\frac{d\xi d\beta d\alpha}{\alpha^2} = C_{\psi_1,\psi_2}^{\Omega_r^a}\langle f, g\rangle, \quad (15)$$

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where

$$C^{\Omega^a_r}_{\psi_1,\psi_2} = \int_{\mathbb{R}_+} \overline{\mathcal{Q}^{a,b,c}_{r,s}\left[e^{-ia(.)^2-ir(.)}\psi_1\right]}(\alpha\omega)\mathcal{Q}^{a,b,c}_{r,s}\left[e^{-ia(.)^2-ir(.)}\psi_2\right](\alpha\omega)(\alpha^{-1})d\alpha < \infty.$$

Proof. We have

$$\begin{aligned}\left(\mathbb{W}P^{\Omega^{a,b,c}_{r,s}}_{\psi_1} f\right)(\xi,\beta,\alpha) &= \sqrt{\frac{b\alpha}{2\pi i}}\int_{\mathbb{R}} e^{-i\Omega^{a,b,c}_{r,s}(\omega,\beta)+ic(\alpha\omega)^2+is(\alpha\omega)-ic\omega\xi} \\ &\quad \times \mathcal{Q}^{a,b,c}_{r,s}\left[e^{ia(.)^2+ir(.)}f\right](\omega+\xi)\overline{\mathcal{Q}^{a,b,c}_{r,s}\left[e^{-ia(.)^2-ir(.)}\psi_1\right](\alpha\omega)}d\omega,\end{aligned}$$

and

$$\begin{aligned}\left(\mathbb{W}P^{\Omega^{a,b,c}_{r,se}}_{\psi_2} g\right)(\xi,\beta,\alpha) &= \sqrt{\frac{b\alpha}{2\pi i}}\int_{\mathbb{R}} e^{-i\Omega^{a,b,c}_{r,s}(\eta,\beta)+ic(\alpha\eta)^2+ie(\alpha\eta)-ic\eta\xi} \\ &\quad \times \mathcal{Q}^{a,b,c}_{r,s}\left[e^{ia(.)^2+ir(.)}g\right](\eta+\xi)\overline{\mathcal{Q}^{a,b,c}_{r,s}\left[e^{-ia(.)^2-ir(.)}\psi_2\right](\alpha\eta)}d\eta.\end{aligned}$$

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Now,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left(\mathbb{W}P_{\psi_1}^{\Omega_{r,s}^{a,b,c}}f\right)(\xi,\beta,\alpha)\overline{\left(\mathbb{W}P_{\psi_2}^{\Omega_{r,s}^{a,b,c}}g\right)(\xi,\beta,\alpha)}\frac{d\beta d\alpha d\xi}{\alpha^2}$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big(\sqrt{\frac{b\alpha}{2\pi i}}\int_{\mathbb{R}}e^{-i\Omega_{r,s}^{a,b,c}(\omega,\beta)+ic(\alpha\omega)^2+is(\alpha\omega)-ic\omega\xi}$$

$$\times\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}f\right](\omega+\xi)\overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_1\right](\alpha\omega)}d\omega\Big)$$

$$\times\Big(\sqrt{\frac{b\alpha}{2\pi i}}\int_{\mathbb{R}}e^{-i\Omega_{r,s}^{a,b,c}(\eta,\beta)+ic(\alpha\eta)^2+is(\alpha\eta)-ic\eta\xi}$$

$$\times\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}g\right](\eta+\xi)\overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_2\right](\alpha\eta)}d\eta\Big)\frac{d\beta d\alpha d\xi}{\alpha^2}$$

$$=\frac{-b}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}}e^{ic(\eta^2-\omega^2)-ic\alpha^2(\eta^2-\omega^2)-is(\eta-\omega)-is\alpha(\eta-\omega)+ic(\eta-\omega)\xi}$$

$$\times\Big(\int_{\mathbb{R}}e^{ib(\eta-\omega)\beta}d\beta\Big)\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}f\right](\omega+\xi)\ \overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_1\right](\alpha\omega)}$$

$$\times\ \overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}g\right](\eta+\xi)}\ \mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_2\right](\alpha\eta)d\omega d\eta\Big]\frac{d\alpha}{\alpha}$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}}\Big[e^{ic(\eta^2-\omega^2)-ic\alpha^2(\eta^2-\omega^2)-is(\eta-\omega)-is\alpha(\eta-\omega)+ic(\eta-\omega)\xi}$$

$$\times\ \delta(\eta-\omega)\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}f\right](\omega+\xi)\ \overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_1\right](\alpha\omega)}$$

$$\times\overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}g\right](\eta+\xi)}\ \mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_2\right](\alpha\eta)d\omega d\eta\Big]\frac{d\alpha}{\alpha}.$$

$$=\int_{\mathbb{R}}\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}f\right](\eta+\xi)\ \overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}g\right](\eta+\xi)}$$

$$\times\int_{\mathbb{R}_+}\overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_1\right](\alpha\omega)}\ \mathcal{Q}_{r,s}^{a,b,c}\left[e^{-ia(.)^2-ir(.)}\psi_2\right](\alpha\omega)\frac{d\alpha}{\alpha}\Big]d\omega$$

$$=C_{\psi_1,\psi_2}^{\Omega_r^a}\int_{\mathbb{R}}\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}f\right](\eta+\xi)\ \overline{\mathcal{Q}_{r,s}^{a,b,c}\left[e^{ia(.)^2+ir(.)}g\right](\eta+\xi)}d\omega,$$

$$=C_{\psi_1,\psi_2}^{\Omega_r^a}\langle f,g\rangle.$$

□

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Theorem 3. *If is a wavelet and* $\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)$ *and* $\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} g\right)(\xi,\beta,\alpha)$ *denote the QPFWPT of signal* $f, g \in L^2(\mathbb{R})$, *then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)\overline{\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} g\right)(\xi,\beta,\alpha)}\frac{d\xi d\beta d\alpha}{\alpha^2} = C_{\psi}^{\Omega_r^a}\langle f, g\rangle, \tag{16}$$

where

$$C_{\psi}^{\Omega_r^a} = \int_{\mathbb{R}_+}\left|\mathcal{Q}_{r,s}^{a,b,c}\big[e^{-ia(.)^2-ir(.)}\psi\big]\right|^2(\alpha\omega)(\alpha^{-1})d\alpha < \infty.$$

Proof. Proof is similar to Theorem (2) and it can be deduced by setting $\psi_1 = \psi_2 = \psi$. □

Remark (1) If $f = g$, in (2), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left|\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)\right|^2\frac{d\xi d\beta d\alpha}{\alpha^2} = C_{\psi}^{\Omega_r^a}||f||^2. \tag{17}$$

Theorem 4. *If* $f \in L^2(\mathbb{R})$, *then* f *can be reconstructed by using the formula,*

$$\begin{aligned} f(x) &= \frac{1}{C_{\psi}^{\Omega_r^a}}\sqrt{\left(\frac{b}{2\pi i}\right)}\, e^{-i\Omega_{r,s}^{a,b,c}(\xi,x)} \\ &\times \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)\psi_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}(x)\frac{d\xi d\beta d\alpha}{\alpha^2}. \end{aligned} \tag{18}$$

Proof. For any $f, g \in L^2(\mathbb{R})$, we have

$$\begin{aligned} C_{\psi}^{\Omega_r^a}\langle f, g\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)\overline{\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} g\right)(\xi,\beta,\alpha)}\frac{d\xi d\beta d\alpha}{\alpha^2} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha) \\ &\times \Big[\int_{\mathbb{R}}\sqrt{\left(\frac{b}{2\pi i}\right)}\, e^{-i\Omega_{r,s}^{a,b,c}(\xi,x)}\ \overline{g(x)}\quad {}_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}(x)\Big]dx\frac{d\xi d\beta a\alpha}{\alpha^2} \\ &= \Big\langle\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\left(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}} f\right)(\xi,\beta,\alpha)\sqrt{\left(\frac{b}{2\pi i}\right)}e^{-i\Omega_{r,s}^{a,b,c}(\xi,x)}\psi_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}(x)\frac{d\xi d\beta d\alpha}{\alpha^2}, g(x)\Big\rangle. \end{aligned}$$

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Therefore,

$$f(x)=\frac{1}{C_{\psi}^{\Omega_r^a}}\sqrt{\left(\frac{b}{2\pi i}\right)}\,e^{-i\Omega_{r,s}^{a,b,c}(\xi,x)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}}f\Big)(\xi,\beta,\alpha)\psi_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}(x)\frac{d\xi d\beta d\alpha}{\alpha^2}.$$

□

Theorem 5. *If $f\in L^2(R)$, then*

$$\int_{\mathbb{R}}\Big(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}}f\Big)(\xi,\beta,\alpha)\overline{\Big(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}}g\Big)(\xi,\beta,\alpha)}d\beta=\frac{2\pi}{b}\Big\langle\mathcal{X}^{\Omega_{r,s}^{a,b,c}},\ \mathcal{Y}^{\Omega_{r,s}^{a,b,c}}\Big\rangle,$$

where $\mathcal{X}^{\Omega_{r,s}^{a,b,c}}=e^{ic(\alpha\omega)^2+is(\alpha\omega)}\mathcal{Q}_{r,s}^{a,b,c}\Big[e^{-ia(.)^2-ir(.)}\psi\Big](\alpha\omega)\hat{f}_{\xi}^{\Omega_{r,s}^{a,b,c}}(\omega)$
and $\mathcal{Y}^{\Omega_{r,s}^{a,b,c}}=e^{ic(\alpha\omega)^2+is(\alpha\omega)}\mathcal{Q}_{r,s}^{a,b,c}\Big[e^{-ia(.)^2-ir(.)}\psi\Big](\alpha\omega)\hat{g}_{\xi}^{\Omega_{r,s}^{a,b,c}}(\omega)$

Proof. From (13) we have

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}}f\Big)(\xi,\beta,\alpha)\overline{\Big(\mathbb{W}P_{\psi}^{\Omega_{r,s}^{a,b,c}}g\Big)(\xi,\beta,\alpha)}d\beta\\
&=\int_{\mathbb{R}}\Big\langle\hat{f}_{\xi}^{\Omega_{r,s}^{a,b,c}},\hat{\psi}_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}\Big\rangle\overline{\Big\langle\hat{g}_{\xi}^{\Omega_{r,s}^{a,b,c}},\hat{\psi}_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}\Big\rangle}d\beta\\
&=\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\hat{f}_{\xi}^{\Omega_{r,s}^{a,b,c}}\overline{\hat{\psi}_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}}d\omega\Big)\Big(\int_{\mathbb{R}}\overline{\hat{g}_{\xi}^{\Omega_{r,s}^{a,b,c}}}\hat{\psi}_{\beta,\alpha}^{\Omega_{r,s}^{a,b,c}}d\eta\Big)d\beta\\
&=\alpha\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\overline{e^{i\Omega_{r,s}^{a,b,c}(\omega,\beta)-ic(\alpha\omega)^2-is(\alpha\omega)}\mathcal{Q}_{r,s}^{a,b,c}\Big[e^{-ia(.)^2-ir(.)}\psi\Big](\alpha\omega)\overline{\hat{f}_{\xi}^{\Omega_{r,s}^{a,b,c}}}}d\omega\Big)\\
&\times\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}e^{i\Omega_{r,s}^{a,b,c}(\eta,\beta)-ic(\alpha\eta)^2-is(\alpha\eta)}\mathcal{Q}_{r,s}^{a,b,c}\Big[e^{-ia(.)^2-ir(.)}\psi\Big](\alpha\eta)\overline{\hat{g}_{\xi}^{\Omega_{r,s}^{a,b,c}}}d\eta\Big)d\beta\\
&=\alpha\int_{R}\Big(\int_{\mathbb{R}}\overline{e^{i\Omega_{r,s}^{a,b,c}(\omega,\beta)}\overline{\mathcal{X}^{\Omega_{r,s}^{a,b,c}}}}d\omega\Big)\Big(\int_{\mathbb{R}}e^{i\Omega_{r,s}^{a,b,c}(\eta,\beta)}\overline{\mathcal{Y}^{\Omega_{r,s}^{a,b,c}}}d\eta\Big)d\beta\\
&=\frac{2\pi}{b}\Big\langle\mathcal{Q}_{r,s}^{a,b,c}\big[\overline{\mathcal{X}^{\Omega_{r,s}^{a,b,c}}}\big],\ \mathcal{Q}_{r,s}^{a,b,c}\big[\overline{\mathcal{Y}^{\Omega_{r,s}^{a,b,c}}}\big]\Big\rangle\\
&=\frac{2\pi}{b}\Big\langle\overline{\mathcal{X}^{\Omega_{r,s}^{a,b,c}}},\ \overline{\mathcal{Y}^{\Omega_{r,s}^{a,b,c}}}\Big\rangle\\
&=\frac{2\pi}{b}\Big\langle\mathcal{X}^{\Omega_{r,s}^{a,b,c}},\ \mathcal{Y}^{\Omega_{r,s}^{a,b,c}}\Big\rangle.
\end{aligned}$$

□

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4 Conclusion

In this paper, we introduce the quadratic-phase Fourier wave packet transform (QPFWPT) based on the quadratic-phase Fourier wavelet. Our study establishes foundational properties of the transform, notably deriving its inversion formula. Future research may investigate the connections between QPFWPT and frame theory, including its potential formulations as wavelet frames and Parseval frames, thereby broadening its theoretical framework and applications.

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