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BIPOLAR INTUITIONISTIC FUZZY HYPER SOFT SET

M.Divya[1], D. Amsaveni[2]

[1]Research Scholar, [2]Associate Professor

[1,2]PG & Research Department of Mathematics,

Sri Sarada College for Women (Autonomous), Salem-16

Email: divyanaveen1988@gmail.com[1], damsaveni2020@gmail.com[2]

Abstract

In this paper, a novel bipolar intuitionistic fuzzy hyper soft set is defined and it is a hybrid model combining a bipolar intuitionistic fuzzy set with a hyper soft set. Some important basic operations such as union, intersection, complement, subset, equality, conjunction (AND), and disjunction (OR) are formally defined along with illustrative examples and their corresponding theorems. Additionally, De Morgan's laws are examined within this framework. This approach provides a solid foundation for applications in complex, uncertain environments where traditional models are limited.

Keywords: Bipolar Intuitionistic Fuzzy Hyper Soft Set, Intuitionistic Fuzzy Hyper Soft Set, Bipolar Fuzzy Hyper Soft Set, Hyper Soft Set.

1. Introduction

The fuzzy sets theory which was introduced by Zadeh[1] in 1965 and its extensions, such as Intuitionistic Fuzzy Sets (IFS)[2], Pythagorean Fuzzy Sets (PFS) [3][4], Complex Fuzzy Sets (CFS) [5], Neutrosophic Fuzzy Sets (NFS) [6] and Bipolar Fuzzy Sets (BFS) [7] have been developed by several researchers. Soft Set (SS) theory was introduced by Molodtsov [8] and further Maji et al.[9] developed Fuzzy Soft Set (FSS) by combining fuzzy set with soft set theory. Çağman and Karataş [10] incorporated Intuitionistic Fuzzy Soft Set (IFSS) in decision making, while Deli and Broumi [11] explained the relation on Neutrosophic Soft Set (NSS). An advancement of soft sets into Hyper Soft Sets (HS) was done by Smarandache [12] and basic operations on Hyper soft sets was given by Saeed et al.[13]. Later, Yolcu and Ozturk[14] integrated fuzzy set with Hyper Soft Sets and formulated the Fuzzy Hyper Soft Set (FHSS). Intuitionistic Fuzzy Hyper Soft Sets (IFHSS) another combination was developed by Yolcu et al.[15] and also Al-Quran et al.[16] introduced Bipolar Fuzzy Hyper Soft Sets (BFHSS). These extensions all play a major role in decision-making. Extending in the same direction, the Novel Bipolar Intuitionistic Fuzzy Hyper Soft Set (BIFHSS) is introduced with some fundamental operations along with theorems and applications.

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2. Preliminaries

Definition: 1 ([12]) Let $e_1, e_2 \dots . e_n$ for $n \geq 1$, be n distinct attributes and their associated attribute values are $A_1, A_2 \dots A_n$ which are pairwise disjoint. Then the pair $(\mathbb{G}, A_1 \times A_2 \times \dots \times A_n)$ is called a Hyper Soft Set over $\mathfrak{U}$, the universal set and $\mathbb{G} : A_1 \times A_2 \times \dots \times A_n \to \mathrm{P}(\mathfrak{U})$, where $P(\mathfrak{U})$ the power set of $\mathfrak{U}$.

Definition: 2 ([15]) Let $e_1, e_2 \dots . e_n$ be n distinct attributes and its respective attribute values are the set $E_1, E_2 \dots . E_n$, which are pairwise disjoint. Let $A_i \subseteq E_i$ with $A_i \neq \emptyset$, $i = 1,2,3 \dots n$. An IFHSS defined as the pair $(\mathcal{K}, A_1 \times A_2 \times \dots \times A_n)$, where $\mathcal{K}: A_1 \times A \times \dots \times A_n \to \mathrm{IFP}(\mathfrak{U})$ where $\mathrm{IFP}(\mathfrak{U})$ be the intuitionistic fuzzy power set of universal set $\mathfrak{U}$ and

$$\mathcal{K}(A_1 \times A_2 \times \dots \times A_n) = \left\{\langle \xi, \left(\frac{\text{ű}}{\theta_{\mathcal{K}(\xi)}(\text{ű}), \sigma_{\mathcal{K}(\xi)}(\text{ű})}\right)\rangle \middle| \text{ű} \in \mathfrak{U}, \xi \in A_1 \times A_2 \times \dots \times A_n \subseteq E_1 \times E_2 \times \dots \times E_n, \right\}$$

where θ is the membership value and σ is the non-membership value with the condition $0 \leq \theta_{\mathcal{K}(\xi)}(\text{ű}) + \sigma_{\mathcal{K}(\xi)}(\text{ű}) \leq 1$ and $\theta_{\mathcal{K}(\xi)}(\text{ű}), \sigma_{\mathcal{K}(\xi)}(\text{ű}) \in [0,1]$.

Definition: 3 ([16]). Consider v_i be n distinct attributes with associated pairwise disjoint attributes values set $\mathbb{V}_i$. Let $B_{\mathbb{V}_i} \subseteq \mathbb{V}_i$, with $B_{\mathbb{V}_i} \neq \emptyset, \forall i = 1,2,3 \dots n$. A BFHSS $(€, ₮)$ is characterized by the mapping $€: ₮ \to \mathrm{P}(z)$, $\mathrm{P}(z)$ is the power set of BFS and its functional values of BFS, $€(a) = \{z, \mathcal{M}^+_{\Pi(a)}(z), \mathcal{M}^-_{\Pi(a)}(z) | \forall\, z \in \mathbb{Z},\ a \in ₮ \subseteq \mathbb{V}\}$

where $₮ = B_{\mathbb{V}_1} \times B_{\mathbb{V}_2} \times \dots . \times B_{\mathbb{V}_n}$, $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 \times \dots \mathbb{V}_n$

and $\mathcal{M}^+_H : ₮ \to [0,1]$, $\mathcal{M}^-_H : ₮ \to [-1,0]$ represent the positive and negative membership value respectively. Hence, the BFHSS is defined as:

$$(€, ₮) = \left\{\langle a, \left(z, \mathcal{M}^+_{\Pi(a)}(z), \mathcal{M}^-_{\Pi(a)}(z)\right)\rangle \middle| \forall\, z \in \mathbb{Z},\ a \in ₮ \subseteq \mathbb{V}\right\}$$

3. The Concept of Bipolar Intuitionistic Fuzzy Hyper Soft Set (BIFHSS)

Definition:3.1 Bipolar Intuitionistic Fuzzy Hyper Soft Set (BIFHSS)

Let $\mathfrak{U}$ be a universal set and $\mathrm{BIFP}(\mathfrak{U})$ denotes the Bipolar intuitionistic fuzzy power set of $\mathfrak{U}$. Let $\mathfrak{d}_1, \mathfrak{d}_2, \dots \mathfrak{d}_n$ be n-well defined attributes, and their respective attribute values $\mathfrak{D}_1, \mathfrak{D}_2 \dots . . \mathfrak{D}_n$, which are pairwise disjoint. Let $\mathcal{A}_i \subseteq \mathfrak{D}_i$ with $\mathcal{A}_i \neq \emptyset$, $\forall\, i = 1,2 \dots, n$.

A BIFHSS is defined as the pair

$$(\text{Ƕ}, \grave{\Lambda}) = \left\{\langle \propto, \left(\grave{u}, \mathcal{M}^+_{\text{Ƕ}(\propto)}(\grave{u}), \mathcal{M}^-_{\text{Ƕ}(\propto)}(\grave{u}), \aleph^+_{\text{Ƕ}(\propto)}(\grave{u}), \aleph^-_{\text{Ƕ}(\propto)}(\grave{u})\right)\rangle \middle| \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}\right\}$$

where $\text{Ƕ}: \grave{\Lambda} \to \mathrm{BIFP}(\mathfrak{U})$ such that its functional values for BIFS

$$\text{Ƕ}(\propto) = \{\grave{u}, \mathcal{M}^+_{\text{Ƕ}(\propto)}(\grave{u}), \mathcal{M}^-_{\text{Ƕ}(\propto)}(\grave{u}), \aleph^+_{\text{Ƕ}(\propto)}(\grave{u}), \aleph^-_{\text{Ƕ}(\propto)}(\grave{u}) | \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}\} \text{ and}$$

Here $\mathcal{M}^+_{\text{Ƕ}}, \aleph^+_{\text{Ƕ}} : \grave{\Lambda} \to [0,1]$ indicate the positive membership and positive non-membership value respectively and $\mathcal{M}^-_{\text{Ƕ}}, \aleph^-_{\text{Ƕ}} : \grave{\Lambda} \to [-1,0]$ indicate the negative membership and negative non-membership value respectively with $0 \leq \mathcal{M}^+_{\text{Ƕ}} + \aleph^+_{\text{Ƕ}} \leq 1, -1 \leq \mathcal{M}^-_{\text{Ƕ}} + \aleph^-_{\text{Ƕ}} \leq 0$ and For simplicity, we denote $\grave{\Lambda} = \mathcal{A}_1 \times \mathcal{A}_2 \dots . \mathcal{A}_n$, $\grave{\Delta} = \mathfrak{D}_1 \times \grave{\mathfrak{D}}_2 \dots . . \mathfrak{D}_n$. BIFHSS is denoted by $(\text{Ƕ}, \grave{\Lambda})$.

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Example:3.2

Consider $\mathfrak{U} = \{\grave{u}_1, \grave{u}_2, \grave{u}_3\}$ are the set of plastic recycling applications, where $\grave{u}_1$ = Household Waste Recycling, $\grave{u}_2$ = Industrial Plastic Waste Recycling, $\grave{u}_3$ =Mixed Plastic Recovery in Urban Plants and $\mathfrak{d} = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3\}$ represent the set of attributes, where, $\mathfrak{d}_1$ = Plastic Types (different plastic materials that can be recycled), $\mathfrak{d}_2$ =Recycling Methods (the processes used to recycle plastics), $\mathfrak{d}_3$ = Environmental Impact (the consequences of each recycling method in terms of carbon emissions).

Let the corresponding attribute values be $\mathfrak{D} = \{\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3\}$,

$\mathfrak{D}_1 = \{\grave{\gamma}_{11} = \text{PET}, \grave{\gamma}_{12} = \text{HDPE}, \grave{\gamma}_{13} = \text{PVC}, \grave{\gamma}_{14} = \text{LDPE}, \grave{\gamma}_{15} = \text{PP}, \grave{\gamma}_{16} = \text{PS}\}$,

$\mathfrak{D}_2 = \{\grave{\gamma}_{21} = \text{Mechanical Recycling}, \grave{\gamma}_{22} = \text{Chemical Recycling}, \grave{\gamma}_{23} = \text{Energy Recovery}\}$

$\mathfrak{D}_3 = \{\grave{\gamma}_{31} = \text{Low Carbon Foot print}, \grave{\gamma}_{32} = \text{Moderate Carbon Footprint},$
$\grave{\gamma}_{33} = \text{High Carbon Footprint}\}$

Suppose that $\mathcal{A}_1 = \{\grave{\gamma}_{11}, \grave{\gamma}_{12}\}$, $\mathcal{A}_2 = \{\grave{\gamma}_{21}, \grave{\gamma}_{22}\}$, $\mathcal{A}_3 = \{\grave{\gamma}_{31}\}$ then the BIFHSS is defined as the pair$(\text{Ħ}, \grave{\Lambda})$

$$(\text{Ħ}, \grave{\Lambda}) = \{\langle(\grave{\gamma}_{11}, \grave{\gamma}_{21}, \grave{\gamma}_{31}), ((\grave{u}_1, 0.3, -0.2, 0.6, -0.4), (\grave{u}_2, 0.3, -0.3, 0.4, -0.2), (\grave{u}_3, 0.4, -0.2, 0.5, -0.4))\rangle,$$
$$\langle(\grave{\gamma}_{11}, \grave{\gamma}_{22}, \grave{\gamma}_{31}), ((\grave{u}_1, 0.5, -0.6, 0.3, -0.2), (\grave{u}_2, 0.1, -0.2, 0.6, -0.3), (\grave{u}_3, 0.2, -0.2, 0.5, -0.4))\rangle,$$
$$\langle(\grave{\gamma}_{12}, \grave{\gamma}_{21}, \grave{\gamma}_{31}), ((\grave{u}_1, 0.7, -0.3, 0.2, -0.4), (\grave{u}_2, 0.6, -0.2, 0.4, -0.4), (\grave{u}_3, 0.3, -0.2, 0.6, -0.4))\rangle,$$
$$\langle(\grave{\gamma}_{12}, \grave{\gamma}_{22}, \grave{\gamma}_{31}), ((\grave{u}_1, 0.4, -0.1, 0.2, -0.5), (\grave{u}_2, 0.1, -0.2, 0.7, -0.3), (\grave{u}_3, 0.3, -0.2, 0.6, -0.4))\rangle\}$$

Definition: 3.3 A BIFHSS on $\mathfrak{U}$, the universal set is defined as null BIFHSS i.e., $0_{(\text{Ħ},\grave{\Lambda})}$ and is defined as $0_{(\text{Ħ},\grave{\Lambda})} = \{\langle\propto, (\grave{u}, 0, 0, 1, -1)\rangle | \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}\}$

if $\mathcal{M}^+_{\text{Ħ}(\propto)}(\grave{u}) = 0 = \mathcal{M}^-_{\text{Ħ}(\propto)}(\grave{u})$ and $\aleph^+_{\text{Ħ}(\propto)}(\grave{u}) = 1$, $\aleph^-_{\text{Ħ}(\propto)}(u) = -1 \;\; \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}$

Definition: **3.4** A BIFHSS on $\mathfrak{U}$, the universal set is said to be absolute BIFHSS i.e., $1_{(\text{Ħ},\grave{\Lambda})}$ and is defined as $1_{(\text{Ħ},\grave{\Lambda})} = \{\langle\propto, (\grave{u}, 1, -1, 0, 0)\rangle | \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}\}$

if $\mathcal{M}^+_{\text{Ħ}(\propto)}(\grave{u}) = 1$, $\mathcal{M}^-_{\text{Ħ}(\propto)}(\grave{u}) = -1$ and $\aleph^+_{\text{Ħ}(\propto)}(\grave{u}) = 0 = \aleph^-_{\text{Ħ}(\propto)}(\grave{u}) \;\; \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}$

Definition: 3.5 The complement $(\text{Ħ}, \grave{\Lambda})^c$ of BIFHSS $(\text{Ħ}, \grave{\Lambda})$ on $\mathfrak{U}$ is defined as

$$(\text{Ħ}, \grave{\Lambda})^c = \left\{\langle\propto, \left(\grave{u}, 1 - \mathcal{M}^+_{\text{Ħ}(\propto)}(\grave{u}), -1 - \mathcal{M}^-_{\text{Ħ}(\propto)}(\grave{u}), 1 - \aleph^+_{\text{Ħ}(\propto)}(\grave{u}), -1 - \aleph^-_{\text{Ħ}(\propto)}(\grave{u})\right)\rangle \,\middle|\, \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}\right\}$$

In other word,

$$(\text{Ħ}, \grave{\Lambda})^c = \left\{\langle\propto, \left(\grave{u}, \aleph^+_{\text{Ħ}(\propto)}(\grave{u}), \aleph^-_{\text{Ħ}(\propto)}(\grave{u}), \mathcal{M}^+_{\text{Ħ}(\propto)}(\grave{u}), \mathcal{M}^-_{\text{Ħ}(\propto)}(\grave{u})\right)\rangle \,\middle|\, \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\Delta}\right\}$$

Example:3.6 Consider the Example 3.2 and calculate $(\text{Ħ}, \grave{\Lambda})^c$

Suppose that $\grave{E}_1 = \{\grave{\gamma}_{11}\}$, $\grave{E}_2 = \{\grave{\gamma}_{21}, \grave{\gamma}_{22}\}$, $\grave{E}_3 = \{\grave{\gamma}_{31}\}$ then the BIFHSSs is defined as the pair

$$(\text{Ħ}, \grave{\Lambda}) = \{\langle(\grave{\gamma}_{11}, \grave{\gamma}_{21}, \grave{\gamma}_{31}), ((\grave{u}_1, 0.3, -0.4, 0.7, -0.6), (\grave{u}_2, 0.8, -0.3, 0.2, -0.7), (\grave{u}_3, 0.6, -1, 0.4, 0))\rangle,$$
$$\langle(\grave{\gamma}_{11}, \grave{\gamma}_{22}, \grave{\gamma}_{31}), ((\grave{u}_1, 0.2, -0.6, 0.8, -0.4), (\grave{u}_2, 0.1, -0.3, 0.9, -0.7), (\grave{u}_3, 0.5, 0, 0.5, -1))\rangle\}$$

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then $\left(\text{Ң},\grave{\Lambda}\right)^{c} =$
$\{\langle(\grave{\gamma}_{11},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.7,-0.6,0.3,-0.4),(\grave{u}_2,0.2,-0.7,0.8\ ,-0.3),(\grave{u}_3,0.4,0,0.6,-1))\rangle,$
$\langle(\grave{\gamma}_{12},\grave{\gamma}_{22},\grave{\gamma}_{31}),((\grave{u}_1,0.8,-0.4,0.2,-0.6),(\grave{u}_2,0.9,-0.7,0.1\ ,-0.3),(\grave{u}_3\ ,0.5,-1,0.5\ ,0))\rangle\}$

Theorem:3.7

If $\left(\text{Ң},\grave{\Lambda}\right)$ is a BIFHSS on $\mathfrak{U}$ then

i) $\left(\left(\text{Ң},\grave{\Lambda}\right)^{c}\right)^{c} = \left(\text{Ң},\grave{\Lambda}\right)$
ii) $0^{c}{}_{(\text{Ң},\grave{\Lambda})} = 1_{(\text{Ң},\grave{\Lambda})}$
iii) $1^{c}{}_{(\text{Ң},\grave{\Lambda})} = 0_{(\text{Ң},\grave{\Lambda})}$

Proof:

The proofs follow directly from the definitions.

Definition: 3.8 Let $\left(\text{Ң}_1,\grave{\Lambda}_1\right)$ and $\left(\text{Ң}_2,\grave{\Lambda}_2\right)$ be two BIFHSSs on $\mathfrak{U}$. $\left(\text{Ң}_1,\grave{\Lambda}_1\right)$ is a subset of $\left(\text{Ң}_2,\grave{\Lambda}_2\right)$ i.e., $\left(\text{Ң}_1,\grave{\Lambda}_1\right) \subseteq \left(\text{Ң}_2,\grave{\Lambda}_2\right)$ if

i) $\grave{\Lambda}_1 \subseteq \grave{\Lambda}_2$
ii) $\mathcal{M}^{+}_{\text{Ң}_1(\propto)}(\grave{u}) \le \mathcal{M}^{+}_{\text{Ң}_2(\propto)}(\grave{u})$, $\mathcal{M}^{-}_{\text{Ң}_1(\propto)}(\grave{u}) \ge \mathcal{M}^{-}_{\text{Ң}_2(\propto)}(\grave{u})$,
$\aleph^{+}_{\text{Ң}_1(\propto)}(\grave{u}) \le \aleph^{+}_{\text{Ң}_2(\propto)}(\grave{u})$, $\aleph^{-}_{\text{Ң}_1(\propto)}(\grave{u}) \ge \aleph^{-}_{\text{Ң}_2(\propto)}(\grave{u})$, $\forall\ \grave{u}\epsilon\mathfrak{U}, \propto \in \grave{\Lambda}$

Example:3.9 Consider the Example 3.2 of BIFHSSs on $\mathfrak{U}$. Suppose that $\grave{E}_1 = \{\grave{\gamma}_{11},\grave{\gamma}_{12}\}$, $\grave{E}_2 = \{\grave{\gamma}_{21},\grave{\gamma}_{22}\}$, $\grave{E}_3 = \{\grave{\gamma}_{31}\}$ and $\grave{F}_1 = \{\grave{\gamma}_{11},\grave{\gamma}_{12}\}$, $\grave{F}_2 = \{\grave{\gamma}_{21},\grave{\gamma}_{23}\}$, $\grave{F}_3 = \{\grave{\gamma}_{31}\}$ are the subsets of attribute values, then the subsets $\left(\text{Ң}_1,\grave{\Lambda}_1\right)$ and $\left(\text{Ң}_2,\grave{\Lambda}_2\right)$ of BIFHSSs over $\mathfrak{U}$ are defined as follows

$\left(\text{Ң}_1,\grave{\Lambda}_1\right) =$
$\{\langle(\grave{\gamma}_{11},\grave{\gamma}_{21},\grave{\gamma}_{31}),\left((\grave{u}_1,0.2,-0.4,0.3,0),(\grave{u}_2,0.3,-0.2,0.2,-0.1),(\grave{u}_3,0.3,-0.2,0.5,-0.1)\right)\rangle,$
$\langle(\grave{\gamma}_{11},\grave{\gamma}_{22},\grave{\gamma}_{31}),((\grave{u}_1,0.3,-0.1,0.3,-0.2),(\grave{u}_2,0.3,-0.2,0.3,-0.2),(\grave{u}_3,0.4,-0.1,0.3,-0.2))\rangle,$
$\langle(\grave{\gamma}_{12},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.1,-0.3,0.5\ ,-0.1),(\grave{u}_2,0.4,-0.2,0.1,-0.1),(\grave{u}_3,0.4,-0.1,0.4,-0.4))\rangle,$
$\langle(\grave{\gamma}_{12},\grave{\gamma}_{23},\grave{\gamma}_{31}),((\grave{u}_1,0.5,-0.2,0.2\ ,-0.1),(\grave{u}_2,0.3,-0.3,0.4,-0.1),(\grave{u}_3,0.4,-0.1,0.3,0))\rangle\}$

then $\left(\text{Ң}_2,\grave{\Lambda}_2\right) =$
$\{\langle(\grave{\gamma}_{11},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.4,-0.7,0.5,-0.2),(\grave{u}_2,0.5,-0.3,0.3,-0.2),(\grave{u}_3,0.4,-0.4,0.6,-0.4))\rangle,$
$\langle(\grave{\gamma}_{11},\grave{\gamma}_{22},\grave{\gamma}_{31}),((\grave{u}_1,0.6,-0.4,0.4,-0.3),(\grave{u}_2,0.6,-0.5,0.4,-0.5),(\grave{u}_3,0.7,-0.2,0.3,-0.4))\rangle,$
$\langle(\grave{\gamma}_{12},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.3,-0.7,0.6\ ,-0.3),(\grave{u}_2,0.6,-0.3,0.4,-0.2),(\grave{u}_3,0.5,-0.2,0.5,-0.7))\rangle,$
$\langle(\grave{\gamma}_{12},\grave{\gamma}_{23},\grave{\gamma}_{31}),((\grave{u}_1,0.7,-0.7,0.3\ ,-0.2),(\grave{u}_2,0.4,-0.4,0.5,-0.2),(\grave{u}_3,0.5,-0.2,0.5,-0.4))\rangle\}$

Definition: 3.10 Let $\left(\text{Ң}_1,\grave{\Lambda}_1\right)$ and$\left(\text{Ң}_2,\grave{\Lambda}_2\right)$ be the two BIFHSSs on $\mathfrak{U}$. $\left(\text{Ң}_1,\grave{\Lambda}_1\right)$ is equal to $\left(\text{Ң}_2,\grave{\Lambda}_2\right)$ i.e., $\left(\text{Ң}_1,\grave{\Lambda}_1\right) = \left(\text{Ң}_2,\grave{\Lambda}_2\right)$ if

i) $\grave{\Lambda}_1 = \grave{\Lambda}_2$
ii) $\mathcal{M}^{+}_{\text{Ң}_1(\propto)}(\grave{u}) = \mathcal{M}^{+}_{\text{Ң}_2(\propto)}(\grave{u})$, $\mathcal{M}^{-}_{\text{Ң}_1(\propto)}(\grave{u}) = \mathcal{M}^{-}_{\text{Ң}_2(\propto)}(\grave{u})$,
$\aleph^{+}_{\text{Ң}_1(\propto)}(\grave{u}) = \aleph^{+}_{\text{Ң}_2(\propto)}(\grave{u})$, $\aleph^{-}_{\text{Ң}_1(\propto)}(\grave{u}) = \aleph^{-}_{\text{Ң}_2(\propto)}(\grave{u})$, $\forall\ \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda}$

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Theorem:3.11

Let $(Ħ, \grave{\Lambda}_1)$, $(Ł, \grave{\Lambda}_2)$ and $(Ʉ, \grave{\Lambda}_3)$ be the BIFHSSs on $\mathfrak{U}$ then

i) $(Ħ, \grave{\Lambda}_1) \subseteq 1_{(Ħ,\grave{\Lambda})}$

ii) $0_{(Ħ,\grave{\Lambda})} \subseteq (Ħ, \grave{\Lambda}_1)$

iii) $(Ħ, \grave{\Lambda}_1) \subseteq (Ł, \grave{\Lambda}_2) \,\&\, (Ł, \grave{\Lambda}_2) \subseteq (Ʉ, \grave{\Lambda}_3) \Rightarrow (Ħ, \grave{\Lambda}_1) \subseteq (Ʉ, \grave{\Lambda}_3)$

Proof

i) Since $\mathcal{M}^{+}_{Ħ(\propto)}(\grave{u}) \leq \mathcal{M}^{+}_{1_{(Ħ)}(\propto)}(\grave{u}) = 1, \quad \mathcal{M}^{-}_{Ħ(\propto)}(\grave{u}) \geq \mathcal{M}^{-}_{1_{(Ħ)}(\propto)}(\grave{u}) = -1,$

$\aleph^{+}_{Ħ(\propto)}(\grave{u}) \leq \aleph^{+}_{1_{(Ħ)}(\propto)}(\grave{u}) = 0, \quad \aleph^{-}_{Ħ(\propto)}(\grave{u}) \geq \aleph^{-}_{1_{(Ħ)}(\propto)}(\grave{u}) = 0, \ \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda}$

It is obvious that $(Ħ, \grave{\Lambda}_1) \subseteq 1_{(Ħ,\grave{\Lambda})}$

ii) Since $0 = \mathcal{M}^{+}_{0_{(Ħ)}(\propto)}(\grave{u}) \leq \mathcal{M}^{+}_{(Ħ)(\propto)}(\grave{u}), \quad 0 = \mathcal{M}^{-}_{0_{(Ħ)}(\propto)}(\grave{u}) \geq \mathcal{M}^{-}_{Ħ(\propto)}(\grave{u}),$

$1 = \aleph^{+}_{0_{(Ħ)}(\propto)}(\grave{u}) \leq \aleph^{+}_{Ħ(\propto)}(\grave{u}), \quad -1 = \aleph^{-}_{0_{(Ħ)}(\propto)}(\grave{u}) \geq \aleph^{-}_{Ħ(\propto)}(\grave{u}), \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda}$

Therefore $0_{(Ħ,\grave{\Lambda})} \subseteq (Ħ, \grave{\Lambda}_1)$

iii) $(Ħ, \grave{\Lambda}_1) \subseteq (Ł, \grave{\Lambda}_2)$

By the definition of subset,

$\mathcal{M}^{+}_{Ħ(\propto)}(\grave{u}) \leq \mathcal{M}^{+}_{Ł(\propto)}(\grave{u}), \quad \mathcal{M}^{-}_{Ħ(\propto)}(\grave{u}) \geq \mathcal{M}^{-}_{Ł(\propto)}(\grave{u})$

$\aleph^{+}_{Ħ(\propto)}(\grave{u}) \leq \aleph^{+}_{Ł(\propto)}(\grave{u}), \quad \aleph^{-}_{Ħ(\propto)}(\grave{u}) \geq \aleph^{-}_{Ł(\propto)}(\grave{u}), \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda}$

Also $(Ł, \grave{\Lambda}_2) \subseteq (Ʉ, \grave{\Lambda}_3)$

By the definition of subset,

$\mathcal{M}^{+}_{Ł(\propto)}(\grave{u}) \leq \mathcal{M}^{+}_{Ʉ(\propto)}(u), \quad \mathcal{M}^{-}_{Ł(\propto)}(\grave{u}) \geq \mathcal{M}^{-}_{Ʉ(\propto)}(\grave{u})$

$\aleph^{+}_{Ł(\propto)}(\grave{u}) \leq \aleph^{+}_{Ʉ(\propto)}(\grave{u}), \quad \aleph^{-}_{Ł(\propto)}(\grave{u}) \geq \aleph^{-}_{Ʉ(\propto)}(\grave{u}), \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda}$

Therefore,

$\mathcal{M}^{+}_{Ħ(\propto)}(\grave{u}) \leq \mathcal{M}^{+}_{Ʉ(\propto)}(\grave{u}), \quad \mathcal{M}^{-}_{Ħ(\propto)}(\grave{u}) \geq \mathcal{M}^{-}_{Ʉ(\propto)}(\grave{u}),$

$\aleph^{+}_{Ħ(\propto)}(\grave{u}) \leq \aleph^{+}_{Ʉ(\propto)}(\grave{u}), \quad \aleph^{-}_{Ħ(\propto)}(\grave{u}) \geq \aleph^{-}_{Ʉ(\propto)}(\grave{u}), \quad \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda}$

Definition: 3.12. Let $(Ħ_1, \grave{\Lambda}_1)$ and $(Ħ_2, \grave{\Lambda}_2)$ be two BIFHSSs on $\mathfrak{U}$ and its union of two BIFHSSs on $\mathfrak{U}$ is defined as $(Ħ_1, \grave{\Lambda}_1) \cup (Ħ_2, \grave{\Lambda}_2) = (Ħ_3, \grave{\Lambda}_3)$ where $\grave{\Lambda}_3 = \grave{\Lambda}_1 \cup \grave{\Lambda}_2 \ \forall \propto \in \grave{\Lambda}_3$, $\grave{u} \epsilon \mathfrak{U}$ then

for positive membership function

$$\mathcal{M}^{+}_{Ħ_3(\propto)}(\grave{u}) = \begin{cases} \mathcal{M}^{+}_{Ħ_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \mathcal{M}^{+}_{Ħ_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(\mathcal{M}^{+}_{Ħ_1(\propto)}(\grave{u}), \mathcal{M}^{+}_{Ħ_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for negative membership function

$$\mathcal{M}^{-}_{Ħ_3(\propto)}(\grave{u}) = \begin{cases} \mathcal{M}^{-}_{Ħ_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \mathcal{M}^{-}_{Ħ_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(\mathcal{M}^{-}_{Ħ_1(\propto)}(\grave{u}), \mathcal{M}^{-}_{Ħ_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for positive non-membership function

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$$\aleph^{+}_{\text{Ħ}_3(\propto)}(\grave{u}) = \begin{cases} \aleph^{+}_{\text{Ħ}_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \aleph^{+}_{\text{Ħ}_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(\aleph^{+}_{\text{Ħ}_1(\propto)}(\grave{u}), \aleph^{+}_{\text{Ħ}_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for negative non-membership function

$$\aleph^{-}_{\text{Ħ}_3(\propto)}(\grave{u}) = \begin{cases} \aleph^{-}_{\text{Ħ}_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \aleph^{-}_{\text{Ħ}_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(\aleph^{-}_{\text{Ħ}_1(\propto)}(\grave{u}), \aleph^{-}_{\text{Ħ}_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

In other word

$$\left(\text{Ħ}_3, \grave{\Lambda}_3\right) = \left\{ \begin{array}{c} \propto, \left(\grave{u}, max\left(\mathcal{M}^{+}_{\text{Ħ}_1(\propto)}(\grave{u}), \mathcal{M}^{+}_{\text{Ħ}_2(\propto)}(\grave{u})\right), min\left(\mathcal{M}^{-}_{\text{Ħ}_1(\propto)}(\grave{u}), \mathcal{M}^{-}_{\text{Ħ}_2(\propto)}(\grave{u})\right),\right. \\ max\left(\aleph^{+}_{\text{Ħ}_1(\propto)}(\grave{u}), \aleph^{+}_{\text{Ħ}_2(\propto)}(\grave{u})\right), min\left(\aleph^{-}_{\text{Ħ}_1(\propto)}(\grave{u}), \aleph^{-}_{\text{Ħ}_2(\propto)}(\grave{u})\right) \middle| \forall\, \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\triangle} \end{array} \right\}$$

Theorem:3.13 Let $\left(S, \grave{\Lambda}_1\right)$, $\left(\text{Đ}, \grave{\Lambda}_2\right)$ and $\left(\text{Ł}, \grave{\Lambda}_3\right)$ be the BIFHSSs on $\mathfrak{U}$ then

i) $\left(S, \grave{\Lambda}_1\right) \cup \left(S, \grave{\Lambda}_1\right) = \left(S, \grave{\Lambda}_1\right)$

ii) $0_{(\text{Ħ},\grave{\Lambda})} \cup \left(S, \grave{\Lambda}_1\right) = \left(S, \grave{\Lambda}_1\right)$

iii) $\left(S, \grave{\Lambda}_1\right) \cup 1_{(\text{Ħ},\grave{\Lambda})} = 1_{(\text{Ħ},\grave{\Lambda})}$

iv) $\left(S, \grave{\Lambda}_1\right) \cup \left(\text{Đ}, \grave{\Lambda}_2\right) = \left(\text{Đ}, \grave{\Lambda}_2\right) \cup \left(S, \grave{\Lambda}_1\right)$

v) $\left(\left(S, \grave{\Lambda}_1\right) \cup \left(\text{Đ}, \grave{\Lambda}_2\right)\right) \cup \left(\text{Ł}, \grave{\Lambda}_3\right) = \left(S, \grave{\Lambda}_1\right) \cup \left(\left(\text{Đ}, \grave{\Lambda}_2\right) \cup \left(\text{Ł}, \grave{\Lambda}_3\right)\right)$

Proof

The proofs are straightforward.

Definition: 3.14 Let $\left(\text{Ħ}_1, \grave{\Lambda}_1\right)$ and $\left(\text{Ħ}_2, \grave{\Lambda}_2\right)$ be two BIFHSSs and its intersection $\left(\text{Ħ}_3, \grave{\Lambda}_3\right)$ on $\mathfrak{U}$ is defined as

$\left(\text{Ħ}_3, \grave{\Lambda}_3\right) = \left(\text{Ħ}_1, \grave{\Lambda}_1\right) \cap \left(\text{Ħ}_2, \grave{\Lambda}_2\right)$, where $\grave{\Lambda}_3 = \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \ \forall \propto \in \grave{\Lambda}_3, \grave{u} \in \mathfrak{U}$ then

for positive membership function

$$\mathcal{M}^{+}_{\text{Ħ}_3(\propto)}(\grave{u}) = \begin{cases} \mathcal{M}^{+}_{\text{Ħ}_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \mathcal{M}^{+}_{\text{Ħ}_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(\mathcal{M}^{+}_{\text{Ħ}_1(\propto)}(\grave{u}), \mathcal{M}^{+}_{\text{Ħ}_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for negative membership function

$$\mathcal{M}^{-}_{\text{Ħ}_3(\propto)}(\grave{u}) = \begin{cases} \mathcal{M}^{-}_{\text{Ħ}_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \mathcal{M}^{-}_{\text{Ħ}_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(\mathcal{M}^{-}_{\text{Ħ}_1(\propto)}(\grave{u}), \mathcal{M}^{-}_{\text{Ħ}_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for positive non-membership function

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$$\aleph^{+}_{\text{Ӊ}_3(\propto)}(\grave{u}) = \begin{cases} \aleph^{+}_{\text{Ӊ}_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \aleph^{+}_{\text{Ӊ}_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(\aleph^{+}_{\text{Ӊ}_1(\propto)}(\grave{u}), \aleph^{+}_{\text{Ӊ}_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for negative non-membership function

$$\aleph^{-}_{\text{Ӊ}_3(\propto)}(\grave{u}) = \begin{cases} \aleph^{-}_{\text{Ӊ}_1(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \aleph^{-}_{\text{Ӊ}_2(\propto)}(\grave{u}) & if \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(\aleph^{-}_{\text{Ӊ}_1(\propto)}(\grave{u}), \aleph^{-}_{\text{Ӊ}_2(\propto)}(\grave{u})\right) & if \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

In other word

$$\left(\text{Ӊ}_3, \grave{\Lambda}_3\right) = \left\{ \begin{array}{c} \propto, (\grave{u}, min\left(\mathcal{M}^{+}_{\text{Ӊ}_1(\propto)}(\grave{u}), \mathcal{M}^{+}_{\text{Ӊ}_2(\propto)}(\grave{u})\right), max\left(\mathcal{M}^{-}_{\text{Ӊ}_1(\propto)}(\grave{u}), \mathcal{M}^{-}_{\text{Ӊ}_2(\propto)}(\grave{u})\right), \\ min\left(\aleph^{+}_{\text{Ӊ}_1(\propto)}(\grave{u}), \aleph^{+}_{\text{Ӊ}_2(\propto)}(\grave{u})\right), max\left(\aleph^{-}_{\text{Ӊ}_1(\propto)}(\grave{u}), \aleph^{-}_{\text{Ӊ}_2(\propto)}(\grave{u})\right) \middle| \forall \ \grave{u} \in \mathfrak{U}, \propto \in \grave{\Lambda} \subseteq \grave{\triangle} \end{array} \right\}$$

Theorem:3.15

Let $\left(S, \grave{\Lambda}_1\right)$, $\left(\text{Đ}, \grave{\Lambda}_2\right)$ and $\left(\text{Ł}, \grave{\Lambda}_3\right)$ be the BIFHSSs on $\mathfrak{U}$ then

i) $\left(S, \grave{\Lambda}_1\right) \cap \left(S, \grave{\Lambda}_1\right) = \left(S, \grave{\Lambda}_1\right)$

ii) $0_{(\text{Ӊ},\grave{\Lambda})} \cap \left(S, \grave{\Lambda}_1\right) = 0_{(\text{Ӊ},\grave{\Lambda})}$

iii) $\left(S, \grave{\Lambda}_1\right) \cap 1_{(\text{Ӊ},\grave{\Lambda})} = \left(S, \grave{\Lambda}_1\right)$

iv) $\left(S, \grave{\Lambda}_1\right) \cap \left(\text{Đ}, \grave{\Lambda}_2\right) = \left(\text{Đ}, \grave{\Lambda}_2\right) \cap \left(S, \grave{\Lambda}_1\right)$

v) $\left(\left(S, \grave{\Lambda}_1\right) \cap \left(\text{Đ}, \grave{\Lambda}_2\right)\right) \cap \left(\text{Ł}, \grave{\Lambda}_3\right) = \left(S, \grave{\Lambda}_1\right) \cap \left(\left(\text{Đ}, \grave{\Lambda}_2\right) \cap \left(\text{Ł}, \grave{\Lambda}_3\right)\right)$

Proof

The proofs are straightforward

Definition: 3.16

Let $\left(S, \grave{\Lambda}_1\right)$ and $\left(\text{Đ}, \grave{\Lambda}_2\right)$ be the two BIFHSSs and its difference $\left(\text{Ł}, \grave{\Lambda}_3\right)$ on $\mathfrak{U}$ is defined as $\left(\text{Ł}, \grave{\Lambda}_3\right) = \left(S, \grave{\Lambda}_1\right) \backslash \left(\text{Đ}, \grave{\Lambda}_2\right)$, where $\left(S, \grave{\Lambda}_1\right) \cap \left(\text{Đ}, \grave{\Lambda}_2\right)^c = \left(\text{Ł}, \grave{\Lambda}_3\right)$.

Example: 3.17

Consider Example 3.2 of BIFHSS. Suppose that $\left(\text{Ӊ}_1, \grave{\Lambda}_1\right)$ and $\left(\text{Ӊ}_2, \grave{\Lambda}_2\right)$ are two BIFHSSs and its corresponding union, intersection and difference on $\mathfrak{U}$ are defined as follows Suppose that $\grave{E}_1 = \{\grave{\gamma}_{13}, \grave{\gamma}_{14}\}$, $\grave{E}_2 = \{\grave{\gamma}_{21}\}$, $\grave{E}_3 = \{\grave{\gamma}_{31}, \grave{\gamma}_{33}\}$ and $\grave{F}_1 = \{\grave{\gamma}_{13}\}$, $\grave{F}_2 = \{\grave{\gamma}_{21}, \grave{\gamma}_{23}\}$, $\grave{F}_3 = \{\grave{\gamma}_{31}, \grave{\gamma}_{33}\}$ are the subsets of attribute values, then the subsets $\left(\text{Ӊ}_1, \grave{\Lambda}_1\right)$ and $\left(\text{Ӊ}_2, \grave{\Lambda}_2\right)$ of BIFHSSs on $\mathfrak{U}$ are defined as follows

$\left(\text{Ӊ}_1, \grave{\Lambda}_1\right) =$

$\{\langle(\grave{\gamma}_{13}, \grave{\gamma}_{21}, \grave{\gamma}_{31}), ((\grave{u}_1, 0.5, -0.3, 0.2, -0.4), (\grave{u}_2, 0.3, -0.2, 0.2, -0.1), (\grave{u}_3, 0.3, -0.2, 0.5, -0.1))\rangle,$

$\langle(\grave{\gamma}_{13}, \grave{\gamma}_{21}, \grave{\gamma}_{33}), ((\grave{u}_1, 0.4, -0.1, 0.3, -0.1), (\grave{u}_2, 0.3, -0.1, 0.3, -0.2), (\grave{u}_3, 0.4, -0.1, 0.6, -0.2))\rangle,$

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$\langle(\grave{\gamma}_{14},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.2,-0.3,0.4,-0.1),(\grave{u}_2,0.4,-0.2,0.5,-0.1),(\grave{u}_3,0.4,-0.1,0.4,-0.4))\rangle,$
$\langle(\grave{\gamma}_{14},\grave{\gamma}_{21},\grave{\gamma}_{33}),((\grave{u}_1,0.6,-0.4,0.4,-0.1),(\grave{u}_2,0.3,-0.3,0.4,-0.1),(\grave{u}_3,0.4,-0.1,0.3,0))\rangle\}$
and $(\text{Ħ}_2,\grave{\Lambda}_2)=$
$\{\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.3,-0.4,0.2,-0.2),(\grave{u}_2,0.5,-0.3,0.4,-0.2),(\grave{u}_3,0.5,-0.4,0.3,-0.4))\rangle,$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{33}),((\grave{u}_1,0.6,-0.4,0.1,-0.1),(\grave{u}_2,0.4,-0.2,0.4,-0.5),(\grave{u}_3,0.7,-0.2,0.3,-0.4))\rangle,$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{23},\grave{\gamma}_{31}),((\grave{u}_1,0.1,-0.7,0.5,-0.3),(\grave{u}_2,0.6,-0.3,0.4,-0.2),(\grave{u}_3,0.5,-0.2,0.5,-0.7))\rangle,$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{23},\grave{\gamma}_{33}),((\grave{u}_1,0.4,-0.1,0.2,-0.5),(\grave{u}_2,0.4,-0.4,0.5,-0.2),(\grave{u}_3,0.5,-0.2,0.5,-0.4))\rangle\}$
and its union of two BIFHSSs is denoted by
$(\text{Ħ}_1,\grave{\Lambda}_1)\cup(\text{Ħ}_2,\grave{\Lambda}_2)=$
$\{\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.5,-0.4,0.2,-0.4),(\grave{u}_2,0.5,-0.3,0.4,-0.2),(\grave{u}_3,0.5,-0.4,0.5,-0.4))\rangle,$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{33}),((\grave{u}_1,0.6,-0.4,0.3,-0.1),(\grave{u}_2,0.4,-0.2,0.4,-0.5),(\grave{u}_3,0.7,-0.2,0.6,-0.4))\rangle,$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{23},\grave{\gamma}_{31}),((\grave{u}_1,0.1,-0.7,0.5,-0.3),(\grave{u}_2,0.6,-0.3,0.4,-0.2),(\grave{u}_3,0.5,-0.2,0.5,-0.7))\rangle,$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{23},\grave{\gamma}_{33}),((\grave{u}_1,0.4,-0.1,0.2,-0.5),(\grave{u}_2,0.4,-0.4,0.5,-0.2),(\grave{u}_3,0.5,-0.2,0.5,-0.4))\rangle,$
$\langle(\grave{\gamma}_{14},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.2,-0.3,0.4,-0.1),(\grave{u}_2,0.4,-0.2,0.5,-0.1),(\grave{u}_3,0.4,-0.1,0.4,-0.4))\rangle,$
$\langle(\grave{\gamma}_{14},\grave{\gamma}_{21},\grave{\gamma}_{33}),((\grave{u}_1,0.6,-0.4,0.4,-0.1),(\grave{u}_2,0.3,-0.3,0.4,-0.1),(\grave{u}_3,0.4,-0.1,0.3,0))\rangle\}$
$(\text{Ħ}_1,\grave{\Lambda}_1)\cap(\text{Ħ}_2,\grave{\Lambda}_2)=$
$\{\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.3,-0.3,0.2,-0.2),(\grave{u}_2,0.3,-0.2,0.2,-0.1),(\grave{u}_3,0.3,-0.2,0.3,-0.1))\rangle$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{33}),((\grave{u}_1,0.4,-0.1,0.1,-0.1),(\grave{u}_2,0.3,-0.1,0.3,-0.2),(\grave{u}_3,0.4,-0.1,0.3,-0.2))\rangle\}$
$(\text{Ħ}_1,\grave{\Lambda}_1)\backslash(\text{Ħ}_2,\grave{\Lambda}_2)=$
$\{\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.5,-0.3,0.2,-0.4),(\grave{u}_2,0.3,-0.2,0.2,-0.1),(\grave{u}_3,0.3,-0.2,0.5,-0.1))\rangle$
$\langle(\grave{\gamma}_{13},\grave{\gamma}_{21},\grave{\gamma}_{33}),((\grave{u}_1,0.4,-0.1,0.3,-0.1),(\grave{u}_2,0.3,-0.1,0.3,-0.2),(\grave{u}_3,0.3,-0.1,0.6,-0.2))\rangle\}$

Theorem:3.18

Let $(S,\grave{\Lambda}_1)$ and $(\text{Đ},\grave{\Lambda}_2)$ be the two BIFHSSs on $\mathfrak{U}$ .Then the following De-Morgan laws holds

i) $\left((S,\grave{\Lambda}_1)\cup(\text{Đ},\grave{\Lambda}_2)\right)^c=(S,\grave{\Lambda}_1)^c\cap(\text{Đ},\grave{\Lambda}_2)^c$

ii) $\left((S,\grave{\Lambda}_1)\cap(\text{Đ},\grave{\Lambda}_2)\right)^c=(S,\grave{\Lambda}_1)^c\cup(\text{Đ},\grave{\Lambda}_2)^c$

Proof

Consider $(S,\grave{\Lambda}_1)\cup(\text{Đ},\grave{\Lambda}_2)=(K,\grave{\Lambda}_3)$ where $\grave{\Lambda}_3=\grave{\Lambda}_1\cup\grave{\Lambda}_2$ & $\forall\propto\in\grave{\Lambda}_3$

$$\left((S,\grave{\Lambda}_1)\cup(\text{Đ},\grave{\Lambda}_2)\right)^c=(K,\grave{\Lambda}_3)^c=(K^c,\grave{\Lambda}_3)$$

for positive membership function

$$\mathcal{M}^+_{K(\propto)}(\grave{u})=\begin{cases}\mathcal{M}^+_{S(\propto)}(\grave{u}) & if\ \ \propto\in\grave{\Lambda}_1-\grave{\Lambda}_2\\ \mathcal{M}^+_{\text{Đ}(\propto)}(\grave{u}) & if\ \ \propto\in\grave{\Lambda}_2-\grave{\Lambda}_1\\ max\left(\mathcal{M}^+_{S(\propto)}(\grave{u}),\mathcal{M}^+_{\text{Đ}(\propto)}(\grave{u})\right) & if\ \ \propto\in\grave{\Lambda}_1\cap\grave{\Lambda}_2\end{cases}$$

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$$\mathcal{M}^{+}_{K^c(\propto)}(\grave{u}) = \begin{cases} 1-\mathcal{M}^{+}_{S(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_1-\grave{\Lambda}_2 \\ 1-\mathcal{M}^{+}_{Đ(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_2-\grave{\Lambda}_1 \\ 1-max\left(\mathcal{M}^{+}_{S(\propto)}(\grave{u}),\mathcal{M}^{+}_{Đ(\propto)}(\grave{u})\right) & if \ \propto\in \grave{\Lambda}_1\cap\grave{\Lambda}_2 \end{cases}$$

for negative membership function

$$\mathcal{M}^{-}_{K(\propto)}(\grave{u}) = \begin{cases} \mathcal{M}^{-}_{S(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_1-\grave{\Lambda}_2 \\ \mathcal{M}^{-}_{Đ(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_2-\grave{\Lambda}_1 \\ min\left(\mathcal{M}^{-}_{S(\propto)}(\grave{u}),\mathcal{M}^{-}_{Đ(\propto)}(\grave{u})\right) & if \ \propto\in \grave{\Lambda}_1\cap\grave{\Lambda}_2 \end{cases}$$

$$\mathcal{M}^{-}_{K^c(\propto)}(\grave{u}) = \begin{cases} -1-\mathcal{M}^{-}_{S(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_1-\grave{\Lambda}_2 \\ -1-\mathcal{M}^{-}_{Đ(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_2-\grave{\Lambda}_1 \\ -1-min\left(\mathcal{M}^{-}_{S(\propto)}(\grave{u}),\mathcal{M}^{-}_{Đ(\propto)}(\grave{u})\right) & if \ \propto\in \grave{\Lambda}_1\cap\grave{\Lambda}_2 \end{cases}$$

for positive non-membership function

$$\aleph^{+}_{K(\propto)}(\grave{u}) = \begin{cases} \aleph^{+}_{S(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_1-\grave{\Lambda}_2 \\ \aleph^{+}_{Đ(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_2-\grave{\Lambda}_1 \\ max\left(\aleph^{+}_{S(\propto)}(\grave{u}),\aleph^{+}_{Đ(\propto)}(\grave{u})\right) & if \ \propto\in \grave{\Lambda}_1\cap\grave{\Lambda}_2 \end{cases}$$

$$\aleph^{+}_{K^c(\propto)}(\grave{u}) = \begin{cases} 1-\aleph^{+}_{S(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_1-\grave{\Lambda}_2 \\ 1-\aleph^{+}_{Đ(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_2-\grave{\Lambda}_1 \\ 1-max\left(\aleph^{+}_{S(\propto)}(\grave{u}),\aleph^{+}_{Đ(\propto)}(\grave{u})\right) & if \ \propto\in \grave{\Lambda}_1\cap\grave{\Lambda}_2 \end{cases}$$

for negative non-membership function

$$\aleph^{-}_{K(\propto)}(\grave{u}) = \begin{cases} \aleph^{-}_{S(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_1-\grave{\Lambda}_2 \\ \aleph^{-}_{Đ(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_2-\grave{\Lambda}_1 \\ min\left(\aleph^{-}_{S(\propto)}(\grave{u}),\aleph^{-}_{Đ(\propto)}(\grave{u})\right) & if \ \propto\in \grave{\Lambda}_1\cap\grave{\Lambda}_2 \end{cases}$$

$$\aleph^{-}_{K^c(\propto)}(\grave{u}) = \begin{cases} -1-\aleph^{-}_{S(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_1-\grave{\Lambda}_2 \\ -1-\aleph^{-}_{Đ(\propto)}(\grave{u}) & if \ \propto\in \grave{\Lambda}_2-\grave{\Lambda}_1 \\ -1-min\left(\aleph^{-}_{S(\propto)}(\grave{u}),\aleph^{-}_{Đ(\propto)}(\grave{u})\right) & if \ \propto\in \grave{\Lambda}_1\cap\grave{\Lambda}_2 \end{cases}$$

Since $\left(S,\grave{\Lambda}_1\right)^c = \left(S^c,\grave{\Lambda}_1\right)$ & $\left(Đ,\grave{\Lambda}_2\right)^c = \left(Đ^c,\grave{\Lambda}_2\right)$
$\left(\left(S,\grave{\Lambda}_1\right)\cup\left(Đ,\grave{\Lambda}_2\right)\right)^c = \left(S^c,\grave{\Lambda}_1\right)\cap\left(Đ^c,\grave{\Lambda}_2\right) = \left(L,\grave{\Lambda}_3\right)$
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$$\mathcal{M}^{+}_{L(\propto)}(\grave{u}) = \begin{cases} \mathcal{M}^{+}_{S^{c}(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \mathcal{M}^{+}_{Đ^{c}(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(\mathcal{M}^{+}_{S^{c}(\propto)}(\grave{u}), \mathcal{M}^{+}_{Đ^{c}(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

$$= \begin{cases} 1 - \mathcal{M}^{+}_{S(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ 1 - \mathcal{M}^{+}_{Đ(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(1 - \mathcal{M}^{+}_{S(\propto)}(\grave{u}), 1 - \mathcal{M}^{+}_{Đ(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

$$= \begin{cases} 1 - \mathcal{M}^{+}_{S(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ 1 - \mathcal{M}^{+}_{Đ(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ 1 - max\left(\mathcal{M}^{+}_{S(\propto)}(\grave{u}), \mathcal{M}^{+}_{Đ(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for negative membership function

$$\mathcal{M}^{-}_{L(\propto)}(\grave{u}) = \begin{cases} \mathcal{M}^{-}_{S^{c}(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \mathcal{M}^{-}_{Đ^{c}(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(\mathcal{M}^{-}_{S^{c}(\propto)}(\grave{u}), \mathcal{M}^{-}_{Đ^{c}(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

$$= \begin{cases} -1 - \mathcal{M}^{-}_{S(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ -1 - \mathcal{M}^{-}_{Đ(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(-1 - \mathcal{M}^{-}_{S(\propto)}(\grave{u}), -1 - \mathcal{M}^{-}_{Đ(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

$$= \begin{cases} -1 - \mathcal{M}^{-}_{S(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ -1 - \mathcal{M}^{-}_{Đ(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ -1 - min\left(\mathcal{M}^{-}_{S(\propto)}(\grave{u}), \mathcal{M}^{-}_{Đ(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for positive non-membership function

$$\aleph^{+}_{L(\propto)}(\grave{u}) = \begin{cases} \aleph^{+}_{S^{c}(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \aleph^{+}_{Đ^{c}(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(\aleph^{+}_{S^{c}(\propto)}(\grave{u}), \aleph^{+}_{Đ^{c}(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

$$= \begin{cases} 1 - \aleph^{+}_{S(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ 1 - \aleph^{+}_{Đ(\propto)}(\grave{u}) & if \quad \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ min\left(1 - \aleph^{+}_{S(\propto)}(\grave{u}), 1 - \aleph^{+}_{Đ(\propto)}(\grave{u})\right) & if \quad \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

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$$= \begin{cases} 1 - \aleph^{+}_{S(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ 1 - \aleph^{+}_{Đ(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ 1 - max\left(\aleph^{+}_{S(\propto)}(\grave{u}), \aleph^{+}_{Đ(\propto)}(\grave{u})\right) & if \ \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

for negative non-membership function

$$\aleph^{-}_{L(\propto)}(\grave{u}) = \begin{cases} \aleph^{-}_{S^c(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ \aleph^{-}_{Đ^c(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(\aleph^{-}_{S^c(\propto)}(\grave{u}), \aleph^{-}_{Đ^c(\propto)}(\grave{u})\right) & if \ \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

$$= \begin{cases} -1 - \aleph^{-}_{S(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ -1 - \aleph^{-}_{Đ(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ max\left(-1 - \aleph^{-}_{S(\propto)}(\grave{u}), -1 - \aleph^{-}_{Đ(\propto)}(\grave{u})\right) & if \ \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

$$= \begin{cases} -1 - \aleph^{-}_{S(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_1 - \grave{\Lambda}_2 \\ -1 - \aleph^{-}_{Đ(\propto)}(\grave{u}) & if \ \ \propto \in \grave{\Lambda}_2 - \grave{\Lambda}_1 \\ -1 - min\left(\aleph^{-}_{S(\propto)}(\grave{u}), \aleph^{-}_{Đ(\propto)}(\grave{u})\right) & if \ \ \propto \in \grave{\Lambda}_1 \cap \grave{\Lambda}_2 \end{cases}$$

Since $\mathcal{M}^{+}_{\left((S,\grave{\Lambda}_1)\cup(Đ,\grave{\Lambda}_2)\right)^c(\propto)}(\grave{u}) = \mathcal{M}^{+}_{(S^c,\grave{\Lambda}_1)\cap(Đ^c,\grave{\Lambda}_2)(\propto)}(\grave{u})$

$\mathcal{M}^{-}_{\left((S,\grave{\Lambda}_1)\cup(Đ,\grave{\Lambda}_2)\right)^c(\propto)}(\grave{u}) = \mathcal{M}^{-}_{(S^c,\grave{\Lambda}_1)\cap(Đ^c,\grave{\Lambda}_2)(\propto)}(\grave{u})$

$\aleph^{+}_{\left((S,\grave{\Lambda}_1)\cup(Đ,\grave{\Lambda}_2)\right)^c(\propto)}(\grave{u}) = \aleph^{+}_{(S^c,\grave{\Lambda}_1)\cap(Đ^c,\grave{\Lambda}_2)(\propto)}(\grave{u})$

$\aleph^{-}_{\left((S,\grave{\Lambda}_1)\cup(Đ,\grave{\Lambda}_2)\right)^c(\propto)}(\grave{u}) = \aleph^{-}_{(S^c,\grave{\Lambda}_1)\cap(Đ^c,\grave{\Lambda}_2)(\propto)}(\grave{u})$

The operators $(K^c, \grave{\Lambda}_3)$ and $(L, \grave{\Lambda}_3)$ are equivalent.

Therefore $\left((S,\grave{\Lambda}_1) \cup (Đ,\grave{\Lambda}_2)\right)^c = (S,\grave{\Lambda}_1)^c \cap (Đ,\grave{\Lambda}_2)^c$

The second proof, likewise, follows easily.

Definition: 3.19

Let $(S,\grave{\Lambda}_1)$ and $(M,\grave{\Lambda}_1)$ be two BIFHSSs and its 'AND' operator i.e., $(S,\grave{\Lambda}_1)\wedge(M,\grave{\Lambda}_2) = (N,\grave{\Lambda}_1 \times \grave{\Lambda}_2)$ on $\mathfrak{U}$ is given by $(N,\grave{\Lambda}_1 \times \grave{\Lambda}_2)$

$$= \left\{\langle(\propto_1,\propto_2), \left(\grave{u}, \mathcal{M}^{+}_{N(\propto_1,\propto_2)}(\grave{u}), \mathcal{M}^{-}_{N(\propto_1,\propto_2)}(\grave{u}), \aleph^{+}_{N(\propto_1,\propto_2)}(\grave{u}), \aleph^{-}_{N(\propto_1,\propto_2)}(\grave{u})\right)\rangle \,\middle|\, \forall\, \grave{u} \in \mathfrak{U}, \propto_1, \propto_2 \in \grave{\Lambda} \subseteq \grave{\Delta}\right\}$$

where $\mathcal{M}^{+}_{N(\propto_1,\propto_2)}(\grave{u}) = min\left(\mathcal{M}^{+}_{S(\propto_1)}(\grave{u}), \mathcal{M}^{+}_{M(\propto_2)}(\grave{u})\right)$

$\mathcal{M}^{-}_{N(\propto_1,\propto_2)}(\grave{u}) = max\left(\mathcal{M}^{-}_{S(\propto_1)}(\grave{u}), \mathcal{M}^{-}_{M(\propto_2)}(\grave{u})\right)$

$\aleph^{+}_{N(\propto_1,\propto_2)}(\grave{u}) = min\left(\aleph^{+}_{S(\propto_1)}(\grave{u}), \ \aleph^{+}_{M(\propto_2)}(\grave{u})\right)$

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$$\aleph^{-}_{N(\propto_1,\propto_2)}(\grave{u}) = max\left(\aleph^{-}_{S(\propto_1)}(\grave{u}),\ \aleph^{-}_{M(\propto_2)}(\grave{u})\right)$$

Definition: 3.20

Let $(S,\grave{\Lambda}_1)$ and $(M,\grave{\Lambda}_1)$ be two BIFHSSs on $\mathfrak{U}$ and its 'OR' operator i.e., $(S,\grave{\Lambda}_1)\vee(M,\grave{\Lambda}_2) = (N,\grave{\Lambda}_1\times\grave{\Lambda}_2)$ on $\mathfrak{U}$ is given by $(N,\grave{\Lambda}_1\times\grave{\Lambda}_2)$

$$= \left\{\langle(\propto_1,\propto_2)\left(\grave{u},\mathcal{M}^{+}_{N(\propto_1,\propto_2)}(\grave{u}),\mathcal{M}^{-}_{N(\propto_1,\propto_2)}(\grave{u}),\aleph^{+}_{N(\propto_1,\propto_2)}(\grave{u}),\aleph^{-}_{N(\propto_1,\propto_2)}(\grave{u})\right)\rangle \,\middle|\, \forall\ \grave{u}\in\mathfrak{U},\propto_1,\propto_2\in\grave{\Lambda} \subseteq\grave{\Delta}\right\}$$

where $\mathcal{M}^{+}_{N(\propto_1,\propto_2)}(\grave{u}) = max\left(\mathcal{M}^{+}_{S(\propto_1)}(\grave{u}),\mathcal{M}^{+}_{M(\propto_2)}(\grave{u})\right)$

$$\mathcal{M}^{-}_{N(\propto_1,\propto_2)}(\grave{u}) = min\left(\mathcal{M}^{-}_{S(\propto_1)}(\grave{u}),\mathcal{M}^{-}_{M(\propto_2)}(\grave{u})\right)$$

$$\aleph^{+}_{N(\propto_1,\propto_2)}(\grave{u}) = max\left(\aleph^{+}_{S(\propto_1)}(\grave{u}),\ \aleph^{+}_{M(\propto_2)}(\grave{u})\right)$$

$$\aleph^{-}_{N(\propto_1,\propto_2)}(\grave{u}) = min\left(\aleph^{-}_{S(\propto_1)}(\grave{u}),\ \aleph^{-}_{M(\propto_2)}(\grave{u})\right)$$

Example:3.21

Consider Example 3.2 of BIFHSS. Let $(\mathrm{H}_1,\grave{\Lambda}_1)$ and $(\mathrm{H}_2,\grave{\Lambda}_2)$ be two BIFHSSs on $\mathfrak{U}$.

Suppose that $\grave{E}_1 = \{\grave{\gamma}_{11}\}, \grave{E}_2 = \{\grave{\gamma}_{21},\grave{\gamma}_{22}\}, \grave{E}_3 = \{\grave{\gamma}_{31}\}$ and that $\grave{F}_1 = \{\grave{\gamma}_{12}\}, \grave{F}_2 = \{\grave{\gamma}_{21},\grave{\gamma}_{23}\}$, $\grave{F}_3 = \{\grave{\gamma}_{31}\}$ are the subsets of attribute values, Then, BIFHSSs $(\mathrm{H}_1,\grave{\Lambda}_1)$ and $(\mathrm{H}_2,\grave{\Lambda}_2)$ on $\mathfrak{U}$ are defined as follows

$(\mathrm{H}_1,\grave{\Lambda}_1) =$
$\{\langle(\grave{\gamma}_{11},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.5,-0.3,0.2,-0.4),(\grave{u}_2,0.4,-0.1,0.3,-0.1),(\grave{u}_3,0.3,-0.2,0.5,-0.1))\rangle$
$\langle(\grave{\gamma}_{11},\grave{\gamma}_{22},\grave{\gamma}_{31}),((\grave{u}_1,0.2,-0.3,0.4\ ,-0.1),(\grave{u}_2,0.6,-0.4,0.4,-0.1),(\grave{u}_3,0.4,-0.1,0.4,-0.4))\rangle\}$
and $(\mathrm{H}_2,\grave{\Lambda}_2) =$
$\{\langle(\grave{\gamma}_{12},\grave{\gamma}_{21},\grave{\gamma}_{31}),((\grave{u}_1,0.3,-0.4,0.2,-0.2),(\grave{u}_2,0.6,-0.4,0.1,-0.1),(\grave{u}_3,0.5,-0.2,0.5,-0.7))\rangle$
$\langle(\grave{\gamma}_{12},\grave{\gamma}_{23},\grave{\gamma}_{31}),((\grave{u}_1,0.1,-0.7,0.5\ ,-0.3),(\grave{u}_2,0.4,-0.1,0.2\ ,-0.5),(\grave{u}_3,0.7,-0.2,0.3,-0.4))\rangle\}$

Let us assume that $\grave{a}_1 = (\grave{\gamma}_{11},\grave{\gamma}_{21},\grave{\gamma}_{31})$, $\grave{a}_2 = (\grave{\gamma}_{11},\grave{\gamma}_{22},\grave{\gamma}_{31})$ and $\grave{b}_1 = (\grave{\gamma}_{12},\grave{\gamma}_{21},\grave{\gamma}_{31})$, $\grave{b}_2 = (\grave{\gamma}_{12},\grave{\gamma}_{23},\grave{\gamma}_{31})$ then 'AND' and 'OR' operation are defined as follows

$(\mathrm{H}_1,\grave{\Lambda}_1)\wedge(\mathrm{H}_2,\grave{\Lambda}_2) =$
$\{\langle(\grave{a}_1 \times \grave{b}_1),((\grave{u}_1,0.3,-0.3,0.2,-0.2),(\grave{u}_2,0.4,-0.1,0.1,-0.1),(\grave{u}_3,0.3,-0.2,0.5,-0.1))\rangle,$
$\langle(\grave{a}_1\times\grave{b}_2),((\grave{u}_1,0.1,-0.3,0.2,-0.3),(\grave{u}_2,0.4,-0.1,0.2,-0.1),(\grave{u}_3,0.3,-0.2,0.3,-0.1))\rangle,$
$\langle(\grave{a}_2\times\grave{b}_1),((\grave{u}_1,0.2,-0.3,0.2,-0.1),(\grave{u}_2,0.6,-0.4,0.1,-0.1),(\grave{u}_3,0.4,-0.1,0.4,-0.4))\rangle,$
$\langle(\grave{a}_2\times\grave{b}_2),((\grave{u}_1,0.1,-0.3,0.4,-0.1),(\grave{u}_2,0.4,-0.1,0.2,-0.1),(\grave{u}_3,0.4,-0.1,0.3,-0.4))\rangle\}$
$(\mathrm{H}_1,\grave{\Lambda}_1)\vee(\mathrm{H}_2,\grave{\Lambda}_2) =$
$\{\langle(\grave{a}_1 \times \grave{b}_1),((\grave{u}_1,0.5,-0.4,0.2,-0.4),(\grave{u}_2,0.6,-0.4,0.3,-0.1),(\grave{u}_3,0.5,-0.2,0.5,-0.7))\rangle,$
$\langle(\grave{a}_1\times\grave{b}_2),((\grave{u}_1,0.5,-0.7,0.5,-0.4),(\grave{u}_2,0.4,-0.1,0.3,-0.5),(\grave{u}_3,0.7,-0.2,0.5,-0.4))\rangle,$
$\langle(\grave{a}_2\times\grave{b}_1),((\grave{u}_1,0.3,-0.4,0.4,-0.2),(\grave{u}_2,0.6,-0.4,0.4,-0.1),(\grave{u}_3,0.5,-0.2,0.5,-0.7))\rangle,$

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$\langle(\grave{a}_2 \times \grave{b}_2), ((\grave{u}_1, 0.2, -0.7, 0.5, -0.3), (\grave{u}_2, 0.6, -0.4, 0.4, -0.5), (\grave{u}_3, 0.7, -0.2, 0.4, -0.4))\rangle\}$

Theorem:3.22

Let $(S, \grave{\Lambda}_1)$ and $(M, \grave{\Lambda}_1)$ be two BIFHSSs on $\mathfrak{U}$ then

i) $[(S, \grave{\Lambda}_1) \vee (M, \grave{\Lambda}_2)]^c = (S, \grave{\Lambda}_1)^c \wedge (M, \grave{\Lambda}_2)^c$

ii) $[(S, \grave{\Lambda}_1) \wedge (M, \grave{\Lambda}_2)]^c = (S, \grave{\Lambda}_1)^c \vee (M, \grave{\Lambda}_2)^c$

Proof

$$(S, \grave{\Lambda}_1) \vee (M, \grave{\Lambda}_2) = \begin{pmatrix} \grave{u}, max\left(\mathcal{M}^+_{S(\propto_1)}(\grave{u}), \mathcal{M}^+_{M(\propto_2)}(\grave{u})\right), min\left(\mathcal{M}^-_{S(\propto_1)}(\grave{u}), \mathcal{M}^-_{M(\propto_2)}(\grave{u})\right), \\ max\left(\aleph^+_{S(\propto_1)}(\grave{u}), \aleph^+_{M(\propto_2)}(\grave{u})\right), min\left(\aleph^-_{S(\propto_1)}(\grave{u}), \aleph^-_{M(\propto_2)}(\grave{u})\right) \end{pmatrix}$$

$$[(S, \grave{\Lambda}_1) \vee (M, \grave{\Lambda}_2)]^c$$

$$= \begin{pmatrix} \grave{u}, 1 - max\left(\mathcal{M}^+_{S(\propto_1)}(\grave{u}), \mathcal{M}^+_{M(\propto_2)}(\grave{u})\right), -1 - min\left(\mathcal{M}^-_{S(\propto_1)}(\grave{u}), \mathcal{M}^-_{M(\propto_2)}(\grave{u})\right), \\ 1 - max\left(\aleph^+_{S(\propto_1)}(\grave{u}), \aleph^+_{M(\propto_2)}(\grave{u})\right), -1 - min\left(\aleph^-_{S(\propto_1)}(\grave{u}), \aleph^-_{M(\propto_2)}(\grave{u})\right) \end{pmatrix}$$

$$= \begin{pmatrix} \grave{u}, min\left(\aleph^+_{S(\propto_1)}(\grave{u}), \aleph^+_{M(\propto_2)}(\grave{u})\right), max\left(\aleph^-_{S(\propto_1)}(\grave{u}), \aleph^-_{M(\propto_2)}(\grave{u})\right), \\ min\left(\mathcal{M}^+_{S(\propto_1)}(\grave{u}), \mathcal{M}^+_{M(\propto_2)}(\grave{u})\right), max\left(\mathcal{M}^-_{S(\propto_1)}(\grave{u}), \mathcal{M}^-_{M(\propto_2)}(\grave{u})\right) \end{pmatrix}$$

$$(S, \grave{\Lambda}_1)^c = \left(\grave{u}, \aleph^+_{S(\propto_1)}(\grave{u}), \aleph^-_{S(\propto_1)}(\grave{u}), \mathcal{M}^+_{S(\propto_1)}(\grave{u}), \mathcal{M}^-_{S(\propto_{1,})}(\grave{u})\right)$$

$$(M, \grave{\Lambda}_1)^c = \left(\grave{u}, \aleph^+_{M(\propto_2)}(\grave{u}), \aleph^-_{M(\propto_2)}(\grave{u}), \mathcal{M}^+_{M(\propto_2)}(\grave{u}), \mathcal{M}^-_{M(\propto_{2,})}(\grave{u})\right)$$

$$(S, \grave{\Lambda}_1)^c \wedge (M, \grave{\Lambda}_2)^c = \begin{pmatrix} \grave{u}, min\left(\aleph^+_{S(\propto_1)}(\grave{u}), \aleph^+_{M(\propto_2)}(\grave{u})\right), max\left(\aleph^-_{S(\propto_1)}(\grave{u}), \aleph^-_{M(\propto_2)}(\grave{u})\right), \\ min\left(\mathcal{M}^+_{S(\propto_1)}(\grave{u}), \mathcal{M}^+_{M(\propto_2)}(\grave{u})\right), max\left(\mathcal{M}^-_{S(\propto_1)}(\grave{u}), \mathcal{M}^-_{M(\propto_2)}(\grave{u})\right) \end{pmatrix}$$

Hence, $[(S, \grave{\Lambda}_1) \vee (M, \grave{\Lambda}_2)]^c = (S, \grave{\Lambda}_1)^c \wedge (M, \grave{\Lambda}_2)^c$ is proved

Similarly, (ii) $[(S, \grave{\Lambda}_1) \wedge (M, \grave{\Lambda}_2)]^c = (S, \grave{\Lambda}_1)^c \vee (M, \grave{\Lambda}_2)^c$ is obtained.

4. Conclusion

In this paper, the Bipolar Intuitionistic Fuzzy Hyper Soft Set (BIFHSS) is presented as a novel and effective approach for representing and managing uncertainty in complex decision-making problems. By combining the features of bipolar intuitionistic fuzzy sets with hyper soft sets, the model effectively handles multi-attribute information that includes both positive and negative aspects. The definitions, operations, and illustrative examples provided in this study highlight the potential of BIFHSS as a valuable tool for future applications in decision-making environments involving incomplete or bipolar data.

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