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Secure Integer Domination in Graphs

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Abstract

An Integer dominating function on a graph G is a function $f : V(G) \to W$ such that for every vertex $v \in V(G)$, $\sum_{v \in V(G)}(N[v]) \geq k$. For any function $f : V(G) \to W$ and any pair of adjacent vertices with $f(v) = 0$ and $u > 0$, the function g_{uv} is defined by $g_{uv}(l) = 1$, $g_{uv}(l) = f(u) - 1$ and $g_{uv}(l) = f(l)$ if $l \in V - \{u, v\}$. A secure integer dominating function on a graph G is defined as an integer dominating function g which satisfies the condition that for every vertex v with $f(v) = 0$, $\exists$ a neighbor u with $f(u) > 0$ is such that g_{uv} is an integer dominating function. The weight of f is $w(f) = \sum_{v \in V(G)} f(v)$. The minimum weight among all the dominant secure integer functions in G is the number of secure integer domination in G. This paper is devoted to initiating the study of SIDF of a graph. In particular, we have studied about the general bounds for standard graphs like Complete graph, Star graph, wheel graph, path, cycle.

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Key Words and Phrases: Domination, Secure domination, Integer domination.

1 Introduction

Let $G = (V, E)$ be an undirected, finite and simple graphs. For any vertex $v \in V(G)$ open neighborhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup v$. Ore. O [6]A dominating set of a graph G is a set $S \subseteq V(G)$ such that for each $u \in V(G) - S$, there exists an $x \in S$ adjacent to u. The minimum cardinality among all dominating set of G is the domination number denoted by $\gamma(G)$.

Domke et al.[3] introduced the concept of Integer $k-$domination and further integer domination on Vizing's conjecture was studied by Bresar et al.[2]. The vertices of the graph G are labeled with integers such that for every vertex $v \in V(G)$, the sum of its closed neighborhood $N[v] \geq k$. The problem was to find the minimum total value labeled on G. Domke et al.[3] A function $f : V(G) \to W$ (W is the whole numbers) is called integer $k-$ dominating function if the sum of functional value over any closed neighbor is at least k. The weight of integer $k-$ dominating function is the value of $F(v) = \sum_{v \in V(G)} f(v)$. The minimum weight of integer $\{k\}-$ dominating function is denoted as $\gamma_k(G)$ and is called integer domination number. Note that, when $k = 1$ its a general domination.

Theorem 1. *[3] If G is a graph and $k \geq 1$ an integer, then $\gamma_k(G) \leq k\gamma(G)$.*

By positioning one or more guards at each vertex in a chosen subset S, we can protect the graph G, ensuring that every vertex in the closed neighborhood of each vertex is protected. The protection of graphs led to the development of the concept of secure domination, which involves identifying optimal guard placements to achieve maximum security. Place guards at a subset S of $V(G)$

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such that S is a dominating set, ensure that whenever a guard moves from a vertex v along the edge to an unguarded vertex u, the resulting guard configuration remains a dominating set, maintaining the graph security. In other words, [4] A *secure dominating set* S of a graph G is a dominating set with the property that each vertex $u \in V(G) - S$ is adjacent to a vertex $v \in S$ such that $(S - v) \cup u$ is a dominating set. The secure domination number denoted by $\gamma_s(G)$ is the minimum cardinality of a secure dominating set. Further, variations of secure domination and properties of the secure domination number have been studied in [5][7]. Some of the variation of secure domination are Secure weak roman domination[8], Secure Italian domination[7], total weak secure roman domination[9], secure w-domination [10], etc.

Theorem 2. *[7] For any G of order $n \geq 2$, $2 \leq \gamma_I^s(G) \leq n$.*

Theorem 3. *[7] For any integer $n \geq 3$, $\gamma_I^s(C_n) = \lceil \frac{3n}{5} \rceil$.*

Theorem 4. *[7] For any non trivial path P_n, $\gamma_I^s(P_n) = \lceil \frac{3n+2}{5} \rceil$.*

In [8] S. T. Hedetniemi et. al proved that for every graph G, $\gamma(G) \leq \gamma_r(G) \leq 2\gamma(G)$, where $\gamma_r(G)$ denoted the minimum weight of weak roman domination function in G. They also characterize graphs G for which $\gamma_r(G) = \gamma(G)$ and for forests G for which $\gamma_r(G) = 2\gamma(G)$.
In the recent Covid pandemic, hospitals were found to be deficient in oxygen cylinder supply, adequate oxygen cylinder supply hospitals provided to the deficient h ospitals. T his s cenario c an be modeled using a graph.
Suppose in "City A", the hospitals depicted as vertices of a graph G. Let two hospitals(vertices) u and v are close to each other hence, vertices are adjacent in G. Strategies to protect a graph G by placing k oxygen cylinders in its closed neighborhood of a vertex such that every vertex v can protected by k guard. This gave rise to the concept of integer domination. Further, suppose there exists a vertex(hospital) $v \in V(G)$ such that $f(v) = 0$ which is deficient in

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k oxygen cylinder, in case if a patient visit hospital v, then patient will be at risk. In such situation the vertex(hospital) which has oxygen cylinder lends the oxygen cylinder to the hospital which has no oxygen cylinder. i.e., the vertex with $f(u) \geq 0$ where $u, v \in V(G)$ lends oxygen cylinder to the vertex v, then $f(u) = f(u) - 1$ and $f(v) = 1$.

If G satisfy the condition of integer $\{k\}-$ domination, then G is secure. If G doesn't satisfy the condition, then we say G is not secure. When G is not secure, in such condition, in order to make G secure the concept of integer $\{k\}-$domination and secure domination is used to and define secure integer domination.

A function $g : V(G) \to W$, where W is the set of whole numbers and $g(v)$ denotes the number of oxygen cylinder available at each hospital $v \in V(G)$ for $k = 0, 1, 2, .., $ and let $V_k = \{v \in V(G) \mid f(v) = k\}$. By the definition of function g, the number of oxygen cylinders available is assigned to every vertex in G. When a oxygen cylinders at $v \in V - V_0$ moves along an edge $u \in V_0$, the new numbers of oxygen cylinders available at vertices are given by the function g_{uv} where $uv \in E(G)$.

$$g_{uv}(s) = \begin{cases} 1, & \text{if } s = v \\ f(u) - 1, & \text{if } s = u \\ f(s), & \text{if } s \in V - \{u, v\} \end{cases}$$

A secure integer dominating function is one that satisfies the condition of integer dominating function as well as secure domination.

Definition 5. A function $g : V(G) \to \{0, 1, 2, .., k\}$ is called *secure integer dominating function* (SIDF) of G if it satisfies the following:

(1) $\forall\, y \in V(G)$, $\sum_{y \in V(G)} g(N[y]) \geq k$.

(2) $\forall\, y \in V_0\ \exists\, z \in N(y) - V_0$ such that g_{yz} is an integer dominating function on G.

The secure integer dominating function has a weight equal to the value of $w(y) = \sum_{y \in V(G)} g(y)$. The minimum weight of secure

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integer dominating function is denoted by $\gamma_k^s(G)$ is called secure integer domination number (SIDN).

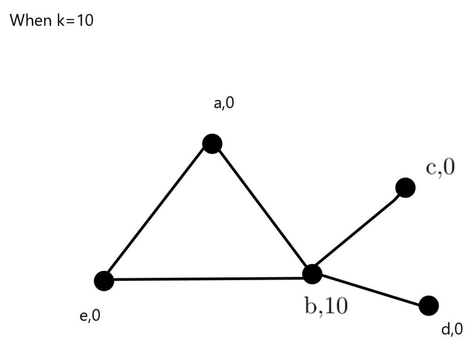

Figure 1: Graph with $\gamma_k(G) = 10$

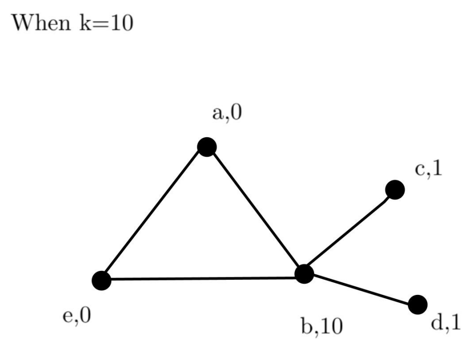

Figure 2: Graph with $\gamma_k^s(G) = 12$

Let $V(G) = \{a, b, c, d, e\}$ and suppose $k = 10$. According to the definition of secure integer domination, the sum of the weights assigned to the closed neighborhood of each vertex must be at least k that is, for every vertex $v \in V(G)$. Consider the weight assignment where $f(b) = 10$ and $f(v) = 0$ for all other vertices $b \neq v$. If all other vertices are adjacent to b, then this assignment satisfies the first condition, since each vertex's closed neighborhood contains b, whose weight is 10. Now, consider the second condition of secure

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integer domination: for any vertex with weight 0, there must exist an adjacent vertex with positive weight that can transfer 1 unit of weight to it, and the resulting function must still satisfy the domination condition. In this case, vertex b, which has weight 10, can transfer 1 unit to any of its neighbors with weight 0. After energy transfer, the condition fails, therefore pedant vertices receives the weight 1. Which implies, SID number to be 12.

Note that, the characteristic function of a secure integer dominating set of G doesn't contain V_0 and G satisfies the condition of (SIDF). If $g(v) = 0, v \in V(G)$ then integer dominating function need not be secure integer dominating function.

2 Terminology

We follow the definition and notation as in D B West [11]. The order of G is denoted by $n =| V(G) |$ and the size of G by $m =| E(G) |$. W denotes whole numbers and p denotes pendant vertices, pendant vertex are vertex with degree one. The distance between two vertices x and y in G is denoted as $d(u, v)$. The diameter, $diam(G)$ is the maximum distance among pairs of vertices in G. The degree of a vertex $v \in V(G)$ is $deg(v) =| N(v) |$. $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively. A complete graph on n vertices is denoted by K_n, complete bipartite with partite sets of size m and n is denoted by $K_{m,n}$. Path on n vertices by P_n and Cycle on n vertices by C_n. A star is the graph $K_{1,n}$ where $n \leq 1$. A double star is formed from two disjoint stars by joining a vertex of degree $n-1$ by an edge. Thus, a double star is a tree with exactly two vertices that are not leaves. We denote double star as $S_{r,s}$.

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3 General bounds

For every graph G, $\gamma_k(G) \leq \gamma_k^s(G)$. Every secure integer dominating function is an integer dominating function. For every graph G, $\gamma_k^s(G) = \gamma_I^s(G)$ if and only if $k = 2$.

Theorem 6. *For every nontrivial, connected graph with $k > 1$, $\gamma_k^s(G) \leq k\gamma_k(G)$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set of G. Suppose $g = (V_0, V_1, ...V_n)$ be the γ_k- function of a graph. Then $S = V_1 \cup V_2 \cup V_3 \cup .... \cup V_n$ is the minimum integer dominating set.
Assign $g : V(G) \to \{0, 1, 2, .., k\}$ such that $v_x \in V(G)$, $N_G[v_x] \geq k$.
There exists at least one vertex $v_i \in V_0$ as G is connected graph. v_i is adjacent to at least one vertex v_j such that $g(v_j) \geq k$.
$g(v_j) = k$ and $g(v_i) = 0$.
$g(v_j) + g(v_i) = k + 0 = k$.
Therefore, G is integer dominating function.
By the applying the definition of SIDF to $g(v_i)$,
We get $g_{v_j v_i}(v_j) = k - 1$ and $g_{v_j v_i}(v_i) = 1$.
$g_{v_j} + g_{v_i} = k - 1 + 1 = k$.
Hence, G is secure.
Thus, G is SIDF.
As $g_{v_j v_i}$ satisfies the definition of SIDF then S will be secure integer dominating set.
We get,
$\gamma_k^s(G) \leq \sum_{n=1}^{n} V_n$.
$\gamma_k^s(G) \leq \sum_{v_x=0}^{n} g(N[v_x])$.
$\gamma_k^s(G) \leq k \mid S \mid$.

If $g_{v_j v_i}$ doesn't satisfies the definition of SIDF then S' will be secure integer dominating set, where $S' = S \cup (v_y)$.
We get, $\gamma_k^s(G) \leq S' + \sum_{v^x=0}^{n} g(N[v_x])$.

$$\gamma_k^s(G) \leq k\gamma_k(G). \qquad \square$$

Implies, $\gamma_k^s(G) \leq k\gamma_k(G)$

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Theorem 7. *For any graphs G, $\gamma_k^s(G) \geq k$.*

Proof. Let G be a simple, connected graph with $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set.
On contrary, assume $\gamma_k^s(G) < k$.
We prove by induction on n.
When $n = 1$.
Let $v_i \in V(G)$, $g(v_i)$ will be k, which contradicts the assumption that $\gamma_k^s(G) < k$.
When $n \geq 1$.
Assign $g(V(G)) \rightarrow \{0, 1, 2, .., k\} \; \forall \; v_i \in V(G)$, such that $N_G[v_i] \geq k$.
If there exists at least one vertex $v_j \in V(G)$ such that $g(v_j) = 0$, then v_j is adjacent to at least one vertex v_p such that $g(v_p) \geq k$.
$g(v_i + g(v_p) = 0 + k = k$.
Therefore, G is integer dominating function.
By applying the definition of secure to v_j,
We get, $g_{v_p v_j}(v_j) = 1$ and $g_{v_p v_j}(v_p) = k - 1$.
$g(N[v_j]) = 1 + k - 1 = k$.
We obtain, G is secure.
Thus, G is secure integer dominating function.
It follows that, $\gamma_k^s(G) > k$, which is a contradiction .
Subsequently, $\gamma_k^s(G) \geq k$. □

Proposition 1. If G be a connected nontrivial graph, let u and v be two non adjacent vertices where $u, v \in V(G)$ then $\gamma_k^s(G)$ will be at least $k + 1$.

Let G be a simple, connected graph with $\delta(G) > 1$, $\exists \; v \in V(G)$ with maximum degree. Then $f(v)$ will be assigned maximum value.

Theorem 8. *If $H_1, H_2, ..., H_r$ are the components of disconnected graph G, then $\gamma_k^s(G) = \gamma_k^s(H_1) + \gamma_k^s(H_2) +, ... + \gamma_k^s(H_r)$.*

Proof. Since G is disconnected, every component of G dominated by $\gamma_k^s(H_i)$. Therefore, we get $\gamma_k^s(G) = \gamma_k^s(H_1) + \gamma_k^s(H_2) + ... + \gamma_k^s(H_r)$. □

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Theorem 9. *Let G be a simple connected graph of order n. Then $\gamma_k^s(G) = k$ if and only if $G = K_n$.*

Proof. To prove $\gamma_k^s(G) = k$ if $G = K_n$.
Case 1: On contrary, assume $\gamma_k^s(G) < k$. By theorem[3.2], we know that $\gamma_k^s(G)$ is at least k, which contradicts our assumption.
Case 2: Assume $\gamma_k^s(G) > k$. Since every vertex is of *deg* $n-1$. Assign value to $V(G)$ such that $\forall\ v_i \in V(G)\ N[v_i] \geq k$. Assign $g(v_1) = k$ and $g(v_i) = 0$, $2 \leq i \leq n$.
$g(N[v_1]) = k + 0 = k$.
G is integer dominating function.
$g_{v_1 v_i}(v_i) = 1$ and $g_{v_1 v_i}(v_1) = k - 1$.
$g(v_1) + g(v_i) = 1 + k - 1 = k$.
G is secure.
Thus, G is SIDF.
Since we get $\gamma_k^s(G) = k$, which contradicts our assumption that $\gamma_k^s(G) > k$.
Case 3: To prove $G = K_n$ if $\gamma_k^s(G) = k$. Assume $G \neq K_n$. Then there exists non adjacent u and v. Since u and v are non adjacent, by observation[1] γ_k^s will be at least $k+1$, which is a contradiction. □

Theorem 10. *If G is a simple graph $\Delta(G) = n - 1$ then $\gamma_k^s(G) \leq k + (n-2)$.*

Proof. Let $V(G) = \{u, v_1, v_2, ..., v_n\}$ be the vertex set. Let u be a vertex with degree $n-1$. By lemma[3.6] $g(u)$ can be either k or $k-1$. Since $\gamma_k^s(G)$ is minimum and $N[u] \geq k$. When $g(u) = k$, then $g(v_j) < k$.
There exists at least one vertex $v_i \in V(G)$ such that $g(v_i) = 0$.
Since G is connected, v_i is adjacent to either u or v_j.
$g(N[v_i]) = g(u) + g(v_i) = k + 0 = k$.
Therefore, G is integer dominating function.
Applying the definition of SIDF to v_i.
$g_{uv_i}(u) = k - 1$, $g_{uv_i}(v_i) = 1$ and $g_{uv_i}(v_j) < k$, if $g(v_j) = 1$ then $g(u) + g(v_i) + g(v_j) = k - 1 + 1 + \sum_{n=1}^{n-2} 1 \geq k$.

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As a result, G is secure.
Implies that, G is secure integer dominating function.
$g(u) + g(v_i) + g(V(G) - \{u, v_i\})$.
$\leq k + 0 + \sum_{n=1}^{n-2} 1$.
$\leq k + (n-2)$.
When $g(u) = k-1$, $g(v_j) = 1$.
$g(N[v_j]) = 1 + k - 1 \geq k$.
So, G is integer dominating function.
We know that, $g(v) \neq 0, v \in V(G)$ then integer dominating function will also be secure integer dominating function.
If $g(u) \neq k$ or $k-1$ then $g(V(G) - u) \to \{0, 1, 2, .., k\}$.
If $V(G) - u$ is disconnected, Then $g(V(G) - u) > 1$.
$g(N[u]) = g(u) + g(V(G) - u) \geq k + k(n-1)$.
Since $\gamma_k^s(G)$ is minimum, $g(u) = k$ or $k-1$.
It follows that, $\gamma_k^s(G) \leq k + (n-2)$. □

Theorem 11. *If G connected nontrivial is a star $K_{1,r}$, then $\gamma_k^s(G) = k + r - 1$.*

Proof. Let G be a star graph with $V(G) = \{u, p_1, p_2, ..., p_r\}$. There exists a vertex u with $deg(u) = n-1$.
On contrary, assume $\gamma_k^s(G) > k + r - 1$. Since $\gamma_k^s(G)$ is minimum, $\gamma_k^s(G) = k + r - 1$.
On contrary, assume $\gamma_k^s(G) < k + r - 1$. By lemma[3.6], $g(u)$ can be assigned 0, 1, k or $k-1$.
When $g(u) = k$.
Let $p_i, 1 \leq i \leq r$ be pendent vertices. Assign $g(p_i) = 0$ and $f(p_i) = 1, 2 \leq i \leq r$.
$N[u] = (\sum_{p=1}^{r} p_r) + g(u) \geq k$.
Therefore, G is integer dominating function.
Apply definition of SIDF to p_1.
Since $g(p_1) = 0$ and $N[p_1] = u$.
Which gives, $g_{up1(p_1)=1}$ and $g_{p_1 u}(u) = k-1$.
Hence, $N[p_1] = \mid u \mid + \mid p_1 \mid$.
$g(N[p_1]) = k - 1 + 1$.
$g(N[p_1]) = k$.

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Therefore, G is Secure.
It follows that, G is secure integer dominating function.
We obtain, $\gamma_k^s(G) = k + r - 1$, which contradicts our assumption that $\gamma_k^s(G) \leq k + r - 1$.
When $g(u) = k - 1$.
Then $g(p_i) = 1$.
We know that, if $g(v) \neq 0, v \in V(G)$ then integer dominating function will also be secure integer dominating function.
We obtain, $\gamma_k^s(G) = k - 1 + r$, which contradicts our assumption that $\gamma_k^s(G) \leq k + r - 1$.

□

Theorem 12. *If $G = F_{1,r}, r > 6$ is fan graph, then* $\gamma_k^s(G) = k + \lceil \frac{n}{3} \rceil$.

Proof. Let $V(G) = \{v, r_1, r_2, ...r_n\}$ be the vertex set of a fan graph there exists a vertex v with $\Delta(G) = n-1$. Let $r_i \in V(G)-v$, $1 \leq i \leq r$. Assign $g(v) = k - 1$, $g(r_1) = 0$ $g(r_i) = 1$, $2 \leq i \leq n$.
$N[v] = r_1 + \sum_{r=2}^{n} r_n + v$.
$g(N[v]) = 0 + \sum_{r=2}^{n} 1 + k - 1 > k$.
$N[r_1] = r_1 + v + r_2$.
$g(N_{r_1)=0+k-1+1=k}$.
$N[r_i] = v + r_i + r_{i+1}$.
$g(N[r_i]) = k - 1 + 1 + 1 > k$.
Therefore, we conclude G is integer dominating function.
Applying the definition of SIDF to $g(r_1)$,
$g_{vr_1}(v) = k - 2$, $g_{vr_1}(r_1) = 1$ and $g_{vr_i}(r_i) = 1$.
$g(v) + g(r_1) + g(r_i) = k - 2 + 1 + 1 = k$.
Therefore, G is secure dominating function.
$g(N[v]) = g(v) + \sum_{r=1}^{n} g(r_n)v \geq k$.
Thus, g is integer dominating function. We get, $\gamma_k^s = k + \lceil \frac{n}{3} \rceil$. □

Theorem 13. *If $G = W_{1,r}$ be a wheel graph, then* $\gamma_k^s(W_{1,r}) =$
$$\begin{cases} k + \lfloor \frac{n}{3} \rfloor, & n = 3p + 1,\ p \in N \\ k + \lceil \frac{n}{3} \rceil, & else \end{cases}$$

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$\gamma_k^s(T) = \gamma(T) + k$ if and only if $G = P_3$.

Theorem 14. *Let $n \geq 4$. Then* $\gamma_k^s(P_n) = \begin{cases} k\lceil \frac{n}{3} \rceil, & P_{3q+1}, q \in N \\ k\lceil \frac{n}{3} \rceil + 1, & P_{3q+2}, q \in N \\ k\lceil \frac{n}{3} \rceil + 2, & P_{3q+3}, q \in N \end{cases}$

Proof. Let P_n be a path on $n \geq 4$ vertices with $V(P_n) = \{v_1, v_2, ..., v_n\}$. Let v_1 and v_n be the pendent vertex. If $g(v_1)$ and $g(v_n)$ will be either 0, 1, $k-1$ or k.
When $g(v_i) = k$ where $i = 1, n$ then $g(v_{i\pm1})$ will be 0.
When $g(v_i) = k-1$ where $i = 1, n$ then $g(v_{i\pm1})$ will be 1.
When $g(v_i) = 1$ where $i = 1, n$ then $(v_{i\pm1})$ will be $k-1$.
we know that, $g(v_i) \neq 0$ then integer dominating function will also be secure integer dominating function.
We prove by induction on n.
Case 1: Let $n = 3q + 1$ where $q \in Z^+$.
When $n = 4$. Let $\{v_1, v_2, v_3, v_4\} \in V(G)$. As v_1 and v_4 are pendent vertices, it can be assigned either k, 0, $k-1$ or 1.
Assign $g(v_i) = k$ where $i = 1, n$ and $g(v_{i\pm1}) = 0$.
We obtain, $N[v_i] = k + 0 = k$.
G is integer dominating function.
By applying the definition of SIDF,
We get, $g_{v_i v_{i\pm1}}(v_i) = k-1$, $g_{v_i v_{i\pm1}}(v_{i\pm1}) = 1$.
$g(v_1) + g(v_2) + g(v_3) + g(v_4) = k + 0 + 0 + k \geq k$.
Hence, G is secure integer dominating function.
Assign $g(v_i) = k-1$, $i = 1, n$ and $g(v_{i+1}) = 1$.
We know that, $g(v_i) \neq 0$ then integer dominating function will also be secure integer dominating function.
We get $\gamma_k^s(P_n) = 2k = k\lceil \frac{n}{3} \rceil$.
When $n \geq 5$. Pendant vertices can be assigned either k or $k-1$ or 1 or 0.
Assign $g(v_i) = k-1$ and $g(v_{i+1}) = 1$ $i = 1, n$.
Let $V(G) - \{v_1, v_n\}$ are internal vertices. Internal vertices with $n = 3p+1$ can be assigned $g(v_{3p+1}) \to \{0, 1, ...k\}$. Assign $g(v_{3p+1}) =$

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$k-2$, and all other vertices can be assigned 1.
$g([N_{3p+1}]) = k-2+1+1 = k$.
G is integer dominating function.
We get, $g(v_{3p-1}) + g(v_{3p+1}) + g(v_{3p+2})$.
$= 1+k-2+1 = k$. Therefore G is secure.
We know that, $g(v_i) \neq 0$ then integer dominating function will also be secure integer dominating function.
Therefore, G is secure integer dominating function.
Assign $g(v_i) = 0$, $g(v_{i+1}) = k$ $i = 1, n$.
Assign $g(v_{3p+1}) = k-2$ all other vertices can be assigned 1.
$g(N[v_{3p+1}) = 1+k-2+1 = k$.
Thus, G is integer dominating function.
Applying the definition of SIDF to v_i.
We get, $g_{v_{i+1}v_i}(v_i) = 1$ and $g_{v_{i+1}v_i}(v_{i+1}) = k-1$.
$g(N[v_i]) = 1+k-1 = k$.
It follows that, G is secure.
Implies that, G is secure integer dominating function.
Thus, $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil$.
Case 2: Let $n = 3q+2$ where $q \in Z^+$.
When $n = 5$. Let $\{v_1, v_2, v_3, v_4, v_5\} \in V(G)$. v_1 and v_5 be pendent vertices. Pendent vertices can be assigned either 0, 1, $k-1$ or k.
$g(v_i) = 1$ and $g(v_{i\pm1}) = k-1$.
Since v_1, v_2, v_4, v_5 are assigned, we get $g(v_3) = 1$ as v_3 is adjacent to v_2 and v_4.
$g(v_i) + g(v_{i+1}) = 1+k-1 = k$.
$g(N[v_3]) = k-1+1+k-1 \geq k$.
G is integer dominating function. We have observed that, $g(v_i) \neq 0$ then integer dominating function will also be secure integer dominating function.
Thus, G is secure integer dominating function.
When $n > 5$. Let $\{v_1, v_2, ..., v_n\} \in V(G)$. v_1 and v_n be pendent vertices, then $g(v_1)$ and $g(v_n)$ will be either 0, 1, $k-1$ or k.
Assign $g(v_i) = k-1$ and $g(v_{i+1}) = 1$ where $i = 1, n$.
Let $V(G) - \{v_1, v_n\}$ are internal vertices. Internal vertices with $n = 3n+1$ can be assigned as $g(v_{3n+1}) \rightarrow \{0, 1, ...k\}$. Assign

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$g(v_{3n+2}) = k - 2$ all other internal vertices can be assigned 1.
$g(N[3n+1]) = 1 + k - 2 + 1 = k$.
This leads, G is integer dominating function.
We have observed that, $g(v_i) \neq 0$ then integer dominating function will also be secure integer dominating function.
Therefore, G is secure integer dominating function.
Assign $g(v_i) = 0$ and $g(v_{i+1}) = k$, $i = 1, n$.
$g(N[v_i]) = v_i + v_{i+1} = 0 + k = k$.
Therefore, G is integer dominating function.
Applying the definition of SIDF to $g(v_i)$.
$g_{v_{i+1}v_i}(v_i) = 1$ and $g_{v_{i+1}v_i}(v_{i+1}) = k - 1$.
$g(N[v_i]) = 1 + k - 1 = k$.
G is secure.
G is secure integer dominating function.
Let $v_i \in (3q+2) - (3q+1)$.
If v_i is a pendant vertex then $g(v_i) = k$ or $k - 1$ or 1.
We get, $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + k$.
or $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + k - 1$.
or $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + 1$.
By definition $\gamma_I^s(G)$ is minimum, so we get $\gamma_I^s(G) = k\lceil \frac{n}{3} \rceil + 1$.
If v_i is internal vertex. If $g(v_i) = f(v_{3n+1}) = k - 2$.
We get, $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + (k - 2)$.
If $g(v_i) \neq g(3n+2)$.
We get$f(v_i) = 1$.
Therefore, $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + 1$.
Case 3: Let $n = 3q + 3$ where $q \in Z^+$.
As we have already assigned the value for all the vertices. We need to prove for the case when $n = 3q + 3$.
Let $v_i \in (3q+3) - (3q+2)$.
If v_i is pendant vertex then $g(v_i) = k$ or $k - 1$ or 1.
We get, $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + 1 + k$.
or $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + 1 + (k - 1)$.
or $\gamma_k^s(P_n) = k\lceil \frac{n}{3} \rceil + 1 + 1 = k\lceil \frac{n}{3} \rceil + 2$.
By definition $\gamma_k^s(G)$ is minimum, we get $\gamma_k^s(G) = k\lceil \frac{n}{3} \rceil + 2$.
If v_i is internal vertex. If $g(v_i) = g(v_{3n+2}) = k - 2$.

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We obtain, $\gamma_k^s(P_n) = k\lceil\frac{n}{3}\rceil + (k-2)$.
If $f(v_i) \neq f(3n+2)$ implies $f(v_i) = 1$.
We find, $\gamma_k^s(P_n) = k\lceil\frac{n}{3}\rceil + 1 + 1 = k\lceil\frac{n}{3}\rceil + 2$. □

Theorem 15. *Let $n \geq 3$. Then* $\gamma_k^s(C_n) = \begin{cases} k\lceil\dfrac{n}{3}\rceil - 6, & C_{3q+1}, q \in N \\ k\lceil\dfrac{n}{3}\rceil - 3, & C_{3q+2}, q \in N \\ k\lceil\dfrac{n}{3}\rceil, & C_{3q+3}, q \in N \end{cases}$

Proof. Let C_n be a cycle on $n \geq 4$ with $V(C_n) = \{v_1, v_2, ..., v_n\}$.
By the definition of SIDF $f(N[v_i) \geq k$.
Every vertex of C_n has degree two. For all v_i, we have $g(v_{i-1}) + g(v_i) + g(v_{i+1}) \geq k$.
Assign $g(v_i) = k-6$, all other vertices as 3.
we get $(v_{i-1}) + g(v_i) + g(v_{i+1})$
$= 3 + k - 6 + 3 = k$
G is secure integer dominating function.
Case 1: Let $n = 3q+1$ where $q \in Z^+$.
When $n = 4$. We have $V(G) = \{v_1, v_2, v_3, v_4\}$. If for any $g(v_i) = k-6$, then by the definition of SIDF, we get $g(v_{i-1})+g(v_i)+g(v_{i+1})$
$= 3 + k - 6 + 3 = k$
G is secure integer dominating function.
Thus, $\gamma_k^s(C_4) = k\lceil\frac{n}{3}\rceil - 6$.
Case 2: Let $n = 3q+2$ where $q \in Z^+$.
Let $v_i \in (3q+2) - (3q+1)$. Then by the definition of SIDF, $g(v)$ sill be almost k.
Since every vertex in G has a degree 2. v dominated itself and its neighbor. Then by the definition of SIDF, $g(v)$ will have at most k.
When $g(v_i) = k$, we get $g(v_{i-1}) + g(v_i) + g(v_{i+1}) \geq k$
$0 + k + 0 = k$
Assign $v_i = k-6$.
We obtain, $\gamma_k^s(G) = k\lceil\frac{n}{3}\rceil - 6 + (k-6)$
When $v_i = 3$.
We secure, $\gamma_k^s(G) = k\lceil\frac{n}{3}\rceil - 6 + 3 = \gamma_k^s(G) = k\lceil\frac{n}{3}\rceil - 3$.

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By definition of SIDF we get, $\gamma_k^s(G)$ is minimum. Thus, we obtain $\gamma_k^s(G) = k\lceil\frac{n}{3}\rceil - 3$.
Case 3: When $n = 3q + 3$ where $q \in Z^+$.
Let $v_i \in (3q + 3) - (3q + 2)$.
When $v_i = k - 6$.
We attain, $\gamma_k^s(G) = k\lceil\frac{n}{3}\rceil - 3 + (k - 6)$
When $v_i = 3$.
We get, $\gamma_k^s(G) = k\lceil\frac{n}{3}\rceil - 3 + 3 = \gamma_k^s(G) = k\lceil\frac{n}{3}\rceil$.
By definition of SIDF we obtain, $\gamma_k^s(G)$ is minimum. Thus, $\gamma_k^s(G) = k\lceil\frac{n}{3}\rceil$.

□

Conclusion
We introduce the concept of secure integer domination in this article and provide general bounds for several types of graphs, such as complete, star, wheel, fan, path, and cycle graphs. Further secure integer domination of grid graph, product of graphs can be explored.

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