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SOME INTERESTING COLORED PARTITIONS ARISING FROM MODULAR EQUATIONS

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Abstract

In this paper, we obtained four new identities on colored partitions and provide proofs for them. These partition identities presented in the paper depends only on the q-identities.

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1 Introduction

In this paper, we presume that $|q| < 1$. For any positive integer n, we apply the classic notation

$$(x;q)_n := \prod_{k=1}^{n}(1-xq^{k-1}) \qquad \textit{and} \qquad (x;q)_\infty := \prod_{k=0}^{\infty}(1-xq^{k}).$$

For $|\alpha\beta| < 1$, Ramanujan's theta function $f(\alpha,\beta)$ is defined by

$$f(\alpha,\beta) := \sum_{n=-\infty}^{\infty} \alpha^{n(n+1)/2}\beta^{n(n-1)/2}.$$

It directly follows from the famous Jacobi's triple product identity that

$$f(\alpha,\beta) := (-\alpha;\alpha\beta)_\infty(-\beta;\alpha\beta)_\infty(\alpha\beta;\alpha\beta)_\infty.$$

The following essential facts of $f(\alpha,\beta)$ is defined by Ramanujan[2, p. 36]:

$$\begin{aligned}
\varphi(q) &:= f(q,q) = (-q;q^2)^2{}_\infty(q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty} = \sum_{n=-\infty}^{\infty} q^{n^2},\\
\psi(q) &:= f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} == \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}},\\
f(-q) &:= f(-q,-q^2) = (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.
\end{aligned}$$

For simplicity, we write $f(-q^n)$ by f_n and infer the following.

$$\varphi(-q) = \frac{f_1^2}{f_2} \qquad \text{and} \qquad \psi(q) = \frac{f_2^2}{f_1}.$$

For our convinience, we follow the typical representation:

$$(\alpha_1,\alpha_2,\ldots,\alpha_n;q)_\infty := \prod_{i=1}^{n}(\alpha_i;q)_\infty = (\alpha_1;q)_\infty(\alpha_2;q)_\infty\ldots(\alpha_n;q)_\infty.$$

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and define for $\alpha, \beta \in \mathbb{N}$

$$(q^{\alpha}; q^{\beta})_{\infty}(q^{\beta-\alpha}; q^{\beta})_{\infty} = (q^{\alpha}, q^{\beta-\alpha}; q^{\beta})_{\infty} = (q^{\alpha\pm}; q^{\beta})_{\infty} \qquad \alpha < \beta. \tag{1}$$

For an illustration, $(q^{2\pm}; q^{8})_{\infty}$ means $(q^{2}, q^{6}; q^{8})_{\infty}$ which is $(q^{2}; q^{8})_{\infty}$ $(q^{6}; q^{8})_{\infty}$. First, we recall the definition of a colored partition and required generating function. This concept is first introduced by A. K. Agarwal and G. E. Andrews [1]. Further S. -S. Huang [3] continued this work on establishing the modular relations between Göllnitz-Gordan functions. For the recent work on colored partitions, one may see [5, 7]. Now, we define colored partition as defined in [1].

"A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions". For an illustration, if 2 is permitted to assign two different colors, say l (lavender) and v (violet), then $4, 3+1, 2_l+2_v, 2_l+2_v, 2_l+2_l, 2_v+1+1, 2_l+1+1, 1+1+1+1$ are the various colored partitions of 4. Also, for the total number of partitions of n

$$\sum_{n=0}^{\infty} \omega(n)q^{n} = \frac{1}{(q^{l}; q^{m})_{\infty}^{m}},$$

represents the generating function for all the parts which are congruent to x modulo y with m various colors with $\omega(0) = 1$. Look at the provided modular equation from the literature. A modular equation of n^{th} degree in the theory of signature two, is an equation relating α and β that is accustomed by

$$\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)} = n\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}.$$

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Then, we claim that β is of n^{th} degree over α and call the quotient

$$m := \frac{z_1}{z_n} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}$$

the multiplier where ${}_2F_1$ represents the classical hypergeometric series described as

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k, \qquad \mid z \mid < 1$$

where

$$(m)_k = m(m+1)\ldots(m+k-1).$$

2 Main Results

THEOREM 2.1. *Let $\omega_1(j)$ is chosen to serve as the sum of total partitions of j being partitioned with in the fragments congruent towards $\pm1, \pm2, \pm3, \pm5, \pm7, \pm9, \pm10, \pm11, \pm13, \pm14, \pm15, \pm17$ modulo 36 by 8 colors, $\pm4, \pm8, \pm16$ modulo 36 by 12 colors, ±6 modulo 36 by 30 colors, ±12 modulo 36 by 36 colors, $+18$ modulo 36 by 40 colors. If $\omega_2(j)$ stand for the total partitions of j being partitioned within the units j into items that are congruent towards $\pm1, \pm5, \pm7, \pm11, \pm13, \pm17$ modulo 36 by 4 colors, $\pm2, \pm10, \pm14, \pm16$ modulo 36 by 10 colors, $\pm3, \pm8, \pm15$ modulo 36 by 12 colors, ±4 modulo 36 by 6 colors, ±6 modulo 36 by 30 colors, ±9 modulo 36 by 16 colors, ±12 modulo 36 by 36 colors, $+18$ modulo 36 by 40 colors. If $\omega_3(j)$ serve as the total partitions of j within the fragments congruent to $\pm2, \pm4, \pm8, \pm10, \pm14, \pm16$ modulo 36 by 12 colors, $\pm3, \pm9, \pm15$ modulo 36 by 16 colors, ±6 modulo 36 by 28 colors, ±12 modulo 36 by 36 colors, $+18$ modulo 36 by 40 colors. Let $\omega_4(j)$ express the sum of partitions of j into elements congruent with $\pm1, \pm5, \pm7, \pm11, \pm13, \pm17$ modulo 36 by*

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4 colors, $\pm 2, \pm 10, \pm 14$ *modulo 36 by 6 colors,* $\pm 3, \pm 15$ *modulo 36 by 12 colors,* $\pm 4, \pm 8, \pm 16$ *modulo 36 by 10 colors,* ± 6 *modulo 36 by 34 colors,* ± 9 *modulo 36 by 16 colors,* ± 12 *modulo 36 by 38 colors,* $+18$ *modulo 36 by 40 colors. Let* $\omega_5(j)$ *represent the sum of partitions of* j *into parts congruent with* $\pm 1, \pm 5, \pm 7, \pm 11, \pm 13, \pm 17$ *modulo 36 by 4 colors,* $\pm 2, \pm 4, \pm 8, \pm 10, \pm 14, \pm 16$ *modulo 36 by 14 colors,* $\pm 3, \pm 9, \pm 15$ *modulo 36 by 12 colors,* ± 6 *modulo 36 by 22 colors,* ± 12 *modulo 36 by 30 colors,* $+18$ *modulo 36 by 40 colors. Then we have*

$$\omega_1(j) - 4\omega_2(j) + \omega_3(j) + 3\omega_4(j) - \omega_5(j) = 0, \ j \geq 0.$$

Proof. Let

$$M = \frac{\varphi^2(-q^3)}{\varphi(-q)\varphi(-q^9)} \qquad \text{and} \qquad N = \frac{\varphi^2(-q^6)}{\varphi(-q^2)\varphi(-q^{18})}.$$

By employing Entry 10(ii) and (iii) of Chapter 17 [2, p. 122], in M and N, one can deduce

$$\frac{M}{N} = \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/8} \qquad \text{and} \qquad \frac{M}{N^2} = \left(\frac{z_1 z_9}{z_3^2}\right)^{1/2}, \tag{2}$$

where β and γ have degrees 3 and 9 over α correspondingly. Also from [2, Entry 3(xii) and (xiii), pp. 352-353], we have

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} = -3\frac{m}{m'} \tag{3}$$

and

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} = \frac{m'}{m} \tag{4}$$

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where β and γ have degree 3 and 9 over α correspondingly and $m = \frac{z_1}{z_3}$ and $m' = \frac{z_3}{z_9}$. On employing (2) in (3) and (4), it is easy to see that

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} = \frac{M^2(N^2+3)}{N^2(M^2-N^2)} \qquad \text{and} \qquad \left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} = \frac{N^2(N^2-1)}{M^2-N^2}.$$

On multiplying the above, we obtain

$$M^4 - 4M^2N^2 + N^4 + 3M^2 - M^2N^4 = 0.$$

Now on redrafting M and N in terms of f_n and then dividing completely by $f_2^4 f_3^{16} f_4^4 f_6^{16} 3 f_{18}^4 f_{36}^4$ and simplifying to the prevalent base q^{36} and further employing (1), we deduce

$$\begin{aligned}
&\frac{1}{\left(q_8^{1\pm,2\pm,3\pm,5\pm,7\pm,9\pm,10\pm,11\pm,13\pm,14\pm,15\pm,17\pm}, q_{12}^{4\pm,8\pm,16\pm}, q_{30}^{6\pm}, q_{36}^{12\pm}, q_{40}^{18+}; q^{36}\right)_\infty} \\
&- \frac{4}{\left(q_4^{1\pm,5\pm,7\pm,11\pm,13\pm,17\pm}, q_{10}^{2\pm,10\pm,14\pm,16\pm}, q_{12}^{3\pm,8\pm,15\pm}, q_6^{4\pm}, q_{30}^{6\pm}, q_{16}^{9\pm}, q_{36}^{12\pm}, q_{40}^{18+}; q^{36}\right)_\infty} \\
&+ \frac{1}{\left(q_{12}^{2\pm,4\pm,8\pm,10\pm,14\pm,16\pm}, q_{16}^{3\pm,9\pm,15\pm}, q_{28}^{6\pm}, q_{36}^{12\pm}, q_{40}^{18+}; q^{36}\right)_\infty} \\
&+ \frac{3}{\left(q_4^{1\pm,5\pm,7\pm,11\pm,13\pm,17\pm}, q_6^{2\pm,10\pm,14\pm}, q_{12}^{3\pm,15\pm}, q_{10}^{4\pm,8\pm,16\pm}, q_{34}^{6\pm}, q_{16}^{9\pm}, q_{38}^{12\pm}, q_{40}^{18+}; q^{36}\right)_\infty} \\
&- \frac{1}{\left(q_4^{1\pm,5\pm,7\pm,11\pm,13\pm,17\pm}, q_{14}^{2\pm,4\pm,8\pm,10\pm,14\pm,16\pm}, q_{22}^{6\pm}, q_{30}^{12\pm}, q_{40}^{18+}; q^{36}\right)_\infty} = 0.
\end{aligned}$$

The atop identity executes the five generating functions and consequently, we have

$$\sum_{j=0}^{\infty} \omega_1(j)q^j - 4\sum_{j=0}^{\infty} \omega_2(j)q^j + \sum_{j=0}^{\infty} \omega_3(j)q^j + 3\sum_{j=0}^{\infty} \omega_4(j)q^j - \sum_{j=0}^{\infty} \omega_5(j)q^j = 0,$$

In the above on collecting the powers of q^j, we get the relevant result. $\square$

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$\omega_1(2) = 44:$	$1_w + 1 + w, 1_b + 1_b, 1_r + 1_r, 1_o + 1_o, 1_{bl} + 1_{bl}, 1_y + 1_y, 1_g + 1_g, 1_v + 1_v, 1_w + 1_b, 1_w + 1_c, 1_w + 1_o, 1_w + 1_{bl}, 1_w + 1_y, 1_w + 1_g, 1_w + 1_v, 1_b + 1_r, 1_b + 1_o, 1_b + 1_{bl}, 1_b + 1_y, 1_b + 1_g, 1_b + 1_v, 1_r + 1_o, 1_r + 1_{bl}, 1_r + 1_y, 1_r + 1_g, 1_r + 1_v, 1_o + 1_{bl}, 1_o + 1_y, 1_o + 1_g, 1_o + 1_v, 1_{bl} + 1_y, 1_{bl} + 1_g, 1_{bl} + 1_v, 1_y + 1_g, 1_y + 1_h, 1_g + 1_v, 2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y, 2_g, 2_v$
$\omega_2(2) = 20:$	$1_w + 1_w, 1_b + 1_b, 1_r + 1_r, 1_o + 1_o, 1_w + 1_b, 1_w + 1_r, 1_w + 1_o, 1_b + 1_r, 1_b + 1_o, 1_r + 1_o, 2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y, 2_g, 2_v, 2_p, 2_m$
$\omega_3(2) = 12:$	$2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y, 2_g, 2_v, 2_p, 2_m, 2_{br}, 2_l$
$\omega_4(2) = 16:$	$1_w + 1_w, 1_b + 1_b, 1_r + 1_r, 1_o + 1_o, 1_w + 1_b, 1_w + 1_r, 1_w + 1_o, 1_b + 1_r, 1_b + 1_o, 1_r + 1_o, 2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y$
$\omega_5(2) = 24:$	$1_w + 1_w, 1_b + 1_b, 1_r + 1_r, 1_o + 1_o, 1_w + 1_b, 1_w + 1_r, 1_w + 1_o, 1_b + 1_r, 1_b + 1_o, 1_r + 1_o, 2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y, 2_g, 2_v, 2_p, 2_m, 2_{br}, 2_l, 2_p, 2_s$

EXAMPLE 2.1. *For $j = 2$, above table authenticates the theorem.*

THEOREM 2.2. *Let $\omega_1(j)$ is chosen to serve as the sum of partitions of j into components which are congruent with $\pm 2, \pm 9, \pm 10, \pm 14, \pm 15$ modulo 36 by 8 colors, $\pm 4, \pm 8$, modulo 36 by 2 colors, $\pm 3, \pm 16$ modulo 36 by 4 colors, ± 12 modulo 36 by 6 colors. If $\omega_2(j)$ is taken to identify the sum of partitions of j into items that are congruent with $\pm 1, \pm 3, \pm 4, \pm 5, \pm 7, \pm 8, \pm 11, \pm 13, \pm 17$ modulo 36 by 2 colors, $\pm 2, \pm 10, \pm 14$ modulo 36 by 4 colors, $\pm 12, \pm 15$ modulo 36 by 6 colors, ± 9 modulo 36 by 8 colors. If $\omega_3(j)$ is taken to be the sum of partitions of j into factor congruent to $\pm 1, \pm 5, \pm 7, \pm 11, \pm 13, \pm 15, \pm 16, \pm 17$ modulo 36 by 4 colors, $\pm 4, \pm 8$, modulo 36 by 2 colors, $\pm 6, \pm 9$ modulo 36 by 8 colors, ± 12 modulo 36 by 6 colors. Let $\omega_4(j)$ express the sum of partitions of j into components congruent towards $\pm 1, \pm 5, , \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 17$ modulo 36 by 4 colors, $\pm 2, \pm 10, \pm 14, \pm 16$ modulo 36 by 2 colors, ± 6 modulo 36 by 6 colors. Let $\omega_5(j)$ represent the sum of partitions of j into items con-*

gruent with $\pm 2, \pm 10, \pm 14$ *modulo 36 by 6 colors,* $\pm 3, \pm 4, \pm 8, \pm 12$ *modulo 36 by 4 colors,* $\pm 6, \pm 16$ *modulo 36 by 2 colors,* ± 9 *modulo 36 by 8 colors,* ± 15 *modulo 36 by 18 colors. Then we have*

$$\omega_1(j-2) - 4\omega_2(j-1) + \omega_3(j) + 3\omega_4(j-2) - \omega_5(j) = 0,\ j \geq 2.$$

Proof. Let

$$M = \frac{\psi(q^3)}{q^{1/2}\psi(q)\psi(q^9)} \qquad \text{and} \qquad N = \frac{\psi^2(q^6)}{q\psi(q^2)\psi(q^{18})}.$$

On using [2, p. 122, Entry 10(ii) and (iii)] of Chapter 17 in the above, we see that

$$\frac{M}{N} = \left(\frac{\alpha\gamma}{\beta^2}\right) \qquad \text{and} \qquad \frac{M^2}{N} = \left(\frac{z_3^2}{z_1 z_9}\right)^{1/2}. \tag{5}$$

Employing (5) in (3) and (4), it is easy to see that

$$\left\{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right\}^{1/4} = \frac{N^2(3+M^2)}{M^2(N^2-M^2)}$$

and

$$\left\{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right\}^{1/4} = \frac{M^2(M^2-1)}{N^2-M^2}.$$

On multiplying the above, we obtain

$$M^4 - 4M^2N^2 + N^4 + 3N^2 - M^4N^2 = 0.$$

Now on redrafting M and N interms of f_n and then dividing completely by $f_1^4 f_2^4 f_6^{16} f_9^4 f_{12}^{16} f_{18}^4$ and simplifying to the prevalent base

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q^{36} and further employing (1), we deduce

$$\begin{aligned}
&\frac{q^2}{\left(q_8^{2\pm,9\pm,10\pm,14\pm,15\pm},q_4^{3\pm,16\pm},q_2^{4\pm,8\pm,},q_6^{12\pm};q^{36}\right)_\infty}\\
&-\frac{4q}{\left(q_2^{1\pm,3\pm,4\pm,5\pm,7\pm,8\pm,11\pm,13\pm,17\pm},q_4^{2\pm,10\pm,14\pm},q_8^{9\pm},q_6^{12\pm,15\pm};q^{36}\right)_\infty}\\
&+\frac{1}{\left(q_4^{1\pm,5\pm,7\pm,11\pm,13\pm,15\pm,16\pm,17\pm},q_2^{4\pm,8\pm},q_8^{6\pm,9\pm},q_6^{12\pm};q^{36}\right)_\infty}\\
&+\frac{3q^2}{\left(q_4^{1\pm,5\pm,7\pm,9\pm,11\pm,13\pm15\pm,17\pm},q_2^{2\pm,10\pm,14\pm,16\pm},q_6^{6\pm};q^{36}\right)_\infty}\\
&-\frac{1}{\left(q_6^{2\pm,10\pm,14\pm},q_4^{3\pm,4\pm,8\pm,12\pm},q_2^{6\pm,16\pm},q_8^{9\pm},q_1^{15\pm}8;q^{36}\right)_\infty}=0.
\end{aligned}$$

The atop identity executes the five generating functions and consequently, we have

$$\begin{aligned}
\sum_{j=0}^{\infty}\omega_1(j)q^{j+2}-4\sum_{j=0}^{\infty}\omega_2(j)q^{j+1}+\sum_{j=0}^{\infty}\omega_3(j)q^{j}+3\sum_{j=0}^{\infty}\omega_4(j)q^{j+2}&\\
-\sum_{j=0}^{\infty}\omega_5(j)q^{j}=0,&
\end{aligned}$$

In the above on collecting the powers of q^j, we get the relevant result. ☐

EXAMPLE 2.2. *For $j=2$, below table authenticates the above theorem.*

$\omega_1(2)=1:$	
$\omega_2(2)=2:$	$1_w, 1_b$
$\omega_3(2)=10:$	$1_w+1_w, 1_b+1_b, 1_r+1_r, 1_o+1_o, 1_w+1_b, 1_w+1_c,$ $1_w+1_r, 1_b+1_r, 1_b+1_o, 1_r+1_o$
$\omega_4(2)=1:$	
$\omega_5(2)=6:$	$2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y$

THEOREM 2.3. *Let $\omega_1(j)$ serve as the total sum of partitions of j into parts that are congruent with $\pm1, \pm2, \pm4, \pm5$ modulo 12 by 2 colors, $\pm3, +6$ modulo 12 by 4 colors. If $\omega_2(j)$ is taken to identify the sum of partitions of j into components that are congruent to $\pm1, \pm2, \pm5$ modulo 12 by 6 colors, ±4 modulo 12 by 8 colors, $\pm3, +6$ modulo 12 by 12 colors. If $\omega_3(j)$ is taken to be the sum of partitions of j into items congruent to $\pm3, \pm4, +6$ modulo 12 by 8 colors. Let $\omega_4(j)$ imply the sum of partitions of j into elements congruent to $\pm1, \pm2, \pm3, \pm4, \pm5, +6$ modulo 12 by 8 colors. Let $\omega_5(j)$ represent the sum of partitions of j into component congruent to $\pm1, \pm2, \pm5$ modulo 12 by 4 colors, $\pm3, \pm4, +6$ modulo 12 by 8 colors. Then we have*

$$\omega_1(j) + 16\omega_2(j-2) - \omega_3(j-2) - \omega_4(j) + 6\omega_5(j-1) = 0, \ j \geq 2.$$

Proof. From Lemma 2.1 of [6], we have if

$$M := \frac{\varphi(-q)}{q^{1/4}\psi(q^2)} \qquad \text{and} \qquad N := \frac{\varphi(-q^3)}{q^{3/4}\psi(q^6)}$$

then

$$(MN)^3 + 16(MN) - M^4 - N^4 - 6(MN)^2 = 0.$$

Now on redrafting M and N in terms of f_n and then dividing completely by $f_1^8 f_3^8$ and simplifying to the prevalent base q^{12} and further employing (1), we deduce

$$\frac{1}{\left(q_2^{1\pm,2\pm,4\pm,5\pm}, q_4^{3\pm.6+}; q^{12}\right)_\infty} + \frac{16q^2}{\left(q_6^{1\pm,2\pm5\pm}, q_{12}^{3\pm,6+}, q_8^{4\pm}; q^{12}\right)_\infty} - \frac{q^2}{\left(q_8^{3\pm,4\pm,6\pm}; q^{12}\right)_\infty}$$
$$- \frac{1}{\left(q_8^{1\pm,2\pm,3\pm,4\pm,5\pm,6+}; q^{12}\right)_\infty} + \frac{6q}{\left(q_4^{1\pm,2\pm,5\pm}, q_8^{3\pm,4\pm,6+}; q^{12}\right)_\infty} = 0.$$

The atop identity executes the five generating functions and consequently, we have

$$\sum_{j=0}^{\infty}\omega_1(j)q^j + 16\sum_{j=0}^{\infty}\omega_2(j)q^{j+2} - \sum_{j=0}^{\infty}\omega_3(j)q^{j+2} - \sum_{j=0}^{\infty}\omega_4(j)q^j + 6\sum_{j=0}^{\infty}\omega_5(j)q^{j+1} = 0,$$

In the above on collecting the powers of q^j, we get the relevant result. □

EXAMPLE 2.3. *For $j = 2$, below table authenticates the above theorem.*

$\omega_1(2) = 5:$	$1_w + 1_w, 1_b + 1_b, 1_w + 1_b, 2_w, 2_b$
$\omega_2(0) = 1:$	
$\omega_3(0) = 1:$	
$\omega_4(2) = 44:$	$1_w + 1 + w, 1_b + 1_b, 1_r + 1_r, 1_o + 1_o, 1_{bl} + 1_{bl}, 1_y + 1_y,$ $1_g + 1_g, 1_v + 1_v, 1_w + 1_b, 1_w + 1_c,$ $1_w + 1_o, 1_w + 1_{bl},$ $1_w + 1_y, 1_w + 1_g, 1_w + 1_v, 1_b + 1_r, 1_b + 1_o, 1_b + 1_{bl}, 1_b + 1_y,$ $1_b + 1_g, 1_b + 1_v, 1_r + 1_o, 1_r + 1_{bl}, 1_r + 1_y, 1_r + 1_g, 1_r + 1_v,$ $1_o + 1_{bl}, 1_o + 1_y, 1_o + 1_g, 1_o + 1_v, 1_{bl} + 1_y, 1_{bl} + 1_g, 1_{bl} + 1_v,$ $1_y + 1_g, 1_y + 1_h, 1_g + 1_v, 2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y, 2_g, 2_v$
$\omega_5(2) = 4:$	$1_w, 1_b, 1_r, 1_o$

THEOREM 2.4. *Let $\omega_1(j)$ serve as the total sum of partitions of j into components that are congruent with $\pm1, \pm2, \pm3, \pm4,$ $\pm5, \pm7, \pm8, +9$ modulo 18 by 6 colors, ±6 modulo 18 by 18 colors. If $\omega_2(j)$ is taken to identify the sum of partitions of j into items that are congruent with $\pm2, \pm4, \pm8$ modulo 18 by 6 colors, $\pm3, +9$ modulo 18 by 12 colors, ±6 modulo 18 by 18 colors. If $\omega_3(j)$ is taken to be the sum of partitions of j into elements congruent to $\pm1, \pm5, \pm7$ modulo 18 by 4 colors, $\pm2, \pm3, \pm4, \pm8, +9$ modulo 18 by 8 colors, ±6 modulo 18 by 16 colors. Let $\omega_4(j)$ imply the sum of partitions of j into components congruent with $\pm1, \pm5, \pm7$ modulo*

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18 by 2 colors, $\pm 2, \pm 4, \pm 8$ *modulo 18 by 4 colors,* $\pm 3, +9$ *modulo 18 by 10 colors,* ± 6 *modulo 18 by 20 colors. Then we have*

$$\omega_1(j-1) + \omega_2(j) = \omega_3(j) - 3\omega_4(j-1), \ j \geq 1.$$

Proof. From Theorem 3.1 of [4], we have if

$$M = \frac{f_3^2 f_6^2}{q^{1/2} f_1 f_2 f_9 f_{18}} \qquad \text{and} \qquad N = \frac{q^{1/6} f_2 f_3^2 f_{18}}{f_1 f_6^2 f_9}.$$

then,

$$N^3 + \frac{1}{N^3} = M - \frac{3}{M}.$$

Now on redrafting M and N in terms of f_n and then dividing completely by $f_1^3 f_2^3 f_3^6 f_6^6 f_9^3 f_{18}^3$ and simplifying to the prevalent base q^{18} and further employing (1), we deduce

$$\frac{q}{\left(q_6^{1\pm,2\pm,3\pm,4\pm,5\pm 7\pm,8\pm,9+}, q_{18}^{6\pm}; q^{18}\right)_\infty} + \frac{1}{\left(q_6^{2\pm,4\pm,8\pm}, q_{12}^{3\pm,9+}, q_{18}^{6\pm}; q^{18}\right)_\infty}$$
$$- \frac{1}{\left(q_4^{1\pm,5\pm,7\pm}, q_8^{2\pm,3\pm,4\pm,8\pm,9+}, q_{16}^{6\pm}; q^{18}\right)_\infty}$$
$$+ \frac{3q}{\left(q_2^{1\pm,5\pm,7\pm}, q_4^{2\pm,4\pm,8\pm}, q_{10}^{3\pm,9+}, q_{20}^{6\pm}; q^{18}\right)_\infty} = 0.$$

The atop identity executes the four generating functions and consequently, we have

$$\sum_{j=1}^{\infty} \omega_1(j) q^{j+1} + \sum_{j=1}^{\infty} \omega_2(j) q^{j} - \sum_{j=1}^{\infty} \omega_3(j) q^{j} + 3 \sum_{j=1}^{\infty} \omega_4(j) q^{j+1} = 0,$$

In the above on collecting the powers of q^j, we get the relevant result. □

EXAMPLE 2.4. *For $j = 2$, below table authenticates the above theorem.*

$\omega_1(1) = 6:$	$1_w, 1_b, 1_r, 1_o, 1_{bl}, 1_y$
$\omega_2(2) = 6:$	$2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y$
$\omega_3(2) = 18:$	$1_w + 1_w, 1_b + 1_b, 1_r + 1_r, 1_o + 1_o, 1_w + 1_b, 1_w + 1_r, 1_w + 1_o, 1_b + 1_r, 1_b + 1_o, 1_r + 1_o, 2_w, 2_b, 2_r, 2_o, 2_{bl}, 2_y, 2_g, 2_v$
$\omega_4(1) = 2:$	$1_w, 1_b$

References

[1] A. K. Agarwal, G. E. Andrews, Rogers-Ramanujan identities for partitions with "N copies of N", *J. Combin. Theory Ser. A*, **45(1)** (1987), 40-49.

[2] B. C. Berndt, *Ramanujan's Notebooks, Part III* Springer, New York, (1991).

[3] S. -S. Huang, On modular relations of Göllnitz-Gordan functions with applications to partitions, *J. Number theory* **66** (1998), 178-216.

[4] M. S. M. Naika, S. Chandankumar, B. Hemanthkumar, New modular relations for cubic class invariants, *Note di Matematica* **34**(2) (2014), 75–89.

[5] B. R. Srivatsa Kumar, R. G. Veeresha, Partition identities arising from Somos's theta function identities, *Ann Univ Ferrara* **63** (2017), 303-313.

[6] K. R. Vasuki, B. R. Srivatsa Kumar, Certain identities for Ramanujan-Göllnitz-Gordon continued fraction, *Journal of Computational and Applied Mathematics* **187** (2006), 87-95.

[7] R. R. Zhou, Some new identities for colored partition, *Ramanujan Journal* **40** (2016), 473-490.