International Journal of Applied Mathematics

Volume 38 No. 8s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The Topological Bi-complex valued metric space

Rabaa Almaitah

Department of Mathematics and Statistics,

Faculty of Science - Mutah University,

Al-Karak, P.O.Box 7, Jordan

Email: rabaamaita@mutah.edu.jo

Abstract

The separation axioms are examined within the context of bi-complex valued metric spaces. Additionally, the concepts of first countability, second countability, and separability are explored in detail for these spaces. Furthermore, compactness and Para compactness are analyzed alongside a specific example to illustrate these properties effectively.

Math. Subject Classification: 54D10 , 54D15, 32C15

Key Words and Phrases: Bi-complex metric space, Separation Axioms, Compactness, Paracompact, Separable, First countable, Second countable.

1 Introduction and Preliminaries

Choi J. et al., in their work referenced as [3], extended the concept of complex-valued metric spaces by introducing the notion of bi-complex-valued metric spaces. Within this framework, they established several sufficient conditions that guarantee the existence

Received: August 16 2025

International Journal of Applied Mathematics
Volume 38 No. 8s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

of common fixed points for pairs of mappings that adhere to contraction conditions. Their interest stemmed from the fact that bi complex numbers serve as a commutative substitute for the non commutative algebra of quaternion division, since both are four-dimensional in the real numbers.

Furthermore, in many ways, those numbers are a more refined generalization of complex numbers than quaternions are. Of course, the ring of bi-complex numbers is not a field due to the presence of zero-divisors which make such structure impossible does limit gain in commutativity. Still, it would be reasonable to assume that bi-complex numbers would work well as scalar quantities in function theory and functional analysis, providing at the very least a rough substitute to quaternions (see [2]). The most complete treatment of analysis in bi-complex form is G.B. Price's book [7] where he studied these bi-complex numbers in greater detail, especially the functions defined on the ring of bi-complex numbers which behave like holomorphic functions of a complex variable.

Although the algebra of bi-complex numbers is a four-dimensional real algebra, a more useful perspective is that it serves as a "complexification" of the complex number field. From this angle, one may think of bi-complex algebra as possessing the attributes of a one-dimensional theory situated within four real dimensions. Following that, some papers focused on certain results concepts of bi-complex spaces (For more details see [1] and [4]).

Bi-complex numbers is denoted by:

$$\mathfrak{B}\mathcal{C} = \{\zeta : \ \zeta = \eta + jv \quad where \ \ \eta = \eta_1 + i\,\eta_2 \ \ and \ \ v = v_1 + i\,v_2 \in \mathbb{C}\}$$

which is a generalization of complex numbers. Also, bi-complex number ζ is written in the form

$$\begin{aligned}\zeta = \eta_1 + i\ \eta_2 + j\ \ (v_1 + i\ v_2) &= \eta_1 + i\ \eta_2 + j\ v_1 + ji\ v_2\\ &= \eta_1 + i\ \eta_2 + j\ v_1 + k\ v_2\end{aligned}$$

where $i,\ j$ and k satisfies $i^{\,2} = j^{\,2} = -1, k^{\,2} = 1\ , i\,j = j\,i = k$ and $\omega_1, \omega_2, v_1, v_2 \in \mathbb{R}$.

Received: August 16 2025

International Journal of Applied Mathematics
Volume 38 No. 8s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

For each $\zeta \in \mathfrak{B}\mathcal{C}$ the norm is defined by

$$||\zeta|| = \sqrt{\eta_1{}^2 + \quad \eta_2{}^2 + \upsilon_1{}^2 + \upsilon_2{}^2}$$

This norm satisfies the usual properties of a norm: non-negativity, positive definiteness homogeneity, and triangle inequality.

Definition 1.1. [3] For any $\xi = \xi_1 + j\ \xi_2 \in \mathfrak{B}\mathcal{C}$ and $\omega = \omega_1 + j\omega_2 \in \mathfrak{B}\mathcal{C}$. Define a partial order relation $\precsim_k$ on $\mathfrak{B}\mathcal{C}$ as follows:

$$\xi \precsim_k w \Longleftrightarrow \xi_1 \preccurlyeq_i \omega_1 \ \ and \ \ \xi_2 \preccurlyeq_i \omega_2,$$

i.e., $\xi \precsim_{ij} \omega$ if one of the following conditions is satisfied:

(i) $\xi_1 = \omega_1$ and, $\xi_2 = \omega_2$

(ii) $\xi_1 \prec_i \omega_1$ and $\xi_2 = \omega_2$,

(iii) $\xi_1 = \omega_1$ and $\xi_2 \prec_i \omega_2$

(iv) $\xi_1 \prec_i \omega_1$ and $\xi_2 \prec_i \omega_2$

We can write $\xi \precnsim_{ij} \omega$ if $\xi \precsim_{ij} \omega$ and $\xi \neq \omega$,i.e., one of (ii),(iii) and (iv) is satisfied , and we will write $\xi \prec_k \omega$ if only (iv) is satisfied .

Remark.For any two bi-complex numbers $\xi, \omega \in \mathfrak{B}\mathcal{C}$, one can easily verify the following:

(i) If $0 \precsim_{ij} \xi \precsim_{ij} \omega$,then $||\xi|| \leq ||\omega||$

(ii) $||\xi + \omega|| \leq ||\xi|| + ||\omega||$

(iii) $||\alpha\xi|| \leq \alpha||\xi||$, where α is a non-negative real number.
(see [3], for more details).

Definition 1.2. [2] Let Υ be a nonempty set. Suppose the mapping $\mathfrak{H} : \Upsilon \times \Upsilon \to \mathfrak{B}\mathcal{C}$ satisfies the following conditions for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{s} \in \Upsilon$

(i) $\mathfrak{H}(\mathfrak{p}, \mathfrak{q}) \succsim_{ij} 0$

International Journal of Applied Mathematics
Volume 38 No. 8s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

(ii) $\mathfrak{H}(\mathfrak{p},\mathfrak{q}) = 0$ if and only if $\mathfrak{p} = \mathfrak{q}$

(iii) $\mathfrak{H}(\mathfrak{p},\mathfrak{q}) = \mathfrak{H}(\mathfrak{q},\mathfrak{p})$

(iv) $\mathfrak{H}(\mathfrak{p},\mathfrak{q}) \precsim_{ij} \mathfrak{H}(\mathfrak{p},\mathfrak{s}) + \mathfrak{H}(\mathfrak{s},\mathfrak{q})$

Then bi-complex valued metric space is denoted by $(\Upsilon,\mathfrak{H})$, or more simply by Υ. Sometimes metric space $(\Upsilon,\mathfrak{H})$ together with a partial order relation $\precsim_k$ on Υ is called partially ordered bi-complex valued metric space.

Example 1.1. *Let $\Upsilon = \mathbb{R}$ together with the mapping $\mathfrak{H}_i \Upsilon \times \Upsilon \to \mathfrak{B}\mathcal{C}, i = 1, 2$, for every $x, y \in \Upsilon$ be defined by*

1. $\mathfrak{H}_1(x,y) = (i + j + K)|x - y|$
2. $\mathfrak{H}_2(x,y) = k|x - y|$.

Thus $\Upsilon_{\mathfrak{H}_1}$ and $\Upsilon_{\mathfrak{H}_2}$ are a bi-complex valued metric spaces.

Definition 1.3. [2] Let $\{\zeta_n\}$ be a sequence in a bicomplex-valued metric space $(\Upsilon,\mathfrak{H})$, then

(i) A sequence $\{\zeta_n\}$ converges to $\zeta_0 \in \Upsilon$ if and only if for any $0 \precsim_{i_2} \varepsilon \in \mathfrak{B}\mathcal{C}$, there exists $N \in \mathbb{N}_{\varepsilon}$ such that $\mathfrak{H}(\zeta_n,\zeta_0) \prec_{ij} \varepsilon$, for all $n > N$. It is denoted by $\zeta_n \to \zeta$ as $n \to +\infty$ or $\lim_{n\to+\infty} \zeta_n = \zeta$.

(ii) A sequence $\{\zeta\}$ is Cauchy sequence if and only if for any $0 \prec_{ij} \varepsilon \in \mathbb{BC}$ there exists $N \in \mathbb{N}_{\varepsilon}$ such that $\mathfrak{H}(\zeta_n,\zeta_m) \prec_{ij} \varepsilon,$, for all $n, m > N$.

(iii) $(\Upsilon,\mathfrak{H})$ is said to be complete bi-complex valued metric space if and only if every Cauchy sequence converges in Υ.

Moving forward, we will revisit some lemmas that are essential to our discussion.

Lemma 1.1. *[2] Let $(\Upsilon,\mathfrak{H})$ be a bi complex valued metric space and $\{\zeta_n\}$ be a sequence in $\mathbb{BC}$. Then $\{\zeta_n\}$ is a Cauchy sequence if and only if $||\mathfrak{H}(\zeta_n,\zeta_{n+m})|| \to 0$ as $n \to +\infty$, $m \in \mathbb{N}$.*

International Journal of Applied Mathematics

Volume 38 No. 8s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Lemma 1.2. *[2] Let $(\Upsilon, \mathfrak{H})$ be a bi-complex valued metric space and $\{\zeta_n\}$ be a sequence in Υ. Then $\{\zeta_n\}$ converges to $\zeta \in \Upsilon, \Longleftrightarrow$ $||\mathfrak{H}(\zeta_n, \zeta)|| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Lemma 1.3. *[2] Let $(\Upsilon, \mathfrak{H})$ be a bi-complex valued metric space and $\{\zeta_n\}$ be a sequence in Υ such that $\lim_{n\to\infty}\zeta_n = a$. Then for any $a \in X$, $\lim_{n\to\infty} ||\mathfrak{H}(\zeta_n, a)|| = ||\mathfrak{H}(\zeta, a)||$.*

2 The main results

To demonstrate that a bi-complex-valued metric space generates a topology, it is essential to formally define the concept of open sets in relation to the given metric $\mathfrak{H}$ Subsequently, one must rigorously verify that these open sets adhere to the fundamental axioms required to establish a topological structure. Define open ball centered at $\mathfrak{p} \in \Upsilon$ with radius $r \in \mathfrak{B}\mathcal{C}$ as

$$\wp(\mathfrak{p},\ r) = \{\mathfrak{q} \in \Upsilon : \mathfrak{H}\,(\mathfrak{p}, \mathfrak{q}) \prec_k r \ and \ \ ||\mathfrak{H}\,(\mathfrak{p}, \mathfrak{q})\,|| < ||r||\}$$

To define the topological bi-complex valued metric space $\mathfrak{J}$, let $\mathfrak{J}$ be the collection of open sets of Υ for which $\mathfrak{U} \in \mathfrak{J}$ iff for every $\mathfrak{p} \in \mathfrak{U}$, there exists $r \succ_k 0$ such that $\wp\,(\mathfrak{p},\ r) \subseteq \mathfrak{U}$ i.e. $\mathfrak{J}$ consists of all unions of open balls, the empty set $\phi, \Upsilon \in \mathfrak{J}$ since they are expressed in the form of unions of open balls. Also, if $\mathfrak{U}_i$ for $i \in \mathbb{N}$ is a collection of open sets then $\cup_{i\in\mathbb{N}}\mathfrak{U}_i$ is again open i.e.by the definitions of open sets as unions of open balls. Moreover, if $\mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{J}$ then $\mathfrak{U}_1 \cup \mathfrak{U}_2 \in \mathfrak{J}$, since for any $\mathfrak{p} \in \mathfrak{U}_1 \cup \mathfrak{U}_2$ there exists $r_1, r_2 \succ_k 0$ such that $\wp\,(\mathfrak{p}, r_1) \subseteq \mathfrak{U}_1$ and $\wp\,(\mathfrak{p}, r_2) \subseteq \mathfrak{U}_2$ by taking $r = \min\{r_1, r_2\}$ so that $\wp\,(\mathfrak{p}, r) \subseteq \mathfrak{U}_1 \cup \mathfrak{U}_2$. So that the axioms of topology $\mathfrak{J}$ are satisfied.

Theorem 2.1. *Every bi-complex valued metric space $(\Upsilon, \mathfrak{H})$ is T_2 (Hausdorff) space.*

Proof. Let $\mathfrak{p}, \mathfrak{q}$ be two distinct points in Υ. For a given point $\mathfrak{p}$, define the open ball centered at $\mathfrak{p}$ with radius $r \in \mathbb{BC}$ as:

$$\wp(\mathfrak{p}, r) = \{t \in \Upsilon : \mathfrak{H}\,(\mathfrak{p}, t) \prec_k r\}.$$

Received: August 16 2025

International Journal of Applied Mathematics

Volume 38 No. 8s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Also, for $\mathfrak{q} \in \Upsilon$ with radius $a \in \mathbb{BC}$ define the open ball

$$\wp(\mathfrak{q}, a) = \{ \ t \in \Upsilon : \ \ \mathfrak{H}(\mathfrak{q}, \ t) \prec_k a \ \ \}$$

If there exists a point $s \in \wp(\mathfrak{p}, r) \cap \wp(\mathfrak{q}, a)$, then $\mathfrak{H}(\mathfrak{p}, t) \prec_k r$ and $\mathfrak{H}(\mathfrak{q}, \ t) \prec a$. So, by triangle inequality we have

$$\mathfrak{H}(\mathfrak{p}, \mathfrak{q}) \preccurlyeq_k \ \mathfrak{H}(\mathfrak{p}, t) + \mathfrak{H}(t, \ \mathfrak{q}) \prec_k r + a$$

But by our selection of r and a, we have

$$r + a \prec_k c = \mathfrak{H}(\mathfrak{p}, \mathfrak{q})$$

which leads us to a contradiction, since $\mathfrak{H}(\mathfrak{p}, \mathfrak{q}) \prec_k \mathfrak{H}(\mathfrak{p}, \mathfrak{q})$. □

Theorem 2.2. *Every bi-complex valued metric space $(\Upsilon, \mathfrak{H})$ is T_4 space.*

Proof. To prove the normality of Υ. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two closed disjoint sets in Υ. For each $v \in \mathfrak{S}_1$ and $u \in \mathfrak{S}_2$, there exists $\lambda_v \succ_k 0$ and $\lambda_u \succ_k 0$ respectively, such that $\mathfrak{V}(v, \lambda_v) \bigcap \mathfrak{S}_2 = \emptyset$ (since $\mathfrak{S}_2$ is closed, $v \notin \mathfrak{S}_2$) and $\mathfrak{V}(u, \ \lambda_u) \bigcap \mathfrak{S}_1 = \emptyset$ (since $\mathfrak{S}_1$ is closed, $u \notin \mathfrak{S}_1$).
Define the two open sets : $\Theta_1 = \bigcup_{j \in \Upsilon} \mathfrak{V}(v, \frac{\lambda_v}{2})$ and $\Theta_2 = \bigcup_{i \in \Upsilon} \mathfrak{V}(u, \frac{\lambda_u}{2})$
To prove that Θ_1 and Θ_2 are two disjoint sets:
let $y \in \Theta_1 \cap \Theta_2$ then there exist $v \in \mathfrak{S}_1$ and $u \in \mathfrak{S}_2$ such that $y \in \mathfrak{V}(v, \ \frac{\lambda_v}{2})$ and $y \in \mathfrak{V}(u, \frac{\lambda_u}{2})$. Then $\mathfrak{H}(v, \mathfrak{y}) \prec_k \frac{\lambda_v}{2}$ and $\mathfrak{H}(u, y) \prec_k \frac{\lambda_u}{2}$, Which implies

$$\mathfrak{H}(u, v) \prec_k \mathfrak{H}(v, y) + \mathfrak{H}(y, u) \prec_k \frac{\lambda_v}{2} + \frac{\lambda_u}{2}$$

However, for $v \in \mathfrak{S}_1$ and $u \in \mathfrak{S}_2$ we get :

$$\mathfrak{H}(u, v) \succcurlyeq_k \lambda_v$$

or

$$\mathfrak{H}(u, v) \succcurlyeq_k \lambda_u$$

International Journal of Applied Mathematics
Volume 38 No. 8s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

since $\mathfrak{V}(v, \lambda_v) \bigcap \mathfrak{S}_2 = \emptyset$ and $\mathfrak{V}(u, \lambda_u) \bigcap \mathfrak{S}_1 = \emptyset$, which contradicts the fact:

$$\lambda_v \preccurlyeq_k \frac{\lambda_v}{2} + \frac{\lambda_u}{2}$$

is impossible and thus $\Theta_1 \cap \Theta_2 = \emptyset$.
So that it is shown that for any two closed disjoint sets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ there exist two open disjoint sets Θ_1 and Θ_2 such that $\mathfrak{S}_1 \subseteq \Theta_1$ and $\mathfrak{S}_2 \subseteq \Theta_2$. Thus the bicomplex valued metric space $(\Upsilon, \mathfrak{H})$ is T_4 □

Theorem 2.3. *Every bi-complex valued metric space* $(\Upsilon, \mathfrak{H})$ *is* T_6 *space.*

Proof. To prove that every closed set is a $\mathfrak{S}_o$ set, for any closed set $\mathfrak{S}_o \subseteq X$, where $\mathfrak{S}_o = \bigcap_{i=1}^{\infty} \mathcal{O}_i$ where

$$\mathcal{O}_i = \{x \in \Upsilon : \mathfrak{H}(\mathfrak{p}, \mathfrak{q}) \prec_k \frac{\alpha}{n} \ \ \textit{and} \ \ ||\alpha|| < 1\}$$

and the space is perfectly normal T_4 □

Corollary 2.1. *The bi-complex valued metric space* $(\Upsilon, \mathfrak{H})$ *satisfies the axioms of separations* $T_0, \ T_1, \ T_{2.5}\,, \ T_3, \ T_{3.5}$ *and* T_5 .

Theorem 2.4. *A bi-complex valued metric space* $(\Upsilon, \mathfrak{H})$ *is*

(i) First countable

(ii) Second countable

(iii) Separable

Proof. (i) for any point $v \in \Upsilon$ has a countable sequence of neighborhood base $\mathcal{B}_v = \{\mathfrak{V}(v, \ \frac{1}{\mathfrak{m}}) : \mathfrak{m} \in \mathbb{N}\}$ which is countable for every point. Or it is easier by $\mathfrak{B}\mathcal{C} \cong \mathbb{R}^4$

(ii) Define a collection of open balls with rational bi-complex radii and bi-complex centers in countable dense subsets of Υ . Let $\mathcal{A}$ be the countable dense subset of Υ, then the closure of $\mathcal{A}$

International Journal of Applied Mathematics
Volume 38 No. 8s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

is Υ. For $v \in \mathscr{A}$ there exist $r \in \mathfrak{B}\mathcal{C}$ such that $r \succ_k 0$ and $\mathfrak{V}(v,r) \subset \mathfrak{B}\mathcal{C}$ is the open ball. The collection of open balls forms a base for a topology, so that for an open set Θ that contains v there exist $r \succ_k 0$ such that $\mathfrak{V}(v,r) \subset \Theta$. But $\mathscr{A}$ is a dense subset in Υ, then there exists $u \in \mathscr{A}$ such that $\frac{\lambda}{2} \prec_k r \prec_k \lambda$. Thus $\mathfrak{V}(u,r) \subseteq \mathfrak{V}(v,\lambda) \subseteq \Theta$, since for any $\mathfrak{s} \in \mathfrak{V}(u,r)$:

$$\mathfrak{H}(v,\mathfrak{s}) \prec_k \mathfrak{H}(v,u) + \mathfrak{H}(u,\mathfrak{s}) \prec_k \frac{\lambda}{2} + r \prec_k \lambda$$

Hence for any arbitrary element we obtain $v \in \mathfrak{V}(u,r) \subseteq \Theta$, for every open set Θ that can be written as a union of open balls with centers and rational radii in $\mathscr{A}$. So that the collection of open balls is countable, since $\mathscr{A}$ is countable.

(iii) To prove that a bi-complex valued metric space is separable, we must find countable subset $\mathscr{A} in \Upsilon$ such that the closure of $\mathscr{A}$ is Υ. Consider $\mathscr{A} = \{\mathfrak{r} = \mathfrak{r}_1 i + \mathfrak{r}_2 j + \mathfrak{r}_3 k$, where $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3 \in \mathbb{Q}\}$ which is countable set with a closure equal to Υ

□

Theorem 2.5. *Let $\Upsilon_{\mathfrak{H}}$ be a bi-complex valued metric space. Then it is connected*

Proof. Assume by contradiction that $\Upsilon_{\mathfrak{H}}$ is disconnected, then there exist two disjoint open sets $\mathfrak{U}_1$ and $\mathfrak{U}_2$ such that $\mathfrak{U}_1 \cap \mathfrak{U}_2 = \emptyset$ and $\mathfrak{U}_1 \cup \mathfrak{U}_2 = \Upsilon_{\mathfrak{H}}$. Consider a point $\mathfrak{p} \in \mathfrak{U}_1$ and a point $\mathfrak{q} \in \mathfrak{U}_2$ then there exist open balls $\wp(\mathfrak{p}, r_1) \subseteq \mathfrak{U}_1$ and $\wp(\mathfrak{q}, r_2) \subseteq \mathfrak{U}_2$ with radii $r_1, r_2 \succ_k 0$. Suppose $\psi : [0,1] \to \Upsilon$ such that $\psi(\mathfrak{t}) = \mathfrak{t}\mathfrak{p} + (1-\mathfrak{t})\mathfrak{q}$ is a continuous path from $\mathfrak{p}$ to $\mathfrak{q}$ in bi-complex valued metric space $\Upsilon_{\mathfrak{H}}$. Also, by the continuity of ψ we have $\psi^{-1}(\mathfrak{U}_1)$ and $\psi^{-1}(\mathfrak{U}_2)$ are open sets in [0,1] since $\psi(0) = \mathfrak{q} \in \mathfrak{U}_2$ and $\psi(1) = \mathfrak{p} \in \mathfrak{U}_1$, so that $0 \in \psi^{-1}(\mathfrak{U}_2)$ and $1 \in \psi^{-1}(\mathfrak{U}_1)$. Hence the two sets $\psi^{-1}(\mathfrak{U}_1)$ and $\psi^{-1}(\mathfrak{U}_2)$ are disjoint and their union is [0,1] which is a contradiction since the connected space can't be separated into two disjoint nonempty open sets. □

Received: August 16 2025

International Journal of Applied Mathematics
Volume 38 No. 8s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

It is known that the bi-complex valued metric space $(\Upsilon, \mathfrak{H})$ is compact if for every cover of Υ there is a finite sub cover (or every sequence has a convergent subsequence).
A bi complex-valued metric space does not have to be compact.

Example 2.1. *The space $(\Upsilon, \mathfrak{H})$, defined by $\mathfrak{H}(\mathfrak{v},\ u) = (\mathfrak{v} - u)\, ij$ is a bi-complex valued metric. Consider that the sequence $\mathfrak{v}_n = (i + j + ij)n$ then, $\mathfrak{H}(\mathfrak{v}_n, 0) = \mathfrak{v}_n = (i + j + ij)\ n$ as $n \longrightarrow \infty$ thus, the sequence goes to infinity in the bicomplex plane, and it is not convergent to a point in $\mathfrak{B}C$ and this means that it has not a convergent subsequence.*

Theorem 2.6. *[8] Every compact metric space is complete.*

So that compactness of bi-complex valued metric spaces implies completeness, but the covers is not true. To discuss what are the necessary and sufficient conditions for the bi-complex valued metric space to be compact, let us define the following.

Definition 2.1. A bi-complex valued metric space $\Upsilon_{\mathfrak{H}}$ is called entirely bounded if for every $\gamma \succ_k 0$ there exists a finite number of points $v_1 v_2, \ldots, v_n \in \Upsilon$ such that $\Upsilon \subseteq \bigcup_{i=1}^{n} \mathfrak{V}(v_i, \gamma)$.

A standard metric space, specifically a real-valued metric space, is compact if and only if it is both complete and entirely bounded. This condition represents a broader generalization of the Heine – Borel Theorem. According to this theorem, any closed and bounded subset S of $\mathbb{R}^n, n \geq 1$ is compact and therefore complete.

Theorem 2.7. *The bi-complex valued metric space $(\Upsilon, \mathfrak{H}$ is compact if it is entirely bounded and closed.*

Proof. Since Υ is an entirely bounded then any sequence $\{\zeta_i\}_{i=1}^{\infty}$ in Υ contains a Cauchy subsequence by repeatedly subdividing the space into smaller balls. So that we can consider the subsequence $\{\zeta_{i_k}\}_{k=1}^{\infty}$ in the bi-complex valued metric space Υ which is convergent to some point $\zeta \in \bar{\Upsilon}$ (the closure of Υ), since Υ is a closed $(\Upsilon = \bar{\Upsilon})$. The original sequence $\{\zeta_i\}_{i=1}^{\infty}$ is a Cauchy

International Journal of Applied Mathematics

Volume 38 No. 8s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

convergent sequence and has a convergent subsequence $\{\zeta_{i_k}\}_{k=1}^{\infty}$ that converges to ζ, so it also converges to ζ. To see this, for $\rho \succ_k 0$ there exists $\mathcal{M}_1$ such that $p,\ q > \mathcal{M}_1$ and $\mathfrak{H}(\zeta_p,\ \zeta_q) \prec_k \frac{\rho}{2}$ also, since $\{\zeta_{i_k}\}_{k=1}^{\infty} \longrightarrow \zeta$ there exists $\mathcal{M}_2$ such that $p_k > \mathcal{M}_2$ and $\mathfrak{H}(\zeta_{p_k}, \zeta) \prec_k \frac{\rho}{2}$. Choose $p_k > \max\{\mathcal{M}_1, \mathcal{M}_2\}$. Then for every $p > p_k$: $\mathfrak{H}(\zeta_p, \zeta) \prec_k \mathfrak{H}(\zeta_p, \zeta_{p_k}) + \mathfrak{H}(\zeta_{p_k}, \zeta) \prec_k \frac{\rho}{2} + \frac{\rho}{2} = \rho$. Thus $\{\zeta_i\}_{k=1}^{\infty} \longrightarrow \zeta$. Hence the space Υ is complete. So that the complete entirely bounded space Υ is compact space. $\square$

Also, from Theorem 2.6 and Theorem 2.8, we obtain the next result.

Corollary 2.2. *The bi-complex valued metric space* $(\Upsilon, \mathfrak{H})$ *is compact if and only if it is entirely bounded and complete space.*

A weaker topological concept than a compactness is called para compactness.

Definition 2.2. A bi-complex valued metric space $(\Upsilon, \mathfrak{H})$ is called paracompact topological space if for every collection of open sets $\{\mathcal{U}_p\}$ that covers the space, there exists another collection of open sets $\{\mathcal{V}_q\}$ such that:

(i) $\{\mathcal{U}_p\}$ is a refinement of $\{\mathcal{V}_q\}$.

(ii) $\{\mathcal{U}_p\}$ is locally finite, meaning every point in space has a neighborhood that intersects only finitely many $\mathcal{U}_p$.

Thus, every compact space can be classified a s paracompact; however, the reverse implication does not necessarily hold. Also, Para compactness requires the existence of a locally finite refinement, which is a weaker condition than the existence of a finite subcover that required by compactness. A finite subcover is always locally finite, but a locally finite refinement may not be finite.

Theorem 2.8. *Every bi-complex valued metric space* $(\Upsilon, \mathfrak{H})$ *is paracompact.*

Received: August 16 2025

International Journal of Applied Mathematics
Volume 38 No. 8s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Proof. The proof follows from the fact that bi-complex valued metric space is metrizable, and every metrizable is paracompact. □

References

[1] Bekele G., and Geremew A.some generalized fixed point results on compact metric spaces,*Ethiop. J. Edu. & Sc.*,(2014), 53-61.

[2] Cerroni C, *The theory of congeneric surd equations to Segre's bicomplex numbers*, Historia Mathematica,(2017);232-251.

[3] Choi J., Datta S., Biswas.T,& N. Islam, Some fixed-point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces, *Honam. Math. J*, **39** (2017), 115-126.

[4] Gnanaprakasam A. J., Boulaaras S. M. Mani G, Cherif B. and Idris S. A.,*Solving system of linear equations via bi complex valued metric space*, Demonstration Mathematica (2021); 54: 474–487.

[5] Mani G., Haque S., Gnanaprakasam, A.J., Ege, O., Mlaiki, N., *The Study of Bi Complex-Valued Controlled Metric Spaces with Applications to Fractional Differential Equations*, Mathematics (2023), 11, 2742.

[6] Namdev R., Tiwari R., Singh U., and Bhardwaj R., *Coupled Invariant Point Theorems in Bicomplex Parametric Partial Metric Space Amongst Dominate F* , Comm. in Math. and Appli (2024), 15(2): 511–524.

[7] Price G.*An Introduction to Multicomplex Spaces and Functions*, New York (1991).

[8] Royden H. and Fitzpatrick P., *Real Analysis*, China Machine Press (2010) 4th Ed.

Received: August 16 2025