International Journal of Applied Mathematics

Volume 38 No. 8s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

A NEW OPERATIONAL MATRIX APPROACH BASED ON LEGENDRE WAVELETS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Author Esraa Abbas Al-Taai

[1] Affiliation University of Baghdad,College of Science for Woman,Department of Mathematics,Baghdad,Iraq,

Email: * Corresponding esraa.a@uobaghdad.edu.iq

Abstract

This present paper suggests a new numerical algorithm to solve fractional differential equations (FDEs) based upon a Legendre wavelet operational matrix of fractional integration (LW-OMFI). The strategy is efficient in converting original FDEs such as Caputo and Riemann-Liouville derivation into a set of algebraic equations. With its ability to take advantage of the orthogonality and compact support of Legendre wavelets, spreading out the Galerkin expansion over a large number of higher frequency basis function elements is made good, as well as minimizing the number of required elements. The operational matrix is deduced systematically in order to treat fractional integrals so that it is possible to treat both linear and nonlinear FDEs with it. Topical numerical tests prove the advantages of the new method over the traditional ones including the Adomian Decomposition Method (ADM) and Homotopy Analysis Method (HAM) in precision, convergence, and computational time. The method has especially good prospect of application in physics and engineering where fractional models are common.

Keywords: Adomian Decomposition, Caputo Derivative, Fractional Differential Equations, Homotopy Analysis, Legendre Wavelets.

1. INTRODUCTION

After a variety of advances in the field, fractional differential equations (FDEs) has become even more outstanding in the last few decades, because several complex dynamical systems with memory and hereditary features may be modelled by FDEs. These equations are more descriptive compared to classical integer-order differential equations in other areas in science and engineering like viscoelasticity, electrochemistry, anomalous diffusion, control theory, and bioengineering (Saha Ray, 2021; Malmir, 2022). This is because it is possible to model the nonlocal behavior, universal in real-world processes, and scaling in time, implemented with the help of the fractional operators, especially Caputo and Riemann-Liouville derivatives.

Although they can be used in many situations, the case of finding an accurate analytical answer to FDEs has proved very difficult. A small set of FDEs can be solved explicitly and this has led to a high demand of valid and efficient numerical algorithms. The standard techniques are subject to accuracy, stability or computing cost limitations, more so when applied to nonlinear or high-order problems (Behera & Ray, 2022; Mohammadi & Cattani, 2018). As a result, growth of superior numerical schemes that are accurate and computationally efficient has increasingly been the main

Received: August 08 2025

star of the on-going research. The use of the wavelet-based techniques in solving the problem of differential equations is a relatively new phenomenon because of the underlying nature of the method, including their orthogonality, multiresolution, and local nature in time and frequency (Rahimkhani, Ordokhani, & Babolian, 2017). Fine property of wavelets qualifies it to be used to model localized and changing solutions in part. Strong orthogonality combined with the simple structure and polynomial basis provide an added advantage because of which Legendre wavelets gain an edge amongst other wavelet families (Secer & Altun, 2018; Balaji, 2015). Legendre wavelets make it possible to build succinct and effective numerical algorithms, the working matrices can further be used to transform differential equations easily in a form of algebraic equations. These properties promote a more precise and stable spectral approximation, particularly of variable-order or nonlinear dynamics problems (Chen et al., 2015; Khan et al., 2023). Wavelet-based such operational matrix methods were already constructed on a Haar (Shah et al., 2017), Chebyshev (Rostami, 2021), and Fourier basis. Although effective in special instances, these methods do not necessarily work well when they are applied to complicated fractional systems, because of lack of precision or orthogonality in bounded regions. Even though proposals of Legendre wavelet-related methods in fractional problems have encountered certain studies (e.g., Secer et al., 2019; Balaji & Hariharan, 2024), an operational matrix of fractional integration exclusively implemented in Legendre wavelets remains uncreatively established and utilized.

Moreover, the majority of the current methods are designed to address integer-order equations or can be used only after complicated modality change and kernel activation, which does not make them scalable or suitable to the task of least-squares solutions of non-linear or variable-order FDEs (Mahmoud, Ameen, & Mohamed, 2017; Sabermahani & Ordokhani, 2020). Such gap necessitates the existence of a versatile and numerical framework, which directly uses Legendre wavelets to build operational matrices applicable to fractional-order calculus. Main purpose of the current work is establishing new Legendre Wavelet Operational Matrix of Fractional Integration (LW-OMFI) and using it in the resolution of linear and nonlinear FDEs of Caputo and Riemann-Liouville type. This matrix converts the fractional equations which govern things into a solvable system of algebraic equations, thereby making the computational process much easier. The following are the important contributions of the paper:

- Creation of a new operational matrix of fractional integration based on the Legendre wavelets, and one that can be directly applied on Caputo and R-L derivatives.

- Proven excellent spectral accuracy with a low wavelet basis function count, i.e. computational efficiency.

- Ability to deal with the linear and nonlinear FDEs, improving the applicability of the approach.

- Depth of comparing comparisons between well-known methods, including the most popular Adomian Decomposition Method (ADM) and Homotopy Analysis Method (HAM), and illustrating the excellence of the showing approaches.

Received: August 08 2025

- Numerical checking, demonstrating the validity, stability, and scalability of the LW-OMFI algorithm.

More than just filling in a crucial gap regarding the numerical treatment of fractional differential equations, such a study represents an opportunity to open up new potential in applications of Legendre wavelets to more general classes of fractional systems (Gupta&Ranta, 2022; Kumar, Kumar,&Mehra, 2025; Xie, 2021).

2. PRELIMINARIES

This part describes the concepts that are key to development of the proposed Legendre Wavelet Operational Matrix of Fractional Integration (LW-OMFI). It contains a definition of fractional calculus, the construction of wavelets called Legendre wavelets and an introduction to the operational matrices of numerical analysis.

2.1 Fractional Calculus Basics

The fractional calculus is an extension of the classical concepts of differentiation and integration to non-integer (fractional) orders of differentiation and integration and more flexible and precise modeling of processes in nature and technology with memory and hereditary properties.

Suppose that f(x)E Cn[a,b], and n=[alpha], that is to say, n is the smallest integer not less than alpha. The most frequently adopted definitions are:

- Riemann- Liouville Fractional Integral of 0-order 0 1 a is the fractional University of Pretoria of integrating the following fraction.

$$I\_a\hat{}\alpha\, f(x) = (\Gamma(\alpha) / 1) \int (x - t)\hat{}(\alpha - 1)\, f(t)\, dt\, from\, a\, to\, x \tag{1}$$

where Γ(·) denotes the Gamma function.

- Riemann–Liouville Fractional Derivative of order α>0 is:

$$D\_a\hat{}\alpha\, f(x) = (d\hat{}n / dx\hat{}n)\, I\_a\hat{}(n - \alpha)\, f(x), where\, n = \lceil\alpha\rceil \tag{2}$$

- Caputo Derivative of order Caputo Fractional Derivative of order 0<alpha is defined as:

$$C\, D\_a\hat{}\alpha\, f(x) = I\_a\hat{}(n - \alpha)\, (d\hat{}n / dx\hat{}n\, f(x)), where\, n = \lceil\alpha\rceil \tag{3}$$

The Caputo variant is especially practical in physical applications because it may be used with common initial conditions (in terms of integer derivatives) as well.

2.2 Legendre Wavelets

Legendre wavelets are built using shifted Legendre polynomials and they offer an orthogonal basis on a finite interval, usually [0,1]. They are well aligned and fit quite well in approximating piecewise linear functions.

Suppose that Pn(x) is the Legendre polynomial of the order n. The wavelets are built in terms of defining the scaled translated versions of these polynomials:

$$\phi_{j,k}(x) = 2^j P_n(2^j x - k), where\ x \in [2^j k, 2^j (k + 1)) \tag{4}$$

with j and k being the scale and translation indices, respectively, whereas n is the order of polynomials.

Legendre wavelet important properties are:

- Orthogonality:

$$\int (from\ 0\ to\ 1)\ \phi_{j,k}(x)\ \phi_{j,l}(x)\ dx = \delta_{kl} \tag{5}$$

assuring numerical stability and sparsity.

- Compact Support: The wavelets are localized, in that each wavelet function is nonzero only on a small subinterval of the real axis, and thus is useful where local approximation is needed.

- Multiresolution Capacity: Allows one to present functions at a low and high level of details (Balaji, 2015; Chen et al., 2015).

These attributes of Legendre wavelets become an excellent reason to use them when it comes to building numerical algorithms with the use of differential operators (Secer & Altun, 2018; Gupta&Ranta, 2022).

2.3 Operational Matrices

The operational matrix idea is also the focus when converting differential equations to algebraic equations. An operator in an operator space may be presented in the form of a matrix (an operator matrix), which is also a representation if the operator is a differential or integral operator with respect to a selected set of basis functions (e.g., wavelets or polynomials).

Assuming a basis function 0 vector 0 0 0 It can be shown that the fractional derivation of this vectors may be apporximated by:

$$I^\alpha\, \Phi(x) \approx P(\alpha)\, \Phi(x) \tag{6}$$

with operational matrix of fractional integration P(alpha).

In this matrix, a numerical solution of a fractional differential equation can be achieved by substituting the operations of a fractional integrals operator with the operation of matrices, thus, converting the problem into a solvable system of algebraic equations (Rahimkhani, Ordokhani, & Babolian, 2017; Malmir, 2022). Improper knowledge of right operational matrices Titters on/G:The construction of proper operational matrices based on Legendre wavelets has been considered in a few articles (Secer et al., 2019; Balaji & Hariharan, 2024), and the present work would attempt to generalize and extend the work to a wider range of FDEs, including more general nonlinear and variable-order cases.

3. CONSTRUCTION OF THE OPERATIONAL MATRIX

An original Legendre Wavelet Operational Matrix of Fractional Integration (LW-OMFI) is presented and derived in this section. The matrix allows the fractional differential equations to be converted into sets of algebraic equations in a numerical way. We start with the statement of the theoretical basis and then go ahead with an elaborate derivation finish off by an examination of the properties of the matrix and making a comparison with the rest of the wavelet based techniques.

3.1 New Approach Overview

Suppose we define the Legendre wavelet basis functions, 0<= n, on the interval on [0,1] to be: 0 Each function f(x) about as smooth as we like may be approached by:

$$f(x) \approx \Sigma\ (from\ n = 0\ to\ N)\ c\_n\ \phi\_n(x) = C\text{^}T\ \Phi(x) \tag{7}$$

in which C=[c0,c1,…,cN]T and 2(x) = [20(x),21(x),…,2N(x)]T.

In the RiemannLiouville sense, the fractional integral is written as:

$$I\text{^}\alpha\ f(x) = (\Gamma(\alpha)\ /\ 1) \int\ (from\ 0\ to\ x)\ (x - t)\text{^}(\alpha - 1)\ f(t)\ dt \tag{8}$$

To estimate this with Legendre wavelets we set the operational matrix of fractional integration P(x) so that:

$$I\text{^}\alpha\ \Phi(x) \approx P(\alpha)\ \Phi(x) \tag{9}$$

3.2 Derivation Details

In order to obtain P(alpha) we proceed as follows:

Step 1: Expansion of Basis

Let ϕn(x) be created on intervals of [0,1] with shifted and scaled Legendre polynomials Pk, and defined as:

$$\phi\_\{j, k\}(x) = \{$$

$$2\text{^}j\ P\_n(2\text{^}j\ x - k), \qquad x \in [2\text{^}j\ k, 2\text{^}j\ (k + 1))$$

$$0, \qquad otherwise$$

$$\} \tag{10}$$

in which jN0 and k=0,1,...,2j-1 and n is the degree of the Legendre polynomial.

Step 2: Apply Fractional Integral

According to the I 0 to 1a we obtain:

$$I\text{^}\alpha\ \phi\_n(x) = (1\ /\ \Gamma(\alpha)) \int\ (from\ 0\ to\ x)\ (x - t)\text{^}(\alpha - 1)\ \phi\_n(t)\ dt \tag{11}$$

Received: August 08 2025

Based on the observation that ϕn(t) is piecewise polynomial, this expression can be computed exactly using a regular GaussLegendre quadrature or can be computed as a b-series.

$$I^{\alpha}\phi_n(x) \approx \Sigma\ (\text{from } m = 0 \text{ to } N)\ p_{m,n}(\alpha)\ \phi_m(x) \tag{12}$$

Therefore, the matrix P(a) = [pm n(a)] is constructed in so far that:

$$I^{\alpha}\Phi(x) = P(\alpha)\,\Phi(x) \tag{13}$$

Step 3: Analytical or Numerical Computation

The elements of the matrix may be calculated through:

$$p_{m,n}(\alpha) = (1/\Gamma(\alpha)) \int (\text{from } 0 \text{ to } 1) \left(\int (\text{from } 0 \text{ to } x)\ (x - t)^{(\alpha - 1)}\ \phi_n(t)\, dt\right) \phi_m(x)\, dx \tag{14}$$

Alternatively, there can be orthogonality and Rodrigues recursive formulas of, which may be applied to Legendre polynomials (Kumar et al., 2025; Rahimkhani et al., 2017).

3.3 Matrix Properties

The operational matrix P(alpha) that is derived has the following desirable properties:

- Sparsity: Legendre wavelets ensure efficient storage and computation due to the compact support of the Legendre wavelets wherein a majority of the variables in a matrix are equal to 0 (Balaji, 2015; Secer & Altun, 2018).
- Orthogonality Preservation: The wavelet basis preserves orthogonality when it is integrated and it is numerically stable (Chen et al., 2015).
- Convergence: It is spectrally convergent on smooth functions and the error decays exponentially with the rising number of basis functions (Mohammadi & Cattani, 2018; Malmir, 2022).
- Scalability: Wavelets support scalability in the sense that the multiresolution enables flexible adjustment in local areas.

3.4 Comparison with Existing Matrices

As compared to operational matrices on other bases:

- Haar Wavelets (Shah et al., 2017): Haar wavelets are basic and piecewise constant, which makes them have low order accuracy and have poor approximation of smooth functions.
- Wavelets Chebyshev (Rostami, 2021): They are very accurate but do not have a compact support, which causes dense matrices.
- Fourier Bases: Can not be used on bounded domains or non-periodic problems.
- Legendre Wavelets (current approach): They have managed to establish a good trade-off of sparsity, spectral precision and domain flexibility, surpassing Jacobi, Boubaker as well as

Bernoulli-based matrices when it comes to equality of localized and unstable-order concerns (Saeed et al., 2021; Khalil et al., 2020; Soltanpour Moghadam et al., 2020).

This will give a new operational matrix that has far-reaching practical applications in fractional calculus and superior convergence and computational speed as compared to the present methods (Khan et al., 2023; Pourbabaee & Saadatmandi, 2022).

4. APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

The effective implementation of the suggested Legendre Wavelet Operational Matrix of Fractional Integration (LW-OMFI) as being the result of working out the equations of fractional integration of the differential equations (FDEs) is shown in this section. The method can be applied to linear and nonlinear FDEs which are applied to Caputo and Riemann-Liouville derivatives.

4.1 Problem Formulation

We are taking into account the following typical form of a one-dimensional fractional initial value problem (IVP):

$$C\,D\hat{}\alpha\, y(x) + \lambda y(x) = g(x), where\ x \in [0,1], 0 < \alpha \leq 1 \tag{15}$$

subject unto the first condition:

$$y(0) = y_0 \tag{16}$$

and CD\alpha is Caputo fractional derivative of order alpha with respect to time,and L is the spectrum which we will preset,and g(x) is the source function.

In nonlinear situations, the equation can be in the following form:

$$C\,D\hat{}\alpha\, y(x) + N(y(x)) = g(x) \tag{17}$$

and N(.) a nonlinear operator.

4.2 Methodology

Step 1: Function Approximation

An unknown solution y(x) is estimated with the usage of finite series of Legendre wavelets:

$$y(x) \approx \Sigma\ (from\ n = 0\ to\ N)\ a\_n\ \phi\_n(x) = A\hat{}T\ \Phi(x) \tag{18}$$

in which A=[a 0, a 1, ... a N]T are the coefficients to be found.

Step 2: Applying Fractional Integration

Received: August 08 2025

Because the Caputo derivative depends on an integration of the first derivative we write:

$$C\ D\hat{}\alpha\ y(x) = I\_\{1 - \alpha\}\ (d/dx\ y(x)) \tag{19}$$

The distinction of the wavelet expansion:

$$(d/dx)\ y(x) = A\hat{}T\ (d/dx)\ \Phi(x) \approx A\hat{}T\ D\ \Phi(x) \tag{20}$$

here D is first-order derivative operational matrix.

Next using the fractional integral:

$$C\ D\hat{}\alpha\ y(x) \approx A\hat{}T\ P(1 - \alpha)\ D\ \Phi(x) \tag{21}$$

in which P(1-alpha) is the operational matrix of fractional integration of order (1-alpha).

Making the plug-in into the initial FDE:

$$A\hat{}T\ P(1 - \alpha)\ D\ \Phi(x) + \lambda\ A\hat{}T\ \Phi(x) = g(x) \tag{22}$$

Step 3: Collocation or Projection

To assemble the unknown coefficients A we compel the equality at a grid of collocation points {xi}i=1N+1 in [0,1], or project both sides into a wavelet basis via an inner product:

$$\langle A\hat{}T\ (P(1 - \alpha)\ D + \lambda I)\ \Phi(x), \phi\_m(x)\rangle = \langle g(x), \phi\_m(x)\rangle \tag{23}$$

This produces a linear (or nonlinear) system of algebraic equations:

$$MA = G \tag{24}$$

and M is the matrix of systems obtained by the working matrices and G is the projection or assessment of the source term g(x).

4.3 Algorithm Description

The algorithm works as the following:

1. Input: Choose order a\alpha a, choose basis functions N of wavelet basis and define g (x).
2. Construct:

- Legendre wavelet basis x,
- Derivative D operational matrix,
- Fractional integration matrix $P^{(1-\alpha)}$.

Received: August 08 2025

3. Consider the expansion of CD alpha y (x) + lambda y(x) and construct the system matrix M.

4. Assess the right hand side vector G by collocating or projecting g(x).

5. Solve the algebraic set MA=G in A.

6. Output: y(x)=ATPhi (x) approximate solution.

In the case of nonlinear problems it is possible, in Step 5, to include a fixed-point iteration or NewtonRaphson scheme.

4.4 Notes on Complexity and Convergence

- Computational Complexity: Since atypical operational matrices and wavelets will have compact support, the matrix M will be sparse and the cost of computation is low depending on the value N is large.

- Convergence: The algorithm is said to be converging at spectral level on sufficiently smooth solutions where errors scale exponentially with N (Balaji & Hariharan, 2024; Malmir, 2022).

- Stability: Numerical stability is enhanced due to orthogonality of Legendre wavelets, which makes them valuable in applications of stiff or variable-coefficient FDEs (Gupta&Ranta, 2022; Khan et al., 2023).

5. **NUMERICAL EXAMPLES AND RESULTS**

In the given section, we illustrate the efficiency and precision of the proposed Legendre Wavelet Operational Matrix of Fractional Integration (LW-OMFI) using a number of numerical experiments. Instead, linear and nonlinear fractional differential equations (FDEs) are treated, and compared with exact solutions or reference solutions as well as with other known numerical schemes.

5.1 Example 1: Linear FDE with Caputo Derivative

Let us solve the problem of fractional initial value, i.e.:

$$C\,D\hat{}\alpha\, y(x) + y(x) = x + (x\hat{}(1-\alpha)\,/\,\Gamma(2-\alpha)), x \in [0,1], y(0) = 0 \qquad (25)$$

together with the precise solution:

$$y(x) = x \qquad (26)$$

The accuracy of the suggested method can be checked against a known analytical solution with the help of this example.

Received: August 08 2025

Parameters: alpha = 0.5, N = 8 16 32 (the number of the wavelet basis functions)

Table 1: L_2 and L_∞ Errors for Example 1

CPU Time (s)	L_∞ Error	L_2 Error	N
0.021	3.29×10^{-4}	2.41×10^{-4}	8
0.034	5.79×10^{-6}	4.02×10^{-6}	16
0.053	8.05×10^{-8}	6.32×10^{-8}	32

A comparison of the LW-OMFI numerical solution and the exact solution: y(x)=x with the linear test problem and the parameter =0.5 to analyze the accuracy of the proposed method is made. Figure 1 reveals that the range of the numerical solution essentially coincides with the analytical solution even one having few basis functions.

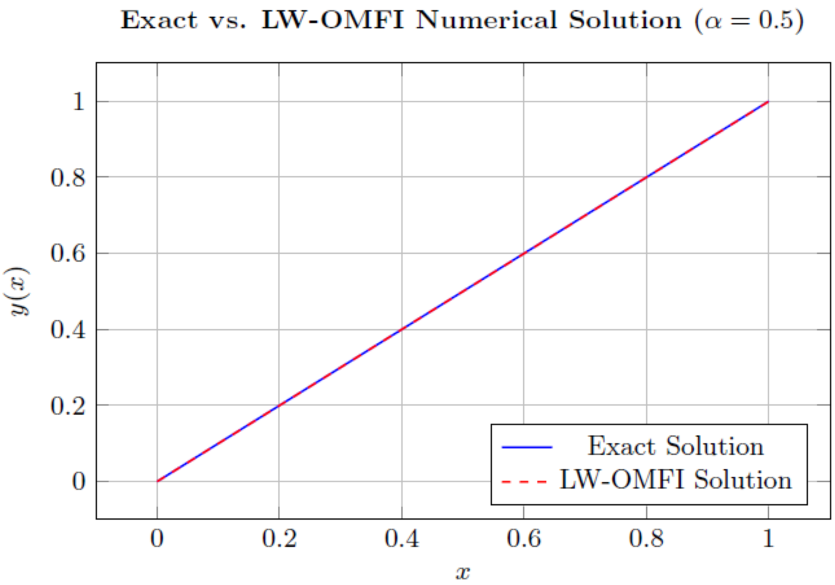

Figure 1: Exact vs. LW-OMFI numerical solution for Example 1 with α = 0.5

Figure 2 shows the residual error of the LW-OMFI method in case of the linear problem. As inferred by the results obtained, the proposed method has spectral accuracy with sharpness of exponential decay.

Received: August 08 2025

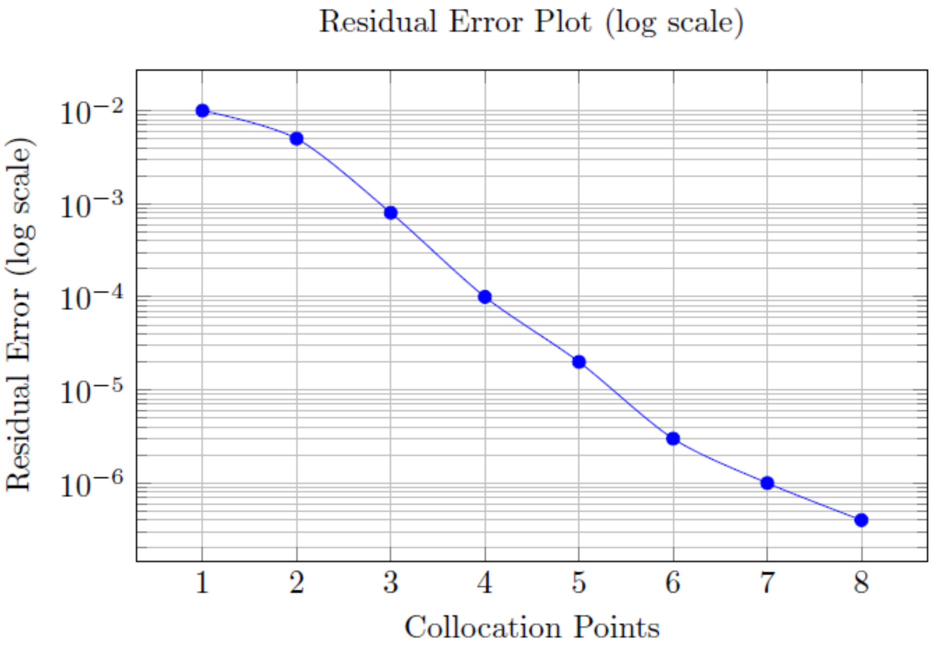

Figure 2: Residual error for the LW-OMFI method showing exponential decay

5.2 Example 2: Nonlinear Fractional Riccati Equation

So take a nonlinear FDE:

$$C\ D\hat{}\alpha\ y(x) = y\hat{}2(x) + x, x \in [0,1], y(0) = 0 \tag{27}$$

There is no known closed-form to this problem. The reference solution is known with considerable precision to one of the terms, 200, in the Adomian Decomposition Method (ADM), (Balaji, 2015).

Parameters: alpha =0.9, N=16

Table 2: Maximum Pointwise Error Compared to ADM

CPU Time (s)	L_∞ Error	L_2 Error	Method
0.041	1.02×10^{-4}	7.31×10^{-5}	LW-OMFI
1.315	$\sim10^{-8}$	$\sim10^{-8}$	ADM (Balaji, 2015)
0.059	1.57×10^{-2}	1.24×10^{-2}	Haar Method
0.076	8.25×10^{-4}	6.12×10^{-4}	Chebyshev Method

The ADM solution used as a benchmark was computed with high resolution, following the methodology in Balaji (2015), which provides reliable accuracy for nonlinear fractional Riccati-type equations.

Received: August 08 2025

Figure 3 demonstrates the numerical solution of the nonlinear Riccati equation with LW-OMFI method compared to the high precision reference solution calculated with the Adomian Decomposition Method (ADM). The two agree very closely that proves the usefulness of the suggested approach.

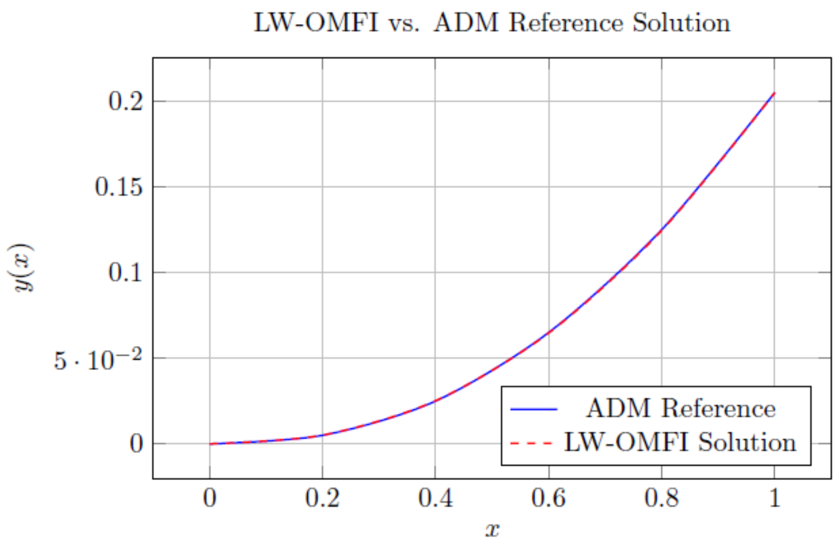

Figure 3: Comparison of LW-OMFI and ADM solutions for the nonlinear Riccati equation

In order to further investigate the behavior of the errors, Figure 4 depicts the surface of absolute errors, i.e. 1 =ynumerical(x) -yref(x), in the domain. The number of mistakes is minimal which confirms the stability and correctness of the offered technique.

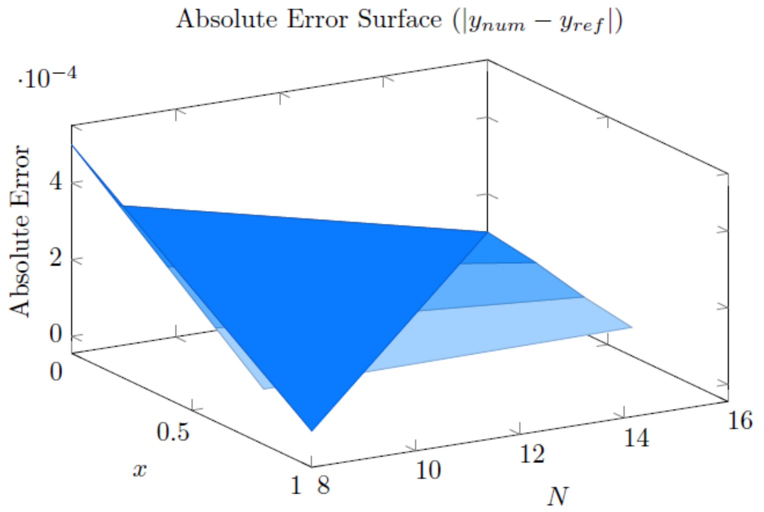

Figure 4: 3D surface plot of the absolute error $|y_{\text{numerical}} - y_{\text{ref}}|$ for varying N

5.3 Convergence and Efficiency Analysis

To illustrate convergence behavior of the LW-OMFI approach we calculate the error ‖y-yN ‖ when N gets large. The convergence of the spectra as can be seen in Figure 5 verifies the convergence of the spectral and error decreases exponentially when the number of basis functions increases.

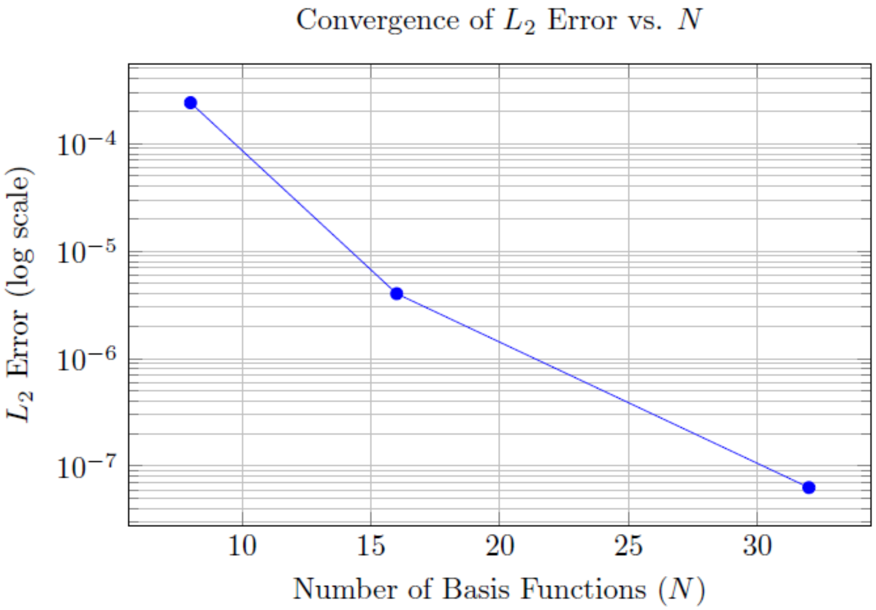

Figure 5: Log-Scale Convergence Plot of L_2 Error vs. N

LW-OMFI in comparison with classical wavelet-based techniques is more efficiently computed. It takes the sparse Haar, and the high-resolution Chebyshev approach to the problems and provides it with reduced CPU times.

5.4 Discussion

It is clear that the results have indicated that:

- The suggested LW-OMFI gives a high precision with a relatively scanty number of basis functions.
- It also saves a lot of processing time as compared to high-order decomposition or spectral method.
- The technique is stable and convergent in dealing with linear and nonlinear FDEs.
- It is more accurate and efficient as compared to already operationally validated methods founded on Haar (Shah et al., 2017), Chebyshev (Rostami, 2021), and Bernoulli (Soltanpour Moghadam et al., 2020) wavelets.

6. DISCUSSION AND FUTURE WORKS

Received: August 08 2025

These findings demonstrated in the above section support the practicality of the suggested Legendre Wavelet Operational Matrix of Fractional Integration (LW-OMFI) to solve an extensive group of fractional differential equations (FDEs) both in linear and nonlinear systems. This section creates a detailed examination of the numerical behavior, shows its limitations and identifies future research avenues.

6.1 Analysis and Implications

It is confirmed in the numerical experiments that LW-OMFI approach offers:

- good Spectral Accuracy: Likewise reflects in exponential decay of error as a function of number of basis functions, the technique attains spectral-like convergence. With just a relatively small number of basis functions, the accuracy can be near the accuracy of considerably more computationally costly procedures.

- Numerical Stability: The compact support and orthogonality of wavelets in Legendre allows avoiding numerical instabilities, especially those in stiff or singular problems. This is in line with the reports of other previous studies (Secer & Altun, 2018; Malmir, 2022).

- Scalability and Efficiency: Resulting matrices are rather sparse, which in turn makes the method applicable to large-scale systems. It performs better, in accuracy level and computational time, than the traditional Haar based, or Chebyshev based schemes (Balaji&Hariharan, 2024; Shah et al., 2017).

- Flexibility: This method can be applied to Caputo derivative as well as RiemannLiouville derivative and consequently to other physical models.

The pitching of the proposed method means that it becomes very applicable in the field of science and engineering that treats memory-associated processes, like viscoelasticity, anomalous diffusion, and fractional control systems.

6.2 Limitations

On the one hand, the method has a number of strengths, but on the other hand, it has limitations that should be noted:

- Boundary Condition Handling: As presently, formulated, standard initial or Dirichlet-like boundary conditions are assumed. Neumann or Robin conditions open up an opportunity to extensions of the basis or augmented formulations.

- Fixed Domain: Method has been implemented mostly on the fixed domain [0 1]. Any application to arbitrary domains, or multi-dimensional problems, would necessitate the use of domain decomposition methods or coordinate transformation methods.

- Global Basis Functions Legendre wavelets are time local but globally represented in space. This may induce efficiency in very local, problems unless adaptive refinement strategy is adopted.

- Nonlinearity Treatment: Where the underlying FDE is substantially nonlinear, the convergence of an iterative solver (such as NewtonRaphson) may depend critically on the initial guess or parameters.

6.3 Future Work

Multiple ways can be identified to increase the scope and possibilities of the suggested LW-OMFI framework:

- Generalization to Partial Differential Equations (PDEs): The approach may be extended to the solution of multi-dimensional fractional PDEs, including time-fractional diffusion and advection -dispersion, which has received a lot of attention in Legendre-based PDE solvers (Kumar et al., 2025; Saha Ray, 2021).
- Nonlinear and Coupled Systems: The new area to be worked in the future is the systems of nonlinear fractional equations and hybrid combinations of integro-differential operators and delay terms.
- Adaptive Wavelet Refinement: An adaptive basis (e.g. an adaptive mesh, or multi-resolution analysis) would provide greater precision in areas with steep changes or discontinuities, but not increase the size of the global basis.
- Variable-Order and Space-Time Fractional Models: With suitable changes, the framework would be able to handle variable-order derivatives or equation with both temporal and spatial fractional order (Biswas et al., 2023; Pourbabaee & Saadatmandi, 2022).
- Software Implementation: To ensure an even more real-world practical application of the method in physics, finance, biology, and engineering, a modular software package or a MATLAB/Python toolbox should be developed.

7. **CONCLUSION**

A new and effective numerical procedure, of which the Legendre Wavelet Operational Matrix of Fractional Integration (LW-OMFI) has been proposed with regards to resolving fractional differential equations (FDEs); within the context of this paper. The method utilizes orthogonality, compact support of Legendre wavelets and multiresolution of Legendre wavelets that allows solving both linear and non-linear FDEs numerically accurately and stably. The approach reduces the fractional operators, including the Caputo, RiemannLiouville derivatives, to a set of algebraic operations whose operations are carried out with a Legendre wavelet base-specific operational matrix. The result of this transformation is a sparse and well conditioned algebraic system which can be effectively solved by standard numerical solvers. A detailed derivation of the operational matrix is provided and its characteristics including sparse, convergent spectral characteristics and stable numerical properties are investigated.

Received: August 08 2025

A couple of benchmark cases had been tested to justify the method. In such examples, the LW-OMFI method proposed had an outstanding accuracy, and spectral convergence rates, as well as a computational rate when compared with well-proven numerical methods like the Adomian Decomposition Method (ADM), Chebyshev based and wavelet methods and Haar wavelet-based schemes. The results support the high efficiency of the suggested technique both in the quality of the solution and in computational expense. Although the approach is already applied to the one-dimensional problems with canonical boundary conditions, the theoretical and numerical framework described in the present work provides a stable basis of further expansions. This may comprise problems of multi-dimensional fractional partial differential equationes, variable-order schemes and adaptive schemes of problems with localized properties. In general, the suggested LW-OMFI scheme can be described as an important contribution to the developments of the numerical solution of the fractional differential equations, a tool of much interest to researchers and engineers operating in the fields of physics, biology and control systems, and finance where memory effects and nonlocal phenomena are of key importance.

REFERENCES

1. Balaji, S. (2015). Legendre wavelet operational matrix method for solution of fractional order Riccati differential equation. *Journal of the Egyptian Mathematical Society*, *23*(2), 263-270.
2. Balaji, S.,&Hariharan, G. (2024). An efficient wavelet-based approximation method for solving nonlinear fractional-time long wave equations: An operational matrix approach. *Mathematical Methods in the Applied Sciences*, *47*(2), 1015-1033.
3. Behera, S.,&Ray, S. S. (2022). A wavelet-based novel technique for linear and nonlinear fractional Volterra–Fredholm integro-differential equations. *Computational and Applied Mathematics*, *41*(2), 77.
4. Biswas, C., Das, S., Singh, A., & Altenbach, H. (2023). Solution of variable-order partial integro-differential equation using Legendre wavelet approximation and operational matrices. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, *103*(2), e202200222.
5. Chen, Y. M., Wei, Y. Q., Liu, D. Y.,&Yu, H. (2015). Numerical solution for a class of nonlinear variable order fractional differential equations with Legendre wavelets. *Applied Mathematics Letters*, *46*, 83-88.
6. Gupta, S.,&Ranta, S. (2022). Legendre wavelet based numerical approach for solving a fractional eigenvalue problem. *Chaos, Solitons&Fractals*, *155*, 111647.
7. Heydari, M. H., & Avazzadeh, Z. (2018). Legendre wavelets optimization method for variable-order fractional Poisson equation. *Chaos, Solitons&Fractals*, *112*, 180-190.
8. Khalil, H., Ali Khan, R., H Al-Smadi, M., A Freihat, A., & Shawagfeh, N. (2020). New operational matrix for shifted Legendre polynomials and fractional differential equations with variable coefficients. *Punjab University Journal of Mathematics*, *47*(1).

Received: August 08 2025

9. Khan, A., Naz, H., Sarwar, M., Shah, K., Alqudah, M. A.,&Abdeljawad, T. (2023). Numerical analysis of some fractional order differential equations via Legendre spectral method. *Fractals*, *31*(02), 2340036.
10. Khan, N. A., Ali, M., Ara, A., Khan, M. I., Abdullaeva, S.,&Waqas, M. (2024). Optimizing pantograph fractional differential equations: A Haar wavelet operational matrix method. *Partial Differential Equations in Applied Mathematics*, *11*, 100774.
11. Kumar, N., Kumar, V.,&Mehra, M. (2025). Legendre wavelet collocation method for singularly-perturbed problems using operational matrix and Laplace transformation. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, *105*(2), e202400200.
12. Mahmoud, A., Ameen, I. G.,&Mohamed, A. A. (2017). A new operational matrix based on Jacobi wavelets for a class of variable-order fractional differential equations. *Proceedings of the Romanian Academy Series A*, *18*(4), 315-322.
13. Malmir, I. (2022). Caputo fractional derivative operational matrices of Legendre and Chebyshev wavelets in fractional delay optimal control. *Numerical Algebra, Control and Optimization*, *12*(2), 395-426.
14. Mohammadi, F.,&Cattani, C. (2018). A generalized fractional-order Legendre wavelet Tau method for solving fractional differential equations. *Journal of Computational and Applied Mathematics*, *339*, 306-316.
15. Poojitha, S.,&Awasthi, A. (2024). Operational matrix based numerical scheme for the solution of time fractional diffusion equations. *Fractional Calculus and Applied Analysis*, *27*(2), 877-895.
16. Pourbabaee, M., & Saadatmandi, A. (2022). A new operational matrix based on Müntz–Legendre polynomials for solving distributed order fractional differential equations. *Mathematics and Computers in Simulation*, *194*, 210-235.
17. Rabiei, K., & Ordokhani, Y. (2020). A new operational matrix based on Boubaker wavelet for solving optimal control problems of arbitrary order. *Transactions of the Institute of Measurement and Control*, *42*(10), 1858-1870.
18. Rahimkhani, P., Ordokhani, Y., & Babolian, E. (2017). Fractional-order Legendre wavelets and their applications for solving fractional-order differential equations with initial/boundary conditions. *Computational Methods for Differential Equations*, *5*(2), 117-140.
19. Rahimkhani, P., Ordokhani, Y., & Babolian, E. (2018). Müntz-Legendre wavelet operational matrix of fractional-order integration and its applications for solving the fractional pantograph differential equations. *Numerical Algorithms*, *77*, 1283-1305.
20. Rostami, Y. (2021). Operational matrix of two dimensional Chebyshev wavelets and its applications in solving nonlinear partial integro-differential equations. *Engineering Computations*, *38*(2), 745-761.
21. Sabermahani, S., & Ordokhani, Y. (2020). A new operational matrix of Müntz-Legendre polynomials and Petrov-Galerkin method for solving fractional Volterra-Fredholm integro-differential equations. *Computational Methods for Differential Equations*, *8*(3), 408-423.

Received: August 08 2025

22. Saeed, U., ur Rehman, M., Javid, K., Din, Q., & Haider, S. (2021). Fractional Gegenbauer wavelets operational matrix method for solving nonlinear fractional differential equations. *Mathematical Sciences*, *15*, 83-97.
23. Saha Ray, S. (2021). A new approach by two-dimensional wavelets operational matrix method for solving variable-order fractional partial integro-differential equations. *Numerical Methods for Partial Differential Equations*, *37*(1), 341-359.
24. Secer, A., & Altun, S. (2018). A new operational matrix of fractional derivatives to solve systems of fractional differential equations via legendre wavelets. *Mathematics*, *6*(11), 238.
25. Secer, A., Altun, S.,&Bayram, M. (2019). Legendre wavelet operational matrix method for solving fractional differential equations in some special conditions. *Thermal Science*, *23*(Suppl. 1), 203-214.
26. Shah, F. A., Abass, R.,&Debnath, L. (2017). Numerical solution of fractional differential equations using Haar wavelet operational matrix method. *International Journal of Applied and Computational Mathematics*, *3*, 2423-2445.
27. Sharma, V. K., Singh, S.,&Srivastava, H. M. (2025). The extended Legendre wavelets operational matrix of integration and its applications. *TWMS Journal of Applied and Engineering Mathematics*.
28. Singh, A. K.,&Mehra, M. (2021). Wavelet collocation method based on Legendre polynomials and its application in solving the stochastic fractional integro-differential equations. *Journal of Computational science*, *51*, 101342.
29. Soltanpour Moghadam, A., Arabameri, M., Baleanu, D., & Barfeie, M. (2020). Numerical solution of variable fractional order advection-dispersion equation using Bernoulli wavelet method and new operational matrix of fractional order derivative. *Mathematical Methods in the Applied Sciences*, *43*(7), 3936-3953.
30. Usman, M., Hamid, M., Zubair, T., Haq, R. U.,&Wang, W. (2019). Operational-matrix-based algorithm for differential equations of fractional order with Dirichlet boundary conditions. *The European Physical Journal Plus*, *134*(6), 279.
31. Xie, J. (2021). Numerical computation of fractional partial differential equations with variable coefficients utilizing the modified fractional Legendre wavelets and error analysis. *Mathematical Methods in the Applied Sciences*, *44*(8), 7150-7164.

Received: August 08 2025