International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Bicomplex Laplace Adomian Approach for ψ-Caputo Fractional Differential Equations

A. Thirumalai [1] , K. Muthunagai [2*]

[1] School of Advanced Sciences,
Vellore Institute of Technology Chennai, Tamil Nadu, India.
e-mail: thirumalai.a2019@vitstudent.ac.in

[2] School of Advanced Sciences,
Vellore Institute of Technology Chennai, Tamil Nadu, India.
e-mail: muthunagai@vit.ac.in

Abstract

In this work, we propose the ψ-Bicomplex Laplace Transform Adomian Decomposition Method to solve fractional differential equations defined with respect to a function ψ in the Caputo sense. This approach integrates the effective Adomian Decomposition Method with a generalized form of the classical bicomplex Laplace transform. We apply the resulting recursive scheme to several numerical examples, including a real-world pharmacokinetic model describing drug concentration in human blood, to evaluate the method's performance. The solutions obtained closely align with known analytical results, while the pharmacokinetic model outcomes closely match experimental data. These findings demonstrate the reliability and effectiveness of the method in solving a broad class of ψ-fractional differential equations.

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Math. Subject Classification: 30G35; 33E12; 28A80; 30C99; 30B99.

Key Words and Phrases: ψ-Caputo derivative; generalized bicmplex fractional Laplace transform; initial value problem; Adomian decomposition method; pharmacokinetics model.

1 Introduction

Fractional-order calculus focuses on the study of derivatives and integrals of noninteger order. In recent years, the field has seen significant advancements, particularly in its applications to various modern scientific and engineering domains such as fluid dynamics, probability theory, control systems, thermodynamics, and dynamical systems [6, 7]. Unlike classical calculus, which follows a unified structure, fractional calculus involves multiple definitions, making it a diverse and multidirectional discipline. Common definitions of fractional derivatives include those proposed by Atangana-Baleanu, Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Riesz and Hilfer. A comprehensive historical overview of the field is provided in [8, 9, 10]. Numerous studies have investigated the application of different fractional derivatives to various types of fractional differential equations (FDEs), demonstrating the utility of fractional calculus in modeling complex phenomena across multiple disciplines.

Almeida proposed a novel form of the fractional derivative [11]. The ψ-Caputo fractional derivative generalizes several other fractional derivative formulations, including the Caputo and Hadamard derivatives. As fundamental components for proving the existence and uniqueness of solutions to various classes of ψ-fractional differential equations (ψ-FDEs), numerous fixed-point theorems are currently in use [12, 13, 14, 15].

Recently, many new fractional integral and derivative operators have been introduced in the field of fractional calculus. A new

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

fractional derivative is typically introduced for one of two reasons: either to incorporate specific mathematical properties that existing fractional derivatives do not capture or to represent physical phenomena that current fractional derivatives are unable to describe effectively. Furthermore, recent research on ψ-Caputo differential operators suggests that ψ-FDEs offer greater flexibility and more suitable outcomes in a variety of situations.

Almeida used the ψ-Caputo derivative to model global population growth, demonstrating that the accuracy of the model is dependent on the appropriate choice of fractional operator and trial function [12, 16]. Since analytical solutions for ψ-fractional differential equations (ψ-FDEs) are not yet available, semi-analytical and numerical methods are essential. Various techniques have been proposed, including operational matrices based on ψ-shifted Legendre polynomials [18] and the ψ-Haar wavelet method for initial value problems [17].

Developing effective analytical and numerical approaches for specific classes of ψ-FDEs remains a significant challenge in fractional calculus. As a result, researchers have increasingly focused on novel methods to model complex physical phenomena. In this regard, the ψ-Bicomplex Laplace Transform (ψ -BLT), a generalized transform introduced in [19], has gained attention for its effective integration with fractional derivatives and integrals in particular types of differential equations [20].

Moreover, the Adomian Decomposition Method (ADM) is a semi-analytical approach that has proven highly effective in solving a wide range of fractional differential equations (FDEs). It finds significant applications across various fields of the biological sciences. Notably, ADM is considered a practical and efficient method, requiring less computational effort compared to many traditional techniques. It simplifies the solution process without the need to linearize the given problem, making the decomposition procedure

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

straightforward and efficient.

The Adomian Decomposition Method (ADM) has proven to be an effective technique for solving various fractional differential equations (FDEs). Its applications include studying generalized fractional Riccati differential equations with respect to another function, ψ [22], and analyzing relaxation-oscillation differential equations involving the ψ-Caputo derivative [21]. ADM has undergone several enhancements that aim to improve accuracy, reduce computational time, and accelerate convergence. As a result of these advancements, the modified ADM offers faster series convergence compared to the classical version. This improved method has demonstrated computational efficiency across a range of models, making it a valuable tool for researchers in applied sciences.

To address non-linear FDEs more effectively, a recent modification of ADM was introduced [23]. Researchers have also explored hybrid approaches by combining ADM with various integral transforms such as the Elzaki, Laplace, Sumudu, and Natural transforms. These combinations have successfully addressed a wide range of FDE models. One prominent example is the Laplace-Adomian decomposition method (LADM), which integrates the Laplace transform with ADM. This hybrid technique replaces traditional solution methods with a decomposition-based approach, yielding accurate and efficient results. Consequently, LADM continues to be widely employed for solving numerous types of FDEs in scientific and engineering applications [24, 25].

This paper aims to apply the standard Adomian Decomposition Method (ADM) in combination with the generalized Bicomplex Laplace transform, known as the ψ-Bicomplex Laplace Adomian Decomposition Method (ψ-BLADM), to address a class of fractional differential equations (FDEs) involving ψ-Caputo fractional derivatives, referred to as ψ-FDEs. The motivation behind this approach is to extend the applicability of recently introduced ψ-fractional

Received: July 22, 2025

operators to more realistic scientific problems. The proposed generalized ψ-BLADM recursive scheme is evaluated through several test cases, including a practical example from pharmacokinetics.

The ψ-BLADM method was selected based on its promising performance in the numerical examples presented later in this paper. Compared to previous studies on ψ-FDEs, our results demonstrate improved accuracy. As shown in [26], this approach has been successfully used to solve various types of ψ-FDEs, including systems of such equations, with excellent results.

Although we encountered some challenges in computing the inverse ψ-Bicomplex Laplace transform for specific functions, the method still yielded highly accurate outcomes and was easy to implement using the Maple 2022 software. We hope this study will contribute to expanding the range of tools available to researchers for tackling different types of fractional problems in the future.

The increasing demand for analytical methods capable of managing multicomponent systems and higher-dimensional data necessitates the extension of effective solution approaches, such as ψ-LTADM, to bicomplex-valued functions. Compared to conventional techniques, the bicomplex structure provides richer algebraic structures that are better able to represent complex relationships.

The solution of ψ-fractional differential equations in bicomplex spaces has received very little attention, despite the fact that there are several analytical and numerical methods for solving classical and even fractional differential equations in real or complex domains. There are not many general-purpose decomposition techniques designed for these kinds of situation, particularly when working with fractional calculus that involves generalized derivatives.

The bicomplex ψ-LTADM created in this work is readily applied to systems described in bicomplex algebra, while retaining the advan-

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

tages of the original approach, such as avoiding linearization, discretization, or perturbation. New approaches to solving ψ-fractional differential equations in hypercomplex spaces are made possible by this study.

The structure of this paper is as follows: Section 2 presents relevant definitions and theorems; Section 3 outlines the formulation of the overall ψ-BLADM scheme; Section 4 demonstrates the proposed approach through test problems, including a real-world application in pharmacokinetics; and Section 5 offers concluding remarks.

2 Preliminaries

This section discusses the fundamental properties of ψ-fractional operators and provides their key definitions. It also covers the ψ-Bicomplex Laplace Transform (BLT) for certain simple functions, introduces the generalized ψ-BLT, and examines the application of the ψ-BLT to ψ-fractional operators and other special functions that are essential for solving ψ-fractional differential equations (ψ-FDEs).

Definition 1. [1] Let $f(t)$ be a real-valued function with exponential order k. The Bicomplex Laplace Transform of $f(t)$ for $t \geq 0$ is defined as:

$$L[f(t);\xi] = \int_0^\infty e^{-t\xi} f(t)\,dt = F(\xi).$$

Here, $F(\xi)$ exists and converges for all $\xi \in D = D_1 \cup D_2 \cup D_3$, or it may have a hypergeometric projection $H_\varrho(\xi)$ in the right half-plane where $a_0 > k + |a_3|$. In the region D, there are an infinite number of values for ξ for which the hyperbolic projection H_ϱ remains the same, as a_1 and a_2 are not subject to any constraints. The regions

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

D_1, D_2, and D_3 are defined as:

$$D_1 = \{\xi = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 : a_0 > k \text{ and } a_3 = 0\}$$
$$D_2 = \{\xi = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 : a_0 > k + a_3 \text{ and } a_3 > 0\}$$
$$D_3 = \{\xi = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 : a_0 > k - a_3 \text{ and } a_3 < 0\}$$

Definition 2. A function y with respect to another function ψ can be expressed as a fractional integral of order $\beta > 0$ using the following equation [11]. Specifically, let $y : I \to \mathbb{R}$ be integrable, where $I = [a, b]$, $\beta \in \mathbb{R}$, $n \in \mathbb{N}$, and $\psi(x) \in C^n(I)$ is a function such that $\psi'(x) \neq 0$ for all $x \in I$. The fractional integral is defined as:

$$I_a^{\beta,\psi} y(x) = \{\Gamma(\beta)\}^{-1} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-1+\beta} y(t)dt. \tag{1}$$

Here, the gamma function is denoted by Γ. In particular, when $\psi(x) = x$ or $\psi(x) = \ln(x)$, the above equation reduces to the Riemann–Liouville and Hadamard fractional integrals, respectively.

Definition 3. Consider an integrable function $y : I \to \mathbb{R}$ defined on the interval $I = [a, b]$. The fractional derivative of y with respect to ψ is defined as follows [6] for a fractional order $\beta > 0$ and a function $\psi \in C^n(I)$ such that $\psi'(x) \neq 0$ for all $x \in I$:

$$\begin{aligned} D_{a^+}^{\beta,\psi} y(x) &= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{n-\beta,\psi} y(x) \\ &= \{\Gamma(n-\beta)\}^{-1} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-1+\beta} y(t)dt. \end{aligned} \tag{2}$$

Here, $n = 1 + [\beta]$. It should be noted that when $\psi(x) = x$ or $\psi(x) = \ln(x)$, equation 1 simplifies t o t he R iemann-Liouville and Hadamard fractional derivatives, respectively.

Definition 4 . [11] L et $I = [a, b]$ b e a n i nterval w here $\beta > 0$ and $n \in \mathbb{N}$. Given that ψ is an increasing function with $\psi'(x) \neq 0$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

for all $x \in I$, let ψ and y be two functions in $C_n(I)$. The ψ-Caputo fractional derivative of order β for the function y is defined as:

$$^{c}D_{a}^{\beta,\psi}y(x) = I_{a^+}^{n-\beta,\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n y(x).$$

Here, $\beta \notin \mathbb{N}$, $n = 1 + \lfloor \beta \rfloor$, and if β is an integer, then $n = \beta$.

$$y_{\psi}^{[n]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n y(x),$$

in its simplified notation. From this definition, it is clear that the ψ-Caputo fractional derivative is:

$$^{c}D_{a}^{\beta,\psi}y(x) = \begin{cases} \Gamma(n-\beta)^{-1}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\beta-1}y_{\psi}^{[n]}(t)\,dt & \text{if } \beta \notin \mathbb{N}, \\ y_{\psi}^{[n]}(x) & \text{if } \beta \in \mathbb{N}. \end{cases}$$

It is important to note that for $\psi(x) = x$ and $\psi(x) = \ln(x)$, this formula simplifies to the classical Caputo derivative and the Caputo–Hadamard fractional derivative, respectively.

In addition, as noted in [11], some important properties of $\psi(x)$-fractional operators are derived by considering the function $y(x) = (\psi(x)-\psi(a))^{\alpha-1}$, where $\alpha \in \mathbb{R}$, $\alpha > n$, and $\beta > 0$. These properties are as follows:
ψ-Caputo fractional derivative:

$$^{C}D_{a^+}^{\beta,\psi}y(x) = \frac{\alpha}{\alpha-\beta}(\psi(x)-\psi(a))^{-\alpha-\beta-1}.$$

ψ-fractional integral:

$$I_{a}^{\beta,\psi}y(x) = \frac{\alpha}{\alpha+\beta}(\psi(x)-\psi(a))^{-\alpha-\beta-1}$$

Relationship between the fractional integral and derivative:

$$I_{a}^{\beta,\psi}\left(^{C}D_{a^+}^{\beta,\psi}y(x)\right) = y(x) - \sum_{k=0}^{n-1}\frac{y_{\psi}^{[k]}(x)}{k!}(\psi(x)-\psi(a))^k.$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Moreover, for $\beta \in (0,1)$, the expression simplifies to:

$$I_a^{\beta,\psi}\left({}^C D_{a^+}^{\beta,\psi} y(x)\right) = y(x) - y(a).$$

These results highlight the key properties of fractional operators with respect to the function ψ, which are essential for solving ψ-fractional differential equations (FDEs). The following definitions and theorems provide specialized functions that are useful in solving these equations.

Definition 5. The Mittag-Leffler function $E_\alpha(z)$, introduced by Gösta Mittag-Leffler, is a well-known function in the theory of fractional calculus. It is expressed as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \, \Re(\alpha) > 0$$

This function plays an important role in fractional calculus and is used to describe solutions to fractional differential equations. Here, $\Gamma(\cdot)$ denotes the Gamma function, which generalizes the factorial function. Additionally, Wiman introduced a two-parameter generalization of the Mittag-Leffler function, which further expands its utility in solving more complex fractional equations. This variant of the Mittag-Leffler function allows for greater flexibility in modeling and solving fractional differential equations with additional parameters.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in C, Re(\alpha) > 0 \tag{3}$$

Similarly, Prabhakar [5] introduced an even more general version of the Mittag-Leffler function known as the three-parameter Prabhakar function or the Prabhakar Mittag-Leffler function. This function generalizes the Mittag-Leffler function even further, and it is defined as follows:

$$E_{\alpha,\beta}^{\aleph}(z) = \frac{1}{\aleph} \sum_{k=0}^{\infty} \frac{\Gamma(\aleph + k) z^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha, \beta, \aleph \in C, Re(\alpha) > 0 \tag{4}$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Definition 6. The Bicomplex Mittag-Leffler $E_\alpha(\xi)$ is defined by

$$E_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k + 1)}, \quad \alpha, \xi \in C_2, \quad |im_j(\alpha)| < |Re(\alpha)| \tag{5}$$

Furthermore, Ritu Agarwal created a two-parameter variant of the Bicomplex Mittag-Leffler function, which is as follows:

$$E_{\alpha,\beta}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta, \xi \in C_2, \quad |im_j(\alpha, \beta)| < |Re(\alpha, \beta)| \tag{6}$$

Similarly, we have defined the three-parameter TM function or the Bicomplex Mittag-Leffler function, which is described as follows:

$$E^{\aleph}_{\alpha,\beta}(\xi) = \frac{1}{\aleph} \sum_{k=0}^{\infty} \frac{\Gamma(\aleph + k)\xi^k}{k!\Gamma(\alpha k + \beta)}, \quad \alpha, \beta, \aleph, \xi \in C_2, |im_j(\alpha, \beta, \aleph)| < |Re(\alpha, \beta, \aleph)| \tag{7}$$

Definition 7. [19, 20] The Laplace transform of the function y with respect to the function ψ is defined as

$$L_\psi[y(x)] = Y(s) = \int_0^\infty e^{-s\psi(x)} \psi'(x)\psi(x)dx, \quad \forall s \in C, \tag{8}$$

where $y : [0, \infty) \to R$ is a real-valued function and ψ is an increasing positive function with $\psi(0) = 0$.

Definition 8. The Bicomplex Laplace transform of the function y with respect to the function ψ is defined as

$$L_\psi[y(x)] = Y(\xi) = \int_0^\infty e^{-\xi\psi(x)} \psi'(x)\psi(x)dx, \quad \forall \xi \in C_2 \tag{9}$$

$y : [0, \infty) \to R$ is a real-valued function and ψ is an increasing positive function with $\psi(0) = 0$.

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 9. *The composition of operations involving ψ or ψ^{-1} can be used to define the generalized Bicomplex Laplace transform, as shown below.*

$$L_\psi = Lo\psi_\psi^{-1} \tag{10}$$

where the definition of the functional operator ψ_ψ is

$$(\psi_\psi f)(x) = f(\psi(x)) \tag{11}$$

Corollary 10. *The inverse generalized Bicomplex Laplace transform can be expressed as:*

$$L_\psi^{-1} = \psi_\psi o L^{-1} \tag{12}$$

or, to put it another way as,

$$L_\psi^{-1}[Y(\xi)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\xi\psi} Y(\xi) d\xi \tag{13}$$

Table 1 provides the general ψ- Bicomplex Laplace transforms of some basic functions.

Table 1: The ψ- Bicomplex Laplace transform of certain functions

$y(x)$	$L_\psi[y(x)] = Y(\xi)$
1	$\frac{1}{\xi}$, $\lvert\xi\rvert > 0$
$e^{a\psi(x)}$	$\frac{1}{\xi-a}$, for $\lvert\xi\rvert > a$
$(\psi(x))^\beta$	$\frac{\Gamma(\beta+1)}{\xi^\beta+1}$, for $\lvert\xi\rvert > 0$
$E_\beta(\lambda(\psi(x))^\beta)$	$\frac{\xi^\beta-1}{\xi^\beta+\lambda}$ for $\mathrm{Re}(\beta) > 0$ and $\lvert\frac{\lambda}{\xi^\alpha}\rvert < 1$
$(\psi(x))^{\beta-1}E_{\beta,\beta}(\lambda(\psi(x))^\beta)$	$\frac{1}{\xi^\beta-\lambda}$ for $\mathrm{Re}(\beta) > 0$ and $\lvert\frac{\lambda}{\xi^\alpha}\rvert < 1$
$(\psi(x))^{\alpha-1}E^\gamma_{\beta,\alpha}(\lambda(\psi(x))^\beta)$	$\frac{\xi^{\beta\gamma-\alpha}}{(\xi^\beta-\lambda)\gamma}$ for $\mathrm{Re}(\beta) > 0$ and $\lvert\frac{\lambda}{\xi^\alpha}\rvert < 1$

Theorem 11. *Let y be a piecewise continuous function defined on each finite interval, with ψ-exponential order. As a result, for every $\beta > 0$,*

$$L_\psi\{I_0^{\beta,\psi(x)} y(x)\} = \xi^{-1}L_\psi(y(x)). \tag{14}$$

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 12. *Assume $\beta > 0$, $n = \lfloor \beta \rfloor + 1$, and let y be a function such that $y(x)$, $I_0^{n-\beta,\psi(x)}y(x)$, $D_0^{1,\psi(x)}I_0^{n-\beta,\psi(x)}y(x)$, $\ldots$, $D_0^{n,\psi(x)}I_0^{n-\beta,\psi(x)}y(x)$ are continuous and exhibit ψ-exponential order over the interval $(0,\infty)$. Additionally, assume that ${}^{R}D_0^{\beta,\psi(x)}y(x)$ is piecewise continuous on $[0,\infty)$. Then,*

$$L_\psi\{{}^{R}D_0^{\beta,\psi(x)}y(x)\} = \xi^\beta L_\psi\{y(x)\} - \sum_{j=0}^{n-1}\xi^{n-1-j}(I_0^{n-j-\beta,\psi(x)}y)(0). \tag{15}$$

Theorem 13. *Let $\beta > 0$ and define $n = \lfloor \beta \rfloor + 1$. Assume that the function $y(x)$, along with its ψ-Caputo fractional derivatives ${}^{C}D_0^{1,\psi(x)}y(x), {}^{C}D_0^{2,\psi(x)}y(x), \ldots, {}^{C}D_0^{n-1,\psi(x)}y(x)$, is continuous and of ψ-exponential order on the interval $[0,\infty)$. Further, suppose that ${}^{C}D_0^{\beta,\psi(x)}y(x)$ is piecewise continuous on $[0,\infty)$. Under these conditions, the ψ-Laplace transform of the ψ-Caputo fractional derivative of order β is given by:*

$$L_\psi\left\{{}^{C}D_0^{\beta,\psi(x)}y(x)\right\} = \xi^\beta L_\psi\{y(x)\} - \sum_{j=0}^{n-1}\xi^{\beta-1-j}\left(D_0^{j,\psi(x)}y\right)(0). \tag{16}$$

3 Fundamental Description of ψ- BLADM for Solving ψ-FDEs

In this section, we will solve initial value problems (IVPs) for ψ-fractional differential equations (FDEs) that involve ψ-Caputo derivatives using the ψ-BLADM. To introduce this method, we begin by considering a generalized IVP for FDEs with respect to an arbitrary function ψ, as outlined below.

$${}^{C}D_{a^+}^{\beta,\psi}y(x) = f(x, y(x)), \quad x \in [a,b], \tag{17}$$

and limited by the basic conditions listed below.

$$y(a) = y_a, \quad y_\psi^{[k]}(a) = y_a^k, k = 1, 2, ..., n-1, \tag{18}$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

where $f(x, y(x))$ is an arbitrary continuous nonlinear function, $y(x)$ is the solution to be known, and ${}^{C}D_{a^+}^{\beta,\psi}$ indicates the ψ-Caputo fractional operator of order β.
The problem presented in 17 is solved using the ψ-BLADM by applying the ψ-Laplace transform (LT) to both sides, resulting in:

$$L_\psi\{{}^{C}D_{a^+}^{\beta,\psi} y(x)\} = L_\psi\{f(x, y(x))\}. \tag{19}$$

Furthermore, when 16 is combined with the specified initial conditions, the result is:

$$\begin{aligned} \xi^\beta L_\psi\{y(x)\} - \sum_{k=0}^{n-1} \xi^{\beta-1-k} y_\psi^{[k]}(a) = L_\psi\{f(x, y(x))\}, \\ L_\psi\{y(x)\} - \sum_{k=0}^{n-1} \frac{1}{\xi^{k+1}} y_\psi^{[k]}(a) = \frac{1}{\xi^\beta} L_\psi\{f(x, y(x))\}. \end{aligned} \tag{20}$$

The inverse ψ-LT applied to both sides of the resulting equation yields:

$$\begin{aligned} L_\psi^{-1}\{L_\psi\{y(x)\}\} - L_\psi^{-1}\{\sum_{k=0}^{n-1} \frac{1}{\xi^{k+1}} y_\psi^{[k]}(a)\} = L_\psi^{-1}\{\frac{1}{\xi^\beta} L_\psi\{f(x, y(x))\}\} \\ y(x) = L_\psi^{-1}\{\sum_{k=0}^{n-1} \frac{1}{\xi^{k+1}} y_\psi^{[}k](a)\} + L_\psi^{-1}\{\frac{1}{\xi^\beta} L_\psi\{f(x, y(x))\}\}. \end{aligned} \tag{21}$$

The following infinite series is used by the standard ADM to express the solution $y(x)$:

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{22}$$

In contrast, the nonlinear term is represented using the decomposed series of Adomian polynomials A_n as follows:

$$y(x, y(x)) = \sum_{n=0}^{\infty} A_n. \tag{23}$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The equation below provides a precise expression for A_n:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N(\sum_{j=0}^{n-1}\lambda^j y_j)\right]_{\lambda=0}, \quad n = 0, 1, 2, .. \tag{24}$$

Equation 21 is thus rewritten as follows using the decomposed series defined earlier.

$$\sum_{n=0}^{\infty} y_n(x) = L_{\psi}^{-1}\{\sum_{k=0}^{n-1}\frac{1}{\xi^{k+1}}y_{\psi}^{[}k](a)\} + L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}}L_{\psi}\{\sum_{n=0}^{\infty}A_n\}\} \tag{25}$$

This leads to the following formal recursive relation:

$$\begin{aligned} y_0(x) &= L_{\psi}^{-1}\{\sum_{k=0}^{n-1}\frac{1}{\xi^{k+1}}y_{\psi}^{[k]}(a)\} \\ y_n(x) &= L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}}L_{\psi}\{A_{n-1}\}\}, \quad n \geq 1. \end{aligned} \tag{26}$$

Finally, considering the following m-term approximations provides a reasonable solution.

$$\psi_n(x) = \sum_{j=0}^{n-1} y_j(x), \tag{27}$$

with the closed-from solution looking like

$$y(x) = \lim_{n\to\infty}\psi_n(x) = \sum_{j=0}^{\infty} y_j(x). \tag{28}$$

4 Applications

This section assesses the performance of the proposed ψ-BLADM scheme on several test problems. It considers both nonlinear and linear ψ-FDEs, with a particular focus on a real-world application in pharmacokinetics, where a mathematical model describing the fluctuation of drug concentration in human blood is analyzed.

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

4.1 Numerical Illustration

The ψ-BLADM technique is further applied to explore a few interesting ψ-FDEs presented as numerical examples in this section. Specifically, both linear and nonlinear initial value problems (IVPs) involving ψ-fractional derivatives are considered.

Example 14. *Consider the IVP for ψ-FDE in the manner described in [12].*

$$ {}^{C}D_{0^+}^{\frac{3}{2},\psi(x)} y(x) = y(x), y(0) = 1, \quad y_{\psi}^{[1]}(0) = 0. \tag{29}$$

We use the ψ-BLT on both sides of the equation to solve 29 with the ψ-BLADM, and when combined with 16, it gives:

$$\begin{aligned} L_{\psi}\{{}^{C}D_{0^+}^{\frac{3}{2},\psi(x)} y(x)\} &= L_{\psi}\{y(x)\} \\ \xi^{\frac{3}{2}} L_{\psi}\{y(x)\} - \xi^{\frac{1}{2}} y(0) &= L_{\psi}\{y(x)\} \\ L_{\psi}\{y(x)\} - \frac{1}{\xi} &= \frac{1}{\xi^{\frac{3}{2}}} L_{\psi}\{y(x)\} \end{aligned} \tag{30}$$

Upon applying the inverse ψ-LT to both sides of the equation, we get:

$$y(x) = L_{\psi}^{-1}\{\frac{1}{\xi}\} + L_{\psi}^{-1}\{\frac{1}{\xi^{\frac{3}{2}}} L_{\psi}\{y(x)\}\}. \tag{31}$$

By applying the ADM, it is transformed into:

$$\sum_{n=0}^{\infty} y_n(x) = L_{\psi}^{-1}\{\frac{1}{\xi}\} + L_{\psi}^{-1}\{\frac{1}{\xi^{\frac{3}{2}}} L_{\psi}\{\sum_{n=0}^{\infty} y_n(x)\}\}. \tag{32}$$

The recursive relations

$$\begin{aligned} y_0(x) &= L_{\psi}^{-1}\{\frac{1}{\xi}\}, \\ y_n(x) &= L_{\psi}^{-1}\{\frac{1}{\xi^{\frac{3}{2}}} L_{\psi}\{y_{n-1}(x)\}\}, \quad n \geq 1. \end{aligned} \tag{33}$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

and the ψ- BLADM provide

$$
\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= L_\psi^{-1}\{\frac{1}{\xi^{\frac{3}{2}}}L_\psi\{1\}\} = \frac{(\psi(x))^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\\
y_2(x) &= L_\psi^{-1}\{\frac{1}{\xi^{\frac{3}{2}}}L_\psi\{\frac{(\psi(x))^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\} = \frac{(\psi(x))^{3}}{\Gamma(4)}\\
&.\\
&.\\
&.
\end{aligned}
$$

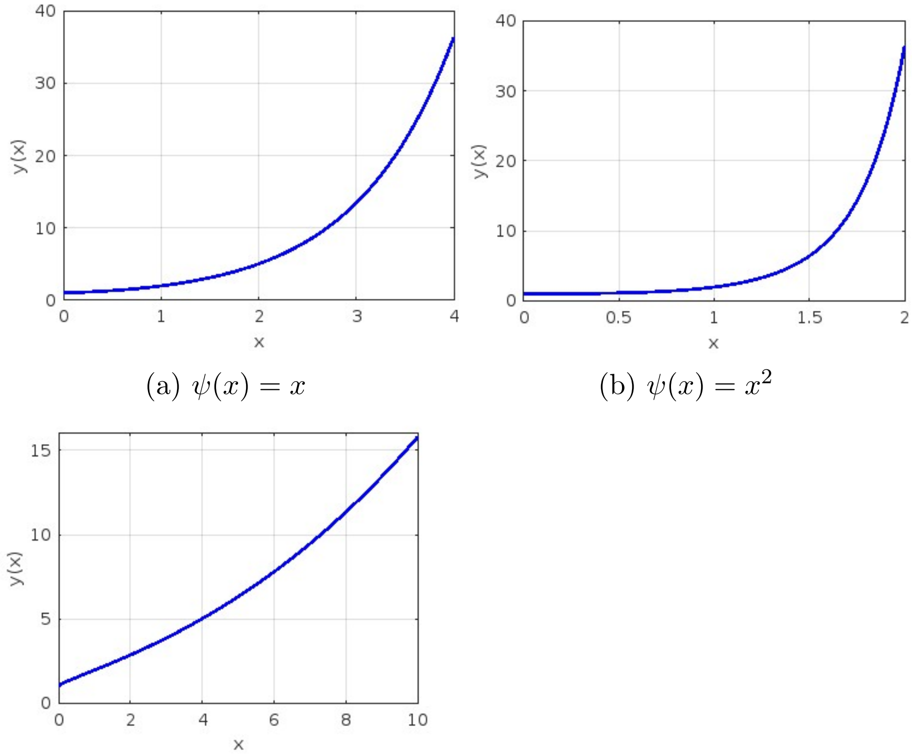
When the aforementioned components are summed, the exact solution to problem 35 is finally obtained as follows.

$$
\begin{aligned}
y(x) &= 1 + \frac{(\psi(x))^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{(\psi(x))^{3}}{\Gamma(4)} + \frac{(\psi(x))^{\frac{9}{2}}}{\Gamma(\frac{11}{2})}...\\
y(x) &= E_{\frac{3}{2}}(\psi(x)-\psi(0))^{\frac{3}{2}}.
\end{aligned}
\tag{34}
$$

where E_β is the Mittag-Leffler function previously mentioned. Furthermore, by calculating the first, third, and fifth terms, Almeida et al. [12] applied the Picard iteration method to approximate the solution of the ψ-FDE model described in 29. A graphical representation of the solution for the model, along with the following kernels, is also displayed in Figure 1: (a) $\psi(x) = x$, (b) $\psi(x) = x^2$, and (c) $\psi(x) = \sqrt{x}$. As is well known, in case (a), the problem 29 simplifies to the Caputo fractional derivative.

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)



(a) $\psi(x) = x$ (b) $\psi(x) = x^2$

(c) $\psi(x) = \sqrt{x}$

Figure 1: ψ-BLADM solutions with respect to different kernels for 29

Example 15. *To analyze the IVP for the ψ-FDE as described in [20].*

$$ {}^{C}D_{0^+}^{\beta,\psi(x)} y(x) - y(x) = 1, \quad 0 < \beta \leq 1, \quad y(0) = 1, \tag{35}$$

We apply the ψ-LT to both sides of the equation 35. Using the ψ-BLADM, which, when combined with 16, results in:

$$\begin{aligned} L_\psi\{{}^{C}D_{0^+}^{\beta,\psi(x)} y(x)\} - L_\psi\{y(x)\} &= L_\psi\{1\} \\ L_\psi\{y(x)\} - \frac{1}{\xi} &= \frac{1}{\xi^\beta} L_\psi\{y(x)\} + \frac{1}{\xi^{\beta+1}} \end{aligned} \tag{36}$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The inverse ψ-BLT is now applied to both sides of the latter equation, and

$$y(x) = 1 + \frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} + L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}} L_{\psi}\{y(x)\}\}. \tag{37}$$

On applying the ADM,

$$\sum_{n=0}^{\infty} y_n(x) = 1 + \frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} + L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}} L_{\psi}\{\sum_{n=0}^{\infty} y_n(x)\}\}. \tag{38}$$

The formal recursive relation and the ψ- BLADM yield

$$\begin{aligned} y_0(x) &= 1 + \frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} \\ y_1(x) &= L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}} L_{\psi}\{1 + \frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)}\}\} = \frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} + \frac{(\psi(x))^{2\beta}}{\Gamma(2\beta+1)} \\ y_2(x) &= L_{\psi}^{-1}\{\frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} + \frac{(\psi(x))^{2\beta}}{\Gamma(2\beta+1)}\} = \frac{(\psi(x))^{2\beta}}{\Gamma(2\beta+1)} + \frac{(\psi(x))^{3\beta}}{\Gamma(3\beta+1)} \; . \\ &. \\ &. \end{aligned}$$

By combining the previously mentioned components, the exact solution to 35 is finally obtained as follows.

$$\begin{aligned} y(x) &= \left(1 + \frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} + \frac{(\psi(x))^{2\beta}}{\Gamma(2\beta+1)} + ...\right) + \left(\frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} + \frac{(\psi(x))^{2\beta}}{\Gamma(2\beta+1)} + ...\right) \\ y(x) &= E_{\beta}(\psi(x))^{\beta} + (\psi(x))^{\beta} E_{\beta,\beta+1}(\psi(x))^{\beta}. \end{aligned} \tag{39}$$

where E_{β} and $E_{\beta,\beta+1}$ represent the Mittag-Leffler fu nctions with one and two parameters, respectively. The following solutions pertaining to the selection of function ψ are presented for specific kernels. Furthermore, Figure 2 presents the graphical representations of the ψ-BLADM solutions for various kernels and fractional-order

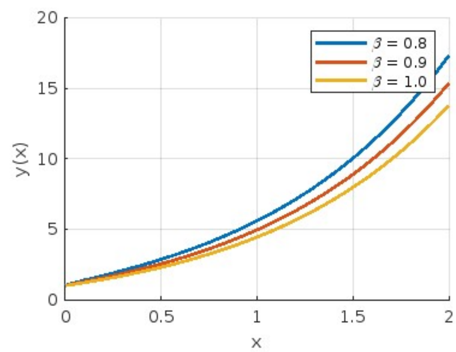
Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

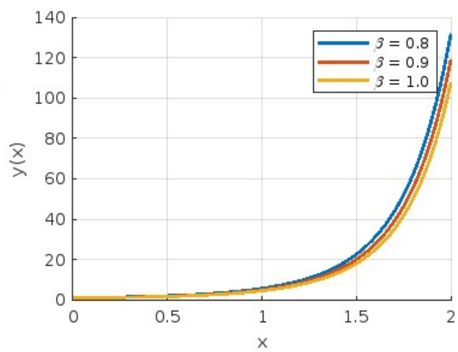
Table 2: Kernels and the corresponding solutions

Kernel	Solution
$\psi(x) = x$	$y(x) = E_\beta(x)^\beta + (x)^\beta E_{\beta,\beta+1}(x)^\beta$
$\psi(x) = \sqrt{x}$	$y(x) = E_\beta(\sqrt{x})^\beta + (\sqrt{x})^\beta E_{\beta,\beta+1}(\sqrt{x})^\beta$
$\psi(x) = x^2$	$y(x) = E_\beta(x^2)^\beta + (x^2)^\beta E_{\beta,\beta+1}(x^2)^\beta$

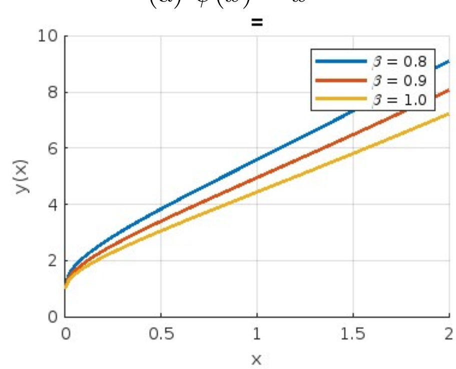
ψ values. Accordingly, we illustrate the solution across a range of parameters. (a) $\psi(x) = x$, (b) $\psi(x) = x^2$, and (c) $\psi(x) = \sqrt{x}$.



(a) $\psi(x) = x$



(b) $\psi(x) = x^2$



(c) $\psi(x) = \sqrt{x}$

Figure 2: ψ-LADM solutions of (35) for different kernels and β.

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Example 16. *To examine the ψ-FDE IVP [17]*

$$
\begin{aligned}
{}^{C}D_{0^+}^{\beta,\psi(x)}y(x) + y(x) = (\psi(x))^4 - \frac{1}{2}(\psi(x))^3 - \frac{3}{\Gamma(4-\beta)}(\psi(x))^{3-\beta} \\
+ \frac{24}{\Gamma(5-\beta)}(\psi(x))^{4-\beta} \qquad (40) \\
y(0) = 0, \quad x \in [0,1], \quad 0 \le \beta \le 1
\end{aligned}
$$

By applying the ψ-Bicomplex Laplace Transform (ψ-BLT) to both sides of the governing equation and utilizing equation (16), *equation* (52) *can be solved using the ψ-BLADM, ultimately leading to the desired solution.*

$$
\begin{aligned}
&L_\psi\{y(x)\} + \frac{1}{\xi^\beta}L_\psi\{y(x)\} \\
&= \frac{1}{\xi^\beta}L_\psi\left\{(\psi(x))^4 - \frac{1}{2}(\psi(x))^3 - \frac{3}{\Gamma(4-\beta)}(\psi(x))^{3-\beta} + \frac{24}{\Gamma(5-\beta)}(\psi(x))^{4-\beta}\right\}
\end{aligned} \tag{41}
$$

This, when applied to the later equation, produces the inverse ψ-BLT

$$
\begin{aligned}
y(x) = &-L_\psi^{-1}\{\frac{1}{\xi^\beta}L_\psi\{y(x)\}\}\frac{1}{\xi^\beta}L_\psi\{y(x)\} \\
&+ L_\psi^{-1}\left\{\frac{1}{\xi^\beta}L_\psi\left\{(\psi(x))^4 - \frac{1}{2}(\psi(x))^3 - \frac{3}{\Gamma(4-\beta)}(\psi(x))^{3-\beta} + \frac{24}{\Gamma(5-\beta)}(\psi(x))^{4-\beta}\right\}\right\}.
\end{aligned} \tag{42}
$$

Additionally, using the conventional ADM, the equation above now becomes

$$
\begin{aligned}
\sum_{n=0}^{\infty} y_n(x) = &-L_\psi^{-1}\{\frac{1}{\xi^\beta}L_\psi\{\sum_{n=0}^{\infty} y_n(x)\}\}\frac{1}{\xi^\beta}L_\psi\{y(x)\} \\
&+ L_\psi^{-1}\left\{\frac{1}{\xi^\beta}L_\psi\left\{(\psi(x))^4 - \frac{1}{2}(\psi(x))^3 - \frac{3}{\Gamma(4-\beta)}(\psi(x))^{3-\beta} + \frac{24}{\Gamma(5-\beta)}(\psi(x))^{4-\beta}\right\}\right\}.
\end{aligned} \tag{43}
$$

Finally, summing the above equation reveals a general recurring

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

pattern, expressed as follows:

$$y_0(x) = L_\psi^{-1}\{\frac{1}{\xi^\beta}L_\psi\{(\psi(x))^4 - \frac{1}{2}(\psi(x))^3 - \frac{3}{\Gamma(4-\beta)}(\psi(x))^{3-\beta} + \frac{24}{\Gamma(5-\beta)}(\psi(x))^{4-\beta}\Big\}\Big\}$$
$$y_n(x) = L_\psi^{-1}\{\frac{1}{\xi^\beta}L_\psi\{y_{n-1}(x)\}\}, \quad n \geq 1. \tag{44}$$

Similarly, a few ψ-BLADM terms are computed using the recurrence techniques described above, as demonstrated below.

$$y_0(x) = (\psi(x))^4 - \frac{1}{2}(\psi(x))^3 + \frac{\Gamma(5)(\psi(x))^{4+\beta}}{\Gamma(5+\beta)} - \frac{3(\psi(x))^{3+\beta}}{\Gamma(4+\beta)}$$
$$y_1(x) = -\frac{\Gamma(5)(\psi(x))^{4+\beta}}{\Gamma(5+\beta)} + \frac{3(\psi(x))^{3+\beta}}{\Gamma(4+\beta)} - \frac{\Gamma(5)(\psi(x))^{4+2\beta}}{\Gamma(5+2\beta)} + \frac{3(\psi(x))^{3+2\beta}}{\Gamma(4+2\beta)}$$
$$\vdots \tag{45}$$

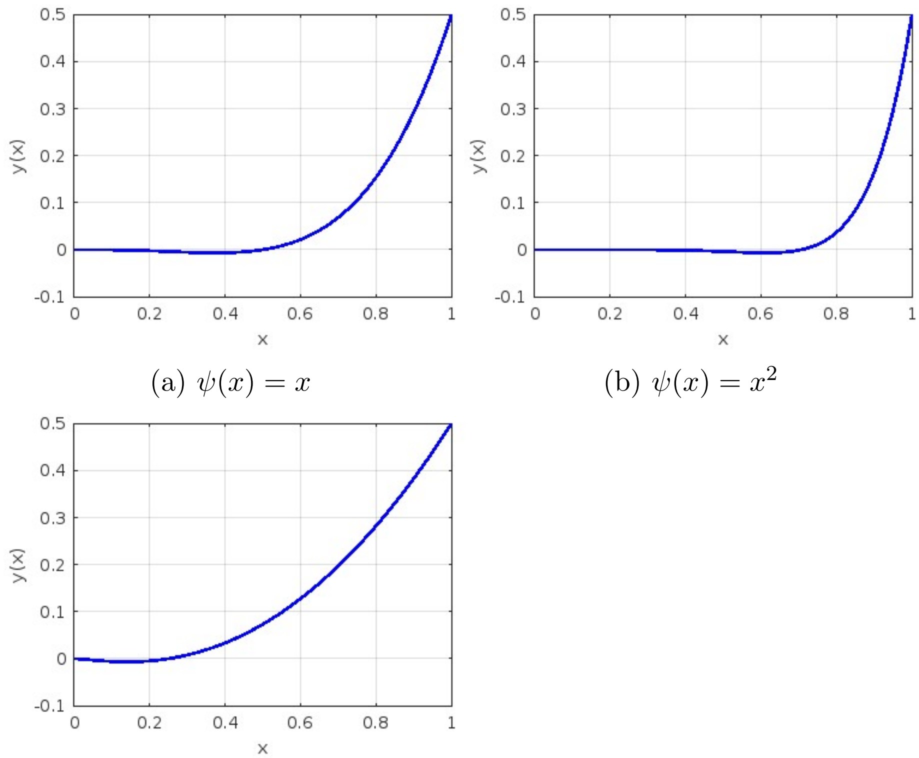
Interestingly, it is important to note that some noise terms appear in the model's recurrence solution, especially with regard to the $y_0(x)$ and $y_1(x)$ components as $\left(\frac{\Gamma(5)(\psi(x))^{4+\beta}}{\Gamma(5+\beta)} - \frac{3(\psi(x))^{3+\beta}}{\Gamma(4+\beta)}\right)$. Therefore, the noise terms will finally be eliminated when just these two elements are added, leaving the precise solution to problem 52 as follows.

$$y(x) = (\psi(x))^4 - \frac{1}{2}(\psi(x))^3. \tag{46}$$

Furthermore, Sunthrayuth et al. [17] solved the linear version of the model represented by equation (52) using the ψ-Haar wavelet operational matrix approach. They obtained an approximate solution by calculating the first six terms. However, by computing just two terms, the current ψ-BLADM yields an exact solution. Additionally, Figure 3 presents the solution graphically for variety of kernels: (a) $\psi(x) = x$, (b) $\psi(x) = x^2$, and (c) $\psi(x) = \sqrt{x}$. The usual Caputo fractional derivative case is represented by (a).

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)



(a) $\psi(x) = x$

(b) $\psi(x) = x^2$

(c) $\psi(x) = \sqrt{x}$

Figure 3: ψ-LADM solutions with respect to the different kernels for 40

Example 17. *Examine the following IVP for the ψ-Caputo Riccati differential equation [17].*

$$ {}^{C}D^{\beta,\psi(x)} = -y^2(x) + 1, \quad 0 < \beta \leq 1, \quad x \in [0,1] y(0) = 1, \quad (47)$$

Consequently, we apply ψ-BLT to both sides of the model in order to solve this by ψ- BLADM, which, with the aid of 16, produces

$$L_\psi\{y(x)\} = -\frac{1}{\xi^\beta} L_\psi\{y^2(x)\} + \frac{1}{s^{\beta+1}} \quad (48)$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Now, the inverse ψ-BLT is applied to both sides of the latter equation, and

$$y(x) = -L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}}L_{\psi}\{y^{2}(x)\}\} + \frac{\psi(x)^{\beta}}{\Gamma(\beta+1)}, \tag{49}$$

such that upon applying the ADM, it reduces to

$$\sum_{n=0}^{\infty} y_n(x) = -L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}}L_{\psi}\{\sum_{n=0}^{\infty} A_n\}\} + \frac{\psi(x)^{\beta}}{\Gamma(\beta+1)}, \tag{50}$$

where A_n are the Adomian polynomials for the nonlinear term $y^2(x)$, which has the following form when calculated iteratively:

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_0y_1, \\ A_2 &= 2y_0y_2 + y_1^2, \\ &. \\ &. \\ &. \end{aligned} \tag{51}$$

Moreover, the following is the general recurrent scheme for the governing model.

$$\begin{aligned} y_0(x) &= \frac{(\psi(x))^{\beta}}{\Gamma(\beta+1)} \\ y_n(x) &= -L_{\psi}^{-1}\{\frac{1}{\xi^{\beta}}L_{\psi}\{A_{n-1}\}\} \end{aligned}$$

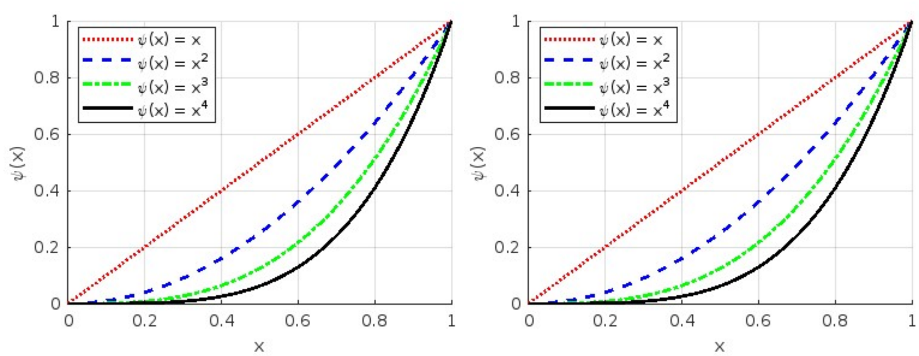
Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

with the following few elements:

$$
\begin{aligned}
y_0(x) &= \frac{(\psi(x))^\beta}{\Gamma(\beta+1)} \\
y_1(x) &= \frac{\Gamma(2\beta+1)(\psi(x))^{3\beta}}{\Gamma(\beta+1)^2\Gamma(3\beta+1)} \\
y_2(x) &= \frac{2\Gamma(4\beta+1)\Gamma(2\beta+1)(\psi(x))^{3\beta}}{\Gamma(\beta+1)^3\Gamma(3\beta+1)\Gamma(5\beta+1)} \\
&\;\;\vdots
\end{aligned}
$$

As a result, the problem's approximate solution is derived from the first three components as follows: $\psi_3 = \sum_{n=0}^{3} y_n(x)$. *Furthermore, Figure 4 illustrates how this solution behaves in relation to different fractional-order* β *values and functions* ψ.



(a) $\beta = 0.9$ (b) $\beta = 1$

Figure 4: ψ-LADM solutions with respect to the different β

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

4.2 Application of Pharmacokinetics

The proposed approach is further extended in this section to explore a practical application involving the ψ-Caputo fractional derivative [27]. The effectiveness of this approach has been demonstrated by Almaida [23] using real-world datasets in several physical applications, including gross domestic product modeling and Newton's law of cooling. The method's application to population growth analysis using the logistic equation [24] is also noteworthy.

This study focuses on pharmacokinetics, which examines how drug concentrations change in human blood. Specifically, we consider a one-compartment open system, as illustrated in Figure 5, which shows how drug concentration changes and is eliminated from the human body. In Figure 5, VD and k represent the drug's volume of distribution and elimination rate, respectively. In this model, the body is treated as a homogeneous entity, allowing the drug to easily enter and exit the body.

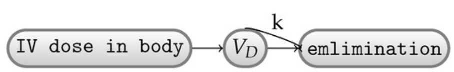


Figure 5: IV bolus, one-compartment open model

Additionally, the following is the form of the mathematical equation that uses classical calculus to describe the whole process:

$$\frac{dy}{dx} = -ky(x), \quad u(0) = u_0 \tag{52}$$

It allows the precise solution that follows.

$$y(x) = y_0 e^{-kx} \tag{53}$$

The exact solution derived above represents the drug level in the human body at any given time x. However, in modern non-classical calculus, the mathematical equation describing the movement of drug concentration in human blood, using the application of the ψ-Caputo fractional derivative from equation (52), takes the following

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

form:

$$D_a^\beta, \psi(x) y(x) = -k y(x) \tag{54}$$

Using the proposed ψ-BLADM, we now turn our attention to the ψ-Caputo fractional equation introduced earlier. The dataset from [30] will be used as a benchmark for the numerical simulations. For more details, refer to the recent study by Awadalla et al. [27], in which the authors applied the ψ- Caputo fractional derivative to fit the dataset from [30]. Accordingly, we examine the following pharmacokinetic situations based on the findings of [27] and [30]:

Case I: Consider the following pharmacokinetics IVP for ψ-FDE:

$${}^C D^{1.13327,x} y(x) = -0.51749 y(x), \quad y(0) = 20, \tag{55}$$

where $\beta = 1.13327, k = 0.51749$, and $\psi(x) = x$, with the body's first dosage being $y(0) = 20$.
With the aid of 16, we apply the ψ-BLT to both sides of the model in order to solve 55 using ψ- BLADM, thus obtaining

$$L_\psi\{y(x)\} - \frac{20}{\xi} = \frac{1}{\xi^{1.13327}} L_\psi\{0.51749 y(x)\}. \tag{56}$$

The following is the result of applying the inverse ψ-BLT to the latter equation.

$$y(x) = L_\psi^{-1}\{\frac{20}{\xi}\} - L_\psi^{-1}\{\frac{1}{\xi^{1.13327}} L_\psi\{0.51749 y(x)\}\}. \tag{57}$$

Using the conventional ADM procedure results in

$$\sum_{n=0}^{\infty} y(x) = L_\psi^{-1}\{\frac{20}{\xi}\} - L_\psi^{-1}\{\frac{1}{\xi^{1.13327}} L_\psi\{0.51749 \sum_{n=0}^{\infty} y(x)\}\}. \tag{58}$$

This demonstrates the following recurring relation

$$\begin{aligned} y_0(x) &= L_\psi^{-1}\{\frac{20}{\xi}\} = 20 \\ y_n(x) &= -L_\psi^{-1}\{\frac{1}{\xi^{1.13327}} L_\psi\{0.51749 y_{n-1}(x)\}\}, \quad n \geq 1. \end{aligned} \tag{59}$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

As a result, we obtain the following results.

$$
\begin{aligned}
y_0(x) &= 20\\
y_1(x) &= -L_\psi^{-1}\{\frac{1}{\xi^{1.13327}}L_\psi\{0.51749(20)\}\}\\
&= -9.728392289(\psi(x)-\psi(0))^{1.13327} = -9.728392289x^{1.13327}\\
y_2(x) &= -L_\psi^{-1}\{\frac{1}{\xi^{1.13327}}L_\psi\{0.51749(-9.728392289x^{1.13327})\}\}\\
&= 2.065825777(\psi(x)-\psi(0))^{2.26654} = 2.065825777x^{2.26654}\\
&.\\
&.\\
&.
\end{aligned}
\tag{60}
$$

which, when added together, provide the same answer 52 as

$$
\begin{aligned}
&y(x) = 20 - 9.728392289x^{1.13327} + 2.065825777x^{2.26654} + ...,\\
&y(x) = y_0E_\beta[(-k)(\psi(x)-\psi(0))^\beta] = 20E_{1.13327}[(-0.51749)(x)^{1.13327})],
\end{aligned}
\tag{61}
$$

where $E_{(.)}$ is the one-parameter Mittag-Leffler function; the derived exact solution is graphically shown in Figure 6a, in comparison with the physical dataset from [30].

Case II: Following is the pharmacokinetics IVP for ψ-FDE:

$$
{}^CD^{1.11080,x+1}y(x) = -0.49621y(x), \quad y(0) = 20, \tag{62}
$$

where $\beta = 1.11080, k = 0.49621$, and $\psi(x) = x+1$, with the body's first dosage being $y(0) = 20$.
With the aid of 16, we apply the ψ-BLT to both sides of the model in order to solve 62 using ψ- BLADM, thus obtaining

$$
L_\psi\{y(x)\} - \frac{20}{\xi} = \frac{1}{\xi^{1.11080}}L_\psi\{0.49621y(x)\}. \tag{63}
$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

The following is the result of applying the inverse ψ-BLT to the latter equation.

$$y(x) = L_{\psi}^{-1}\{\frac{20}{\xi}\} - L_{\psi}^{-1}\{\frac{1}{\xi^{1.11080}}L_{\psi}\{0.49621y(x)\}\}. \tag{64}$$

Additionally, using the conventional ADM procedure results in

$$\sum_{n=0}^{\infty} y(x) = L_{\psi}^{-1}\{\frac{20}{\xi}\} - L_{\psi}^{-1}\{\frac{1}{\xi^{1.11080}}L_{\psi}\{0.49621\sum_{n=0}^{\infty} y(x)\}\}. \tag{65}$$

This demonstrates the subsequent recurring relationship

$$\begin{aligned} y_0(x) &= L_{\psi}^{-1}\{\frac{20}{\xi}\} = 20 \\ y_n(x) &= -L_{\psi}^{-1}\{\frac{1}{\xi^{1.11080}}L_{\psi}\{0.49621y_{n-1}(x)\}\}, \quad n \geq 1. \end{aligned} \tag{66}$$

As a result, we obtain

$$\begin{aligned} y_0(x) &= 20 \\ y_1(x) &= -L_{\psi}^{-1}\{\frac{1}{\xi^{1.11080}}L_{\psi}\{0.49621(20)\}\} \\ &= -9.433446716(\psi(x) - \psi(0))^{1.11080} = -9.433446716x^{1.11080} \\ y_2(x) &= -L_{\psi}^{-1}\{\frac{1}{\xi^{1.11080}}L_{\psi}\{0.49621(-9.433446716x^{1.11080})\}\} \\ &= 1.988051839(\psi(x) - \psi(0))^{2.2216} = 1.9880518397x^{2.2216} \\ &. \\ &. \\ &. \end{aligned} \tag{67}$$

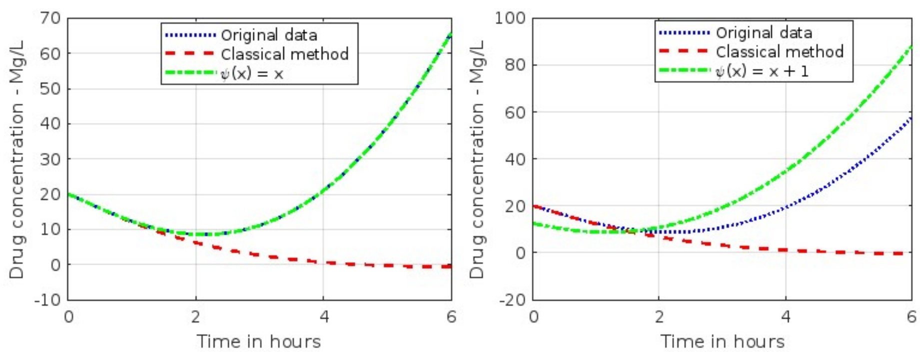
which, when added together, provide the same answer 52 as

$$\begin{aligned} y(x) &= 20 - 9.433446716x^{1.11080} + 1.9880518397x^{2.2216} + ..., \\ y(x) &= y_0 E_{\beta}[(-k)(\psi(x) - \psi(0))^{\beta}] = 20E_{1.11080}[(-0.49621)(x)^{1.11080})], \end{aligned} \tag{68}$$

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

where $E_{(.)}$ represents the one-parameter Mittag-Leffler function; the derived solution is graphically presented in Figure 6a, together with the physical dataset from [30].



(a) (b)

Figure 6: ψ-LADM solutions for the fractional pharmacokinetics IVP

5 Conclusion

Fractional differential equations (FDEs) are powerful tools for modeling a wide range of physical phenomena encountered in the real world, where high-precision solutions are often essential. The incorporation of ψ-Caputo fractional derivatives enhances the flexibility of these models, allowing them to capture hidden dynamics in complex real-world processes. In this work, a ψ-based Laplace Decomposition Method (ψ-BLADM) was proposed for solving ψ-Caputo differential e quations. T he p roposed s cheme w as fi rst validated through several test initial value problems (IVPs) involving fractional differential e quations. Furthermore, t he m ethod w as extended to practical applications, particularly in pharmacokinetic modeling. The results demonstrated a strong agreement with empirical data and yielded solutions that closely matched the known exact solutions in the test cases. In summary, the proposed ap-

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

proach is effective and versatile, making it suitable for a broad class of ψ-FDEs under various initial and boundary conditions.

References

[1] A. Kumar and P. Kumar, Bicomplex version of Laplace transform, *International Journal of Engg. and Tech*, **3** (2011), 225-232.

[2] M. Alsulami, Generalized Laplace Transform with Adomian Decomposition Method for Solving Fractional Differential Equations Involving ψ-Caputo Derivative, *Mathematics*, **12** (2024), 3499.

[3] U. P. Sharma and R. Agarwal, Bicomplex Laplace transform of fractional order, properties and applications, *Journal of Computational Analysis and Applications*, **30**(2022), 370-385.

[4] A. Thirumalai, and K. Muthunagai, Physical Applications of the Bicomplex Fractional Laplace Transform: A Perception, (2024).

[5] R. Agarwal, U. P. Sharma and R. P. Agarwal, Bicomplex Mittag-Leffler function and associated properties, *Journal of Nonlinear Sciences and Applications (JNSA)*, **15**, (2022).

[6] H. Ji-Huan, Fractal calculus and its geometrical explanation, *Results in Physics*, **10** (2018), 272-276.

[7] Q. Wang, Fractal calculus and its application to explanation of biomechanism of polar bear hairs, *Fractals*, **26**, (2018), 1850086.

[8] A. A. Kilbas, M. S. Hari and J. T. Juan, *Theory and applications of fractional differential equations*, elsevier, (2006).

Received: July 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[9] G. S. Teodoro, JA Tenreiro and E. Capelas De Oliveira, A review of definitions of fractional derivatives and other operators, *Journal of Computational Physics*, **388** (2019), 195-208.

[10] C. Milici, D. Gheorghe and J. M. Tenreiro, *Introduction to fractional differential equations*, Springer, (2018).

[11] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Communications in Nonlinear Science and Numerical Simulation*, **44** (2017), 460-481.

[12] R. Almeid, B. M. Agnieszka, and M. T. M. Teresa, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Mathematical Methods in the Applied Sciences*, **41** (2018), 336-352.

[13] C. Derbazi, B. Zidane, and B. Mouffak, Coupled system of ψ-Caputo fractional differential equations without and with delay in generalized Banach spaces, *Results in Nonlinear Analysis*, **5** (2022), 42-61.

[14] A. Boutiara, S. A. Mohamed, and B. Maamar, Existence results for ψ-Caputo fractional neutral functional integro-differential equations with finite delay, *Turkish Journal of Mathematics*, **44** (2020), 2380-2401.

[15] H. Khan, Z. Akbar, and K¿ Ishfaq, Analysis of a Coupled System of ψ-Caputo Fractional Derivatives with Multipoint–Multistrip Integral Type Boundary Conditions, *Qualitative Theory of Dynamical Systems*, **23** (2024), 129.

[16] H. Aydi, J. Mohamed, and S. Bessem, On positive solutions for a fractional thermostat model with a convex–concave source term via ψ-Caputo fractional derivative, *Mediterranean Journal of Mathematics*, **17** (2020), 16.

[17] F. M. Alharbi, et al., Numerical Investigation of Fractional-Order Differential Equations via ψ-Haar-Wavelet Method, *Journal of Function Spaces*, **2021** (2021), 3084110.

Received: July 22, 2025

[18] R. Almeida, J. Mohamed, and S. Bessem, A numerical study of fractional relaxation–oscillation equations involving ψ-Caputo fractional derivative, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, **113** (2019), 1873-1891.

[19] F. Jarad, and A. Thabet, *Generalized fractional derivatives and Laplace transform*, (2020).

[20] H. M. Fahad, Mujeeb Ur Rehman, and F. Arran, On Laplace transforms with respect to functions and their applications to fractional differential equations, *Mathematical Methods in the Applied Sciences*, **46** (2023), 8304-8323.

[21] M. Alsulami, et al., Semi-analytical investigation for the ψ-Caputo fractional relaxation-oscillation equation using the decomposition method, *European Journal of Pure and Applied Mathematics*, **17** (2024), 2311-2398.

[22] A. S. Alsulami, M. Al-Mazmumy, M. A. Alyami, & M. Alsulami, APPLICATION OF ADOMIAN DECOMPOSITION METHOD TO A GENERALIZED FRACTIONAL RICCATI DIFFERENTIAL EQUATION (ψ-FRDE), *Advances in Differential Equations and Control Processes*, **31** (2024), 531-561.

[23] M. Al-Mazmumy, Mariam et al., Efficient Modified Adomian Decomposition Method for Solving Nonlinear Fractional Differential Equations." International Journal of Analysis and Applications 22 (2024): 76-76.

[24] P. B. Dhandapani et al., On a Novel Dynamics of a SIVR Model Using a Laplace Adomian Decomposition Based on a Vaccination Strategy, *Fractal and Fractional*, **7** (2023), 407.

[25] A. O. Yunus et al., Mathematical analysis of fractional-order Caputo's derivative of coronavirus disease model via Laplace Adomian decomposition method,*Beni-Suef University Journal of Basic and Applied Sciences*, **11** (2022), 144.

[26] M. Al-Mazmumy et.al, Reliable computational method for systems of fractional differential equations endowed with ψ-Caputo fractional derivative, *Contemporary Mathematics* (2024), 4991-5011.

[27] M. Awadalla et.al, Modeling Drug Concentration Level in Blood Using Fractional Differential Equation Based on Psi-Caputo Derivative, *Journal of Mathematics*, **2022** (2022), 9006361.

[28] R. Almeida et.al, What is the best fractional derivative to fit data?, *Applicable Analysis and Discrete Mathematics*, **11** (2017), 358-368.

[29] M. Awadalla, Y. N. Yves Yannick and A. A. Kinda, Psi-Caputo Logistic Population Growth Model, *Journal of Mathematics*, **2021** (2021), 8634280.

[30] A. T. Ahmed, *Pharmacokinetics of drugs following IV bolus, IV infusion, and oral administration*, **10**, Rijeka: Intechopen (2015).