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NUMERICAL ALGORITHM FOR SOLVING THE INVERSE PROBLEM OF SOURCE TERM IDENTIFICATION OF THE SUBDIFFUSION DIFFERENTIAL EQUATION UNDER STURM TYPE BOUNDARY CONDITIONS

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Abstract

This work is devoted to the inverse problem of source term identification for the subdiffusion differential equation under Sturm-type boundary conditions. Additional information in the form of the final overdetermination condition is given. We construct a numerical algorithm for solving the inverse problem. The algorithm is based on the biconjugate gradient stabilized iterative method and the Tikhonov regularization method. To solve the forward initial boundary value subproblems at each iteration, we apply the finite difference scheme. We present the results of numerical experiments that confirm the ability of the developed algorithm to solve the inverse problem in the case of perturbations in input data.

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Key Words and Phrases: fractional differential equations, Sturm type boundary conditions, inverse and ill-posed problems, numerical algorithms, finite difference scheme, conjugate gradient method

1 Introduction

Recently, fractional differential equations have become a popular instrument in the construction of mathematical models of various phenomena in science, engineering, and economics [1, 2, 3, 4]. Applications of fractional calculus include anomalous diffusion models, lossy media, viscoelasticity models, ferroelectrics, neuron models, and others.

Forward problems, such as initial boundary value problems for classical and time-fractional differential equations, are a well-developed topic [5, 6, 7, 8, 9, 10]. Inverse problems, including source term identification problems for subdiffusion equations, are less frequently explored. Usually, these problems are ill-posed [11, 12, 13, 14, 15], meaning that the small disturbance for the input data produces large changes in the solution. Thus,

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the regularization methods must be used to achieve a stable solution [16, 17, 18, 19, ?, 21]. Many researchers have implemented and studied various numerical methods for the source term identification problem for time-fractional differential equations. Among them are quasi-boundary value regularization method [22], Landweber iterative method [23], Tikhonov regularization method [24], conjugate gradient method [25, 26], and others.

Note that most of the works on the source identification problems consider the Dirichlet (first type) and Neumann (second type) boundary conditions. Many models of various processes such as diffusion and heat transfer lead to more complex boundary conditions. Work [27] considers the source term identification problem for the time-fractional diffusion equation in the case of Robin boundary conditions.

Nonlocal boundary conditions of Samarskii-Ionkin type arise in models of heat transfer in plasma [28, 29, 30]. One of the approaches [31] to dealing with such nonlocal boundary conditions consists in reducing the problem of not strongly regular nonlocal boundary conditions to a solution of two problems with local boundary conditions, namely, the Dirichlet type and the Sturm type (third type). This approach was adapted to develop the numerical algorithm for solving the forward initial boundary value problem for the subdiffusion equation with nonlocal boundary conditions in work [32].

In this work, we construct a numerical algorithm for solving the inverse problem of source term identification in the subdiffusion equation with Sturm-type boundary conditions. Such a problem arises as a subproblem in solving the source term identification problem in the case of nonlocal boundary conditions. To solve the inverse problem, we apply the iterative biconjugate gradient stabilized (BiCGSTAB) method and Tikhonov regularization method. At each iteration of this method, we need to solve the well-posed forward initial boundary value subproblems. For this task, we use the previously developed numerical algorithm [32] based on a finite difference scheme. To confirm the efficiency of the proposed algorithm, we conduct numerical experiments with disturbed data. The proposed numerical algorithm may be used in numerical algorithms for solving more generalized problem of source term identification for a subdiffusion equation in the case of nonlocal boundary condition.

The rest of the article is structured as follows: Section 2 describes the statement of the inverse problem of source term identification for the subdiffusion differential equation with Sturm-type boundary conditions. Section 3 is devoted to the discretization and finite difference scheme for solving the forward initial boundary value problem for the considered subdiffusion equation. Section 4 describes the numerical algorithm for solving the inverse problem. Section 5 describes the conducted numerical experiments and discusses their results. Section 6 concludes the article.

2 Problem Statement

In this work, we consider numerical algorithm for solving the source term identification inverse problem for the one-dimensional subdiffusion equation. Consider the equation

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) = \psi(x)\eta(x,t) + f(x,t), \tag{1}$$

with initial conditions

$$u(x,0) = 0, \quad 0 \le x \le 1, \tag{2}$$

and the following boundary condition

$$\begin{cases} u_x(0,t) - au(0,t) = 0, \\ u_x(1,t) + au(1,t) = 0, \end{cases} \quad 0 \le t \le T, \tag{3}$$

where $a > 0$.

Here, we consider the Caputo fractional derivative with order $0 < \alpha < 1$ in the form [4]

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$$D_t^{\alpha} f(x,t) = \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t \frac{f^{(m)}(x,s)}{(t-s)^{\alpha-m+1}} ds,$$

with $m = \lceil \alpha \rceil : \alpha \in (m-1, m)$, $m \in N, x > 0$.

When functions $\psi(x)$, $\eta(x,t)$, and $f(x,t)$ are known, the problem of finding the unknown function $u(x,t)$ from equation (1)– (3) is a forward initial boundary value problem.

In the present work, we consider the inverse problem of source term identification. It consists in finding a pair of unknown functions $[u(x,t), \psi(x)]$ when functions $\eta(x,t)$ and $f(x,t)$ are given. To solve it, we have additional final overdetermination data

$$u(x,T) = \varphi(x), 0 \le x \le 1. \tag{4}$$

In real life, the measurement data contains noise. Thus, we assume that we are given function $\varphi_\delta(x)$ such as

$$\|\varphi(x) - \varphi_\delta(x)\| < \delta, \tag{5}$$

where $\|\cdot\|$ denotes the L^2 norm, and $\delta > 0$ is the noise level.

This source term identification problem is an ill-posed one. Thus, we need to use a regulatization method to obtain a stable solution. Theoretical analysis of the inverse problem of source term identification for time-fractional differential equation under more generalized Robin boundary condition can be found in [27]. In the present work, we focus on the numerical solution of the considered inverse problem, rather than the theoretical aspects of stability and convergence of the approximate solution.

3 Discretization and Algorithm for Solving the Forward Problem

Consider the initial boundary value problem in the form

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) = f(x,t), \tag{6}$$

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$$u(x,0)=0, \quad 0\le x\le 1, \tag{7}$$

$$\begin{cases} u_x(0,t)-au(0,t)=0, \\ u_x(1,t)+au(1,t)=0, \end{cases} \quad 0\le t\le T, \tag{8}$$

where $a>0$, $f(x,t)$ is a given function.

Let us introduce regular grids for x and t with $(M+1)$ and $(N+1)$ points, respectively: $i=0,...,M,\ h=1/M,\ x_i=ih$, $n=0,...,N,\ \tau=1/N,\ t_n=n\tau$. Denote the values of grid functions at grid points as $U_{i,n}=u(x_i,t_n)$.

To approximate the Caputo fractional derivative of function $u(x,t)$ at the time level n, we use the L1 formula [6, 4]:

$$\begin{aligned} & D_t^{\alpha}(U_{i,n}) \approx \sigma_{\alpha,\tau}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left(U_{i,n-j+1}-U_{i,n-j}\right), \\ & \sigma_{\alpha,\tau}=\frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}, \quad \omega_j^{(\alpha)}=j^{1-\alpha}-(j-1)^{1-\alpha}, \\ & n=1,\dots,N. \end{aligned} \tag{9}$$

Using formula (9) of order $O(\tau^{2-\alpha})$ and a central difference scheme of order $O(h^2)$ to construct an implicit difference scheme for initial boundary value problem (6)–(8), we obtain the following system of linear algebraic equations in matrix form for points $x_i,\ i=0,...,M$

$$A\cdot\begin{bmatrix} U_{0,n} \\ U_{1,n} \\ \vdots \\ U_{M-1,n} \\ U_{M,n} \end{bmatrix} = \begin{bmatrix} F_{0,n} \\ F_{1,n} \\ \vdots \\ F_{M-1,n} \\ F_{M,n} \end{bmatrix}, \tag{10}$$

where

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$$A = \begin{bmatrix} \left(\sigma_{\alpha,\tau} + \frac{2}{h^2} + \frac{2a}{h}\right) & \left(-\frac{2}{h^2}\right) & \cdots & & \\ \left(-\frac{1}{h^2}\right) & \left(\sigma_{\alpha,\tau} + \frac{2}{h^2}\right) & \left(-\frac{1}{h^2}\right) & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & \left(-\frac{1}{h^2}\right) & \left(\sigma_{\alpha,\tau} + \frac{2}{h^2}\right) & \left(-\frac{1}{h^2}\right) \\ & & \cdots & \left(-\frac{2}{h^2}\right) & \left(\sigma_{\alpha,\tau} + \frac{2}{h^2} + \frac{2a}{h}\right) \end{bmatrix},$$

$$F_{i,n} = \sigma_{\alpha,\tau} \left(U_{i,n-1} - \sum_{j=2}^{n} \omega_j^{(\alpha)} \left(U_{i,n-j+1} - U_{i,n-j} \right) \right) + f_{i,n},$$

$$F_{i,1} = \sigma_{\alpha,\tau} U_{i,0} + f_{i,1}.$$

Thus, to solve the initial boundary value problem (1)–(3), we need to solve SLAEs (10) for $U_{i,n}, \quad i = 0, ..., M$ for each sequential time level $n = 1, 2, ..., N$. Coefficient matrix A is a tridiagonal one. To solve these SLAEs, we utilize the Thomas algorithm (sweep method) [33].

The stability and convergence of the difference scheme (10) is investigated in work [32].

The difference scheme (10) is unconditionally stable with respect to the right-hand part and the following theorem is valid.

Theorem 1. *Suppose* $\{U_{i,n} \mid 0 \le i \le M, \ 0 \le n \le N\}$ *is the solution of the difference scheme* (10). *Then, the following estimate holds:*

$$\|U_n\|_\infty \le \frac{1}{\sigma_{\alpha,\tau} \omega_n^{(\alpha)}} \max_{1 \le m \le n} \|f_m\|_\infty, \quad 1 \le n \le N, \tag{11}$$

where $U_n = \{U_{0,n}, U_{1,n}, ..., U_{M,n}\}$.

The approximation scheme (10) has a truncation error and convergence order of $O(\tau^{2-\alpha} + h^2)$.

Theorem 2. *Suppose* $\{U_{i,n} | \ 0 \le i \le M, \ 0 \le n \le N\}$ *is the exact solution of problem* (6)–(8) *and* $\left\{\overline{U_{i,n}} | \ 0 \le i \le M, \ 0 \le n \le N\right\}$ *is the solution of difference scheme* (10). *Consider the error*

$$e^U{}_{i,n} = U_{i,n} - \overline{U_{i,n}}, \ 0 \le i \le M, \ 0 \le n \le N.$$

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Then, for some $c_1 > 0$, it holds that

$$\left\| e^U{}_n \right\|_\infty \leq c_1 T^\alpha \Gamma(1-\alpha)(\tau^{2-\alpha} + h^2), \quad 1 \leq n \leq N. \tag{12}$$

4 Numerical Algorithm for Inverse Problem

Let us represent the solution $u(x,t)$ of the inverse problem (1)–(4) as a sum of functions

$$u(x,t) = w(x,t) + v(x,t).$$

Here, $v(x,t)$ is a solution of the initial boundary value problem

$$D_t^\alpha v(x,t) - v_{xx}(x,t) = f(x,t), \tag{13}$$

$$v(x,0) = 0, \quad 0 \leq x \leq 1, \tag{14}$$

$$\begin{cases} v_x(0,t) - av(0,t) = 0, \\ v_x(1,t) + av(1,t) = 0, \end{cases} \quad 0 \leq t \leq T. \tag{15}$$

We can obtain $v(x,t)$ numerically using the difference scheme (10). Then, for $w(x,t)$ and $\psi(x)$, we have an inverse problem in form

$$D_t^\alpha w(x,t) - w_{xx}(x,t) = \psi(x)\eta(x,t), \tag{16}$$

$$w(x,0) = 0, \quad 0 \leq x \leq 1, \tag{17}$$

$$\begin{cases} w_x(0,t) - aw(0,t) = 0, \\ w_x(1,t) + aw(1,t) = 0, \end{cases} \quad 0 \leq t \leq T, \tag{18}$$

$$w(x,T) = \phi(x), 0 \leq x \leq 1, \tag{19}$$

where $\phi(x) = \varphi(x) - v$.

We discretize problem (16)–(19) as described above.

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Let us introduce the operator $\mathcal{A} : \mathbb{R}^{(M+1)} \to \mathbb{R}^{(M+1)}$ that represent obtaining the solution $W_{i,N} = w(x_i, T),\ i = 0, ..., M$ of the problem (16)–(18) with assumed values ψ_i.

Then, the inverse problem of finding ψ can be represented in the form of an operator equation.

$$\mathcal{A}\psi = \phi, \tag{20}$$

We implement the Tiknonov regularization method by changing the problem (20) by the regularized one [13]

$$\left(\mathcal{A}^*\mathcal{A} + \beta I\right)\psi = \mathcal{A}^*\varphi, \tag{21}$$

where β is the regularization parameter.

Let us denote $\mathcal{B} = \mathcal{A}^*\mathcal{A} + \beta I$ and $b = \mathcal{A}^*\varphi$. Then, equation (21) takes form

$$\mathcal{B}\psi = b, \tag{22}$$

To choose the appropriate value of the regularization parameter β, we apply Morozov's discrepancy principle [11]. Parameter β is chosen from the relation

$$\beta_{\mathrm{D}} = \beta(\delta) : \quad \left\|\mathcal{A}\psi_{\beta_{\mathrm{D}}} - \varphi_\delta\right\|^2 = \delta^2. \tag{23}$$

To solve equations (20) and (22), we apply the biconjugate gradient stabilized (BiCGSTAB) iterative method [34].

Below is the listing of the iterative BiCGSTAB process for equation (22).

1. (a) Set $s = 0$ as iterative step.
    (b) Compute $r_0 = b - \mathcal{B}\psi_0$, where ψ_0 is some initial approximation.
    (c) Set some arbitrary r^*. We used $r_0^* = r_0$.
    (d) Set $p_0 = r_0$
2. Until convergence ($\|r_s\| < \mu$, $0 < \mu < 1$) do:

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(a) Compute coefficient $\chi_s = (r_s, r_0^*)/(\mathcal{B}p_s, r_0^*)$.
(b) Compute $d_s = r_s - \chi\mathcal{B}p_s$.
(c) Compute coefficient $\xi_s = (\mathcal{B}d_s, d_s)/(\mathcal{B}d_s, \mathcal{B}d_s)$.
(d) Compute $\psi_{s+1} = \psi_s + \chi_s p_s + \xi_s d_s$.
(e) Compute $r_{s+1} = d_s - \xi_s\mathcal{B}d_s$.
(f) Compute coefficient $\zeta_s = \frac{(r_{s+1}, r_0^*)}{(r_s, r_0^*)} \times \frac{\chi_s}{\xi_s}$.
(g) Compute $p_{s+1} = r_{s+1} + \zeta_s(p_s - \xi_s\mathcal{B}p_s)$.
(h) Increment the step counter $s = s + 1$.
(i) Go to step 2(a).

At steps 1(b), 2(a), and 2(e), we need to compute the value of operator $\mathcal{B}$ using difference scheme (10).

5 Numerical Experiments

The test problem is as follows:

$$D_t^\alpha u(x,t) - u_{xx}(x,t) = \psi(x)\eta(x,t) + f(x,t),$$

$$u(x,0) = 0, \quad 0 \le x \le 1,$$

$$\begin{cases} u_x(0,t) - au(0,t) = 0, \\ u_x(1,t) + au(1,t) = 0, \end{cases} \quad 0 \le t \le 1.$$

Obtaining the analytical solution to the forward problem is a difficult task. We use the difference scheme (10) to numerically solve the initial boundary value problem with known functions $\psi(x), \eta(x,t), f(x,t)$. Thus, we obtain the final data $\varphi(x)$. To test the stability, we add the random perturbations to this data:

$$\varphi_\delta = \varphi(1 + \text{rand}(-\epsilon, \epsilon)),$$

where ϵ is the noise level, and $\delta = \|\varphi - \varphi_\delta\|$.

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To assess the accuracy of the approximate solution $\widetilde{\psi}$, we compute the relative error

$$\gamma = \frac{\left\| \psi - \widetilde{\psi} \right\|}{\|\psi\|}.$$

5.1 Example 1

In this example, we used the following functions:

$$f\left(x,t\right) = \frac{\Gamma\left(m+1\right)}{\Gamma\left(m+1-\alpha\right)} t^{m-\alpha} \left(x^2\left(1-x\right)^2 + ax\left(1-x\right)+1\right),$$

$$\eta(x,t) = 2t^m.$$

$$\psi(x) = \sin(4\pi x)\cos(1-\pi x).$$

Values of parameters were $\alpha = 0.5$, $m = 2$, $a = 1$. Computations were performed with grid sizes $M = 2048$, $N = 128$. The number of iterations of the BiCGSTAB method was 20.

Table 1: Results of Numerical Experiments for Example 1

Noise Level ϵ	**Relative Data Error** $\frac{\delta}{\|\varphi\|}$	**Regularization Parameter** β	**Relative Solution Error** γ
0	0	0	0.005
0.001	5.7×10^{-4}	5×10^{-7}	0.035
0.01	5.7×10^{-3}	5×10^{-6}	0.14
0.1	5.7×10^{-2}	5×10^{-5}	0.35

Table 1 shows the values of the regularization parameter β and the relative error γ of the obtained solution with respect to noise

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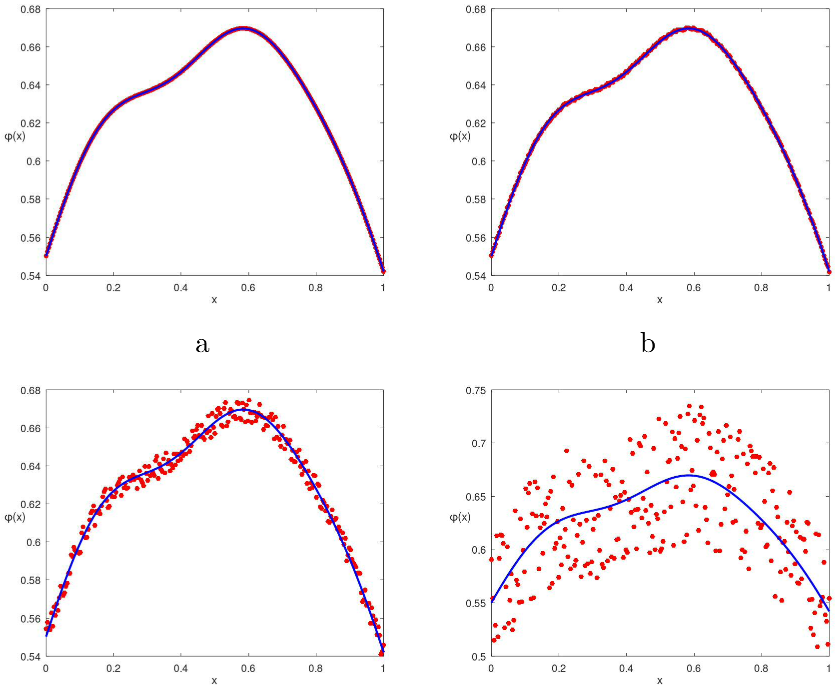
level ϵ and relative data error $\frac{\delta}{\|\varphi\|}$. Values of regularization parameter β were chosen using the discrepancy principle (23) as a maximum value so that inequality $\|A\psi_\beta - \varphi_\delta\| \leq \delta$ holds.

Figures 1(a)–(d) show the graphs for input data $\varphi(x)$ and $\varphi_\delta(x)$ for various noise levels ϵ.



a

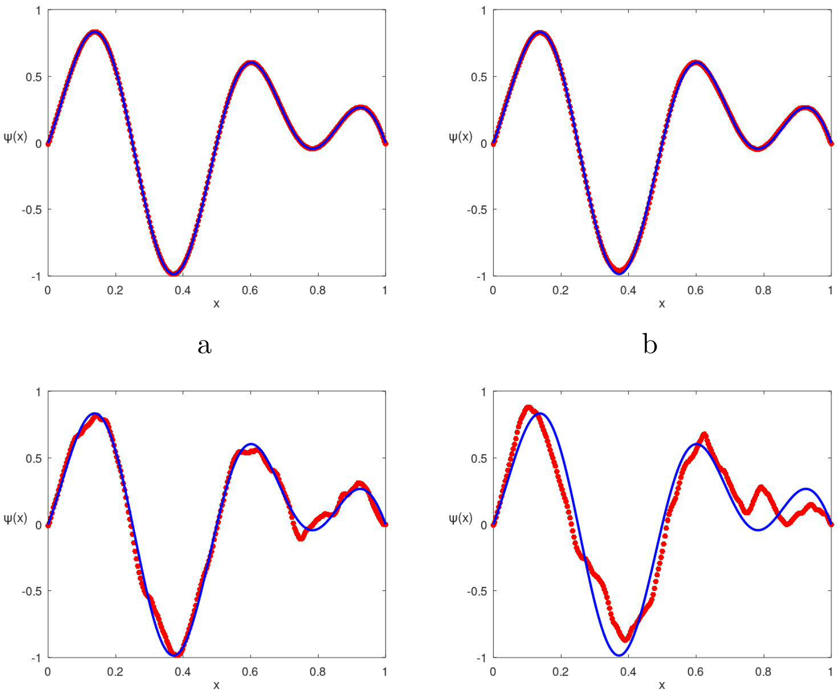
b

c

d

Figure 1: Graphs of input data $\varphi(x)$ (blue lines) and noised data $\varphi_\delta(x)$ (red dots) for Example 1 for various noise levels ϵ. (**a**) $\epsilon = 0$ (no perturbation); (**b**) $\epsilon = 0.001$; (**c**) $\epsilon = 0.01$; (**d**) $\epsilon = 0.1$.

Figures 2(a)–(d) show the graphs of exact solution $\psi(x)$ and approximate solution $\widetilde{\psi}(x)$ for various noise levels ϵ.



a

b

c

d

Figure 2: Graphs of exact solution $\psi(x)$ (blue lines) and approximate solution $\widetilde{\psi}(x)$ (red dots) for Example 1 obtained from the input data $\varphi_\delta(x)$ with various noise levels ϵ. (**a**) $\epsilon = 0$ (no perturbation); (**b**) $\epsilon = 0.001$; (**c**) $\epsilon = 0.01$; (**d**) $\epsilon = 0.1$.

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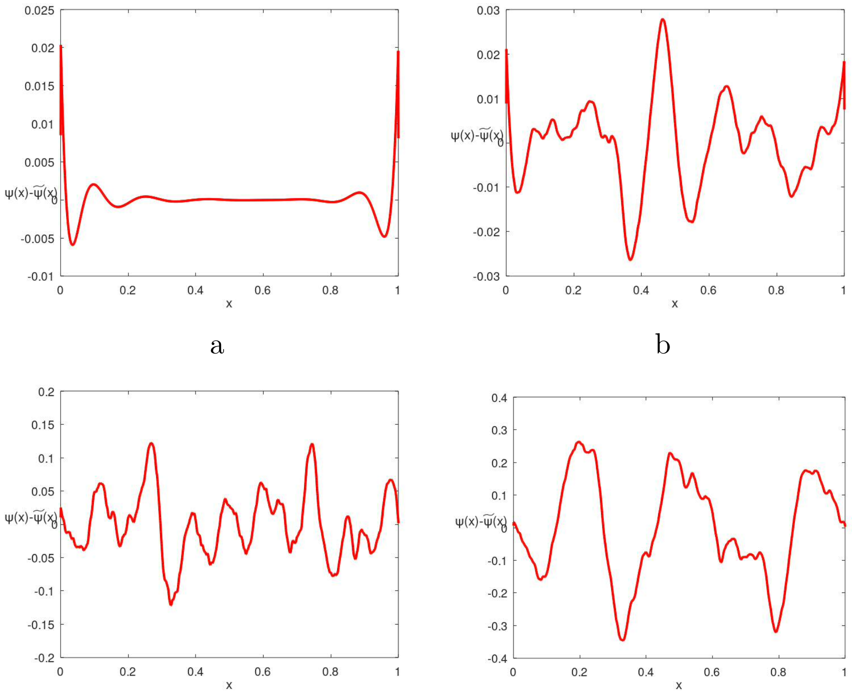
Figures 3(a)–(d) show the graphs of the solution error $\psi(x) - \widetilde{\psi}(x)$ for various noise levels ϵ.



a

b

c

d

Figure 3: Graphs of solution error $\psi(x) - \widetilde{\psi}(x)$ for Example 1 obtained from the input data $\varphi_\delta(x)$ with various noise levels ϵ. (**a**) $\epsilon = 0$ (no perturbation); (**b**) $\epsilon = 0.001$; (**c**) $\epsilon = 0.01$; (**d**) $\epsilon = 0.1$.

5.2 Example 2

In this example, we used the following functions:

$$f(x,t) = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha} \left(x^2(1-x)^2 + ax(1-x) + 1\right),$$

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$$\eta(x,t) = 2t^m.$$

$$\psi(x) = \begin{cases} x, \ 0 \le x \le 0.5, \\ 1 - x, \ 0.5 < x \le 1. \end{cases}$$

Values of parameters were $\alpha = 0.5$, $m = 2$, $a = 1$. Computations were performed with grid sizes $M = 2048$, $N = 128$. The number of iterations of the BiCGSTAB method was 20.

Table 2 shows the values of the regularization parameter β and the relative solution error γ with respect to noise level ϵ and relative data error.

Table 2: Results of Numerical Experiments for Example 1

Noise Level ϵ	**Relative Data Error** $\frac{\delta}{\lVert\varphi\rVert}$	**Regularization Parameter** β	**Relative Solution Error** γ
0	0	0	0.00015
0.001	5.9×10^{-4}	1×10^{-5}	0.025
0.01	5.8×10^{-3}	1×10^{-4}	0.07
0.1	5.8×10^{-2}	1×10^{-3}	0.12

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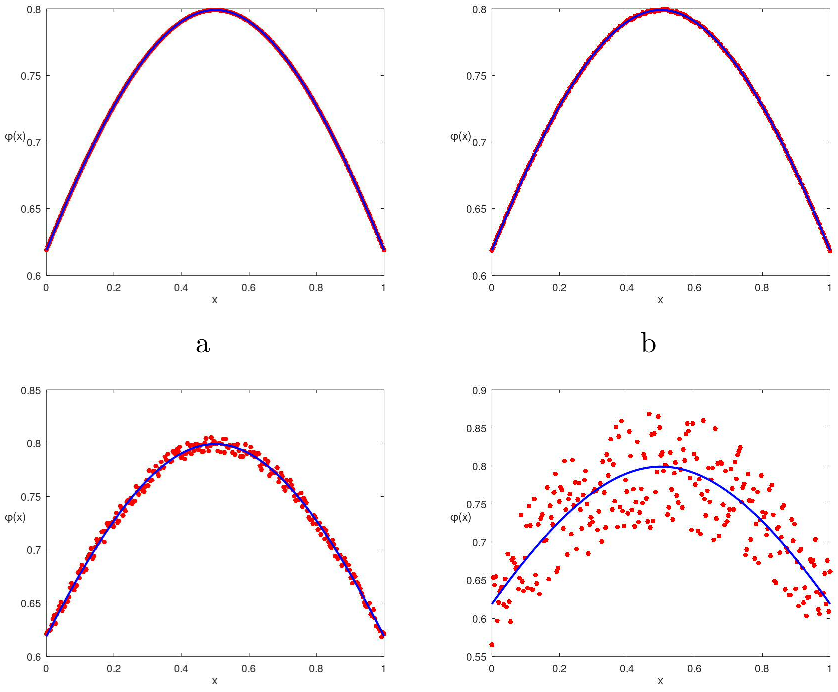
Figures 4(a)–(d) show the graphs for input data $\varphi(x)$ and $\varphi_\delta(x)$ for various noise levels ϵ.

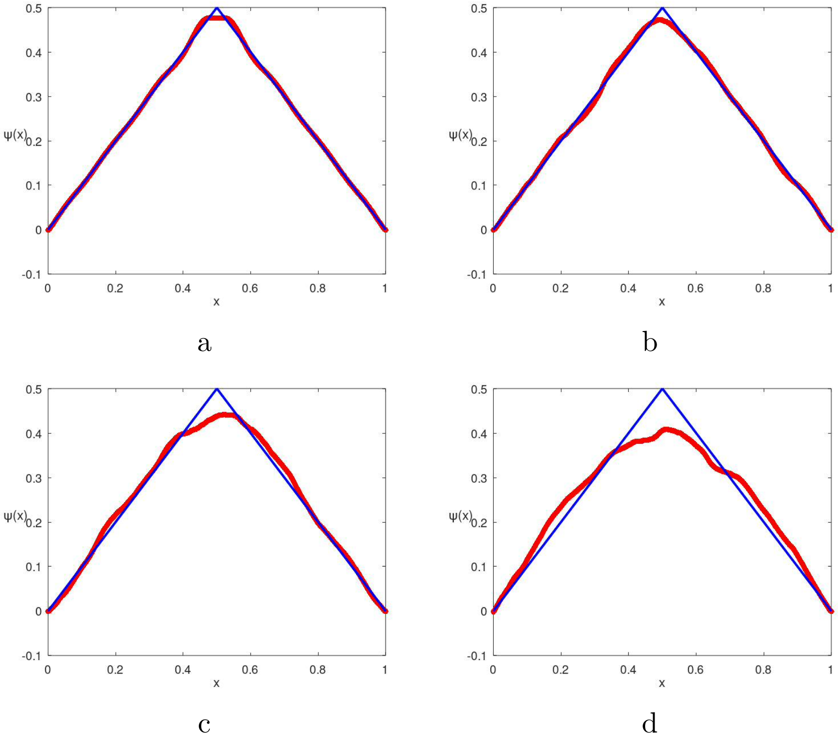


a b

c d

Figure 4: Graphs of input data $\varphi(x)$ (blue lines) and noised data $\varphi_\delta(x)$ (red dots) for Example 2 for various noise levels ϵ. (**a**) $\epsilon = 0$ (no perturbation); (**b**) $\epsilon = 0.001$; (**c**) $\epsilon = 0.01$; (**d**) $\epsilon = 0.1$.

Figures 5(a)–(d) show the graphs of exact solution $\psi(x)$ and approximate solution $\widetilde{\psi}(x)$ for various noise levels ϵ.



a b

c d

Figure 5: Graphs of exact solution $\psi(x)$ (blue lines) and approximate solution $\widetilde{\psi}(x)$ (red dots) for Example 2 obtained from the input data $\varphi_\delta(x)$ with various noise levels ϵ. (**a**) $\epsilon = 0$ (no perturbation); (**b**) $\epsilon = 0.001$; (**c**) $\epsilon = 0.01$; (**d**) $\epsilon = 0.1$.

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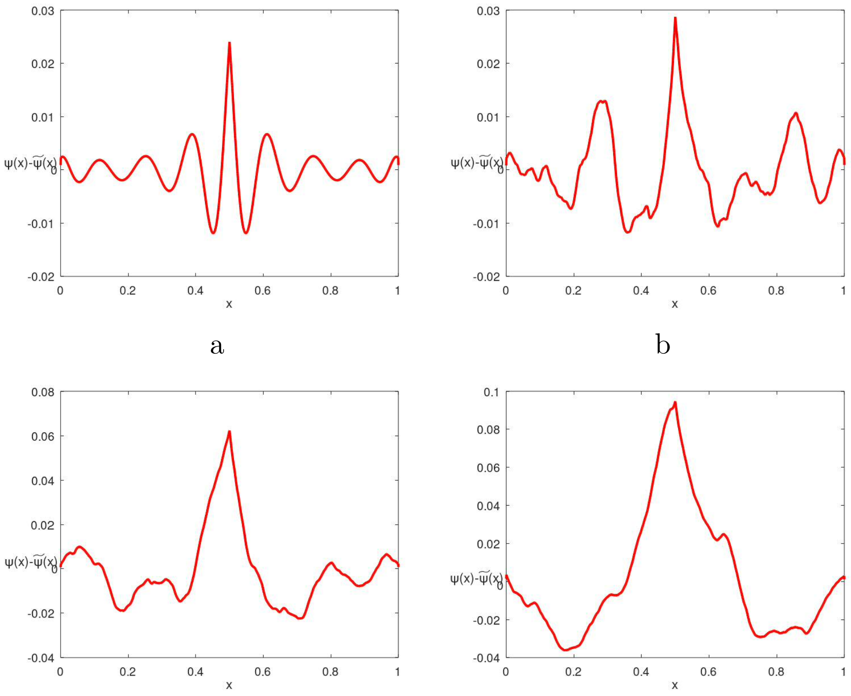
Figures 6(a)–(d) show the graphs of the solution error $\psi(x) - \widetilde{\psi}(x)$ for various noise levels ϵ.



a

b

c

d

Figure 6: Graphs of solution error $\psi(x) - \widetilde{\psi}(x)$ for Example 2 obtained from the input data $\varphi_\delta(x)$ with various noise levels ϵ. (**a**) $\epsilon = 0$ (no perturbation); (**b**) $\epsilon = 0.001$; (**c**) $\epsilon = 0.01$; (**d**) $\epsilon = 0.1$.

5.3 Discussion

From these examples, we can see that the developed numerical algorithm performs satisfactory. The lower the noise level ϵ, the lower the error of the approximate solution. Thus, the numerical experiments confirm the validity of the regularized algorithm. The

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numerical algorithm allows one to obtain the stable solution even in the case of significant perturbations in the input data.

6 Conclusions

In this work, we have constructed a numerical algorithm for solving the inverse problem of source term identification for the subdiffusion equation with Sturm-type boundary conditions. The algorithm is based on the biconjugate gradient stabilized iterative method and Tikhonov regularization. At each iteration of the method, we need to solve the initial boundary value subproblems. To solve these subproblems, we apply the difference scheme. We have performed a series of numerical experiments to study the performance of the developed algorithm. The results of the numerical experiments show that our algorithm allows us to obtain a stable solution in the case of perturbed data.

In the future, we plan to adapt our numerical algorithm to solve the inverse problem of source term identification for the subdiffusion equation with nonlocal not strongly regular boundary conditions. We plan to conduct theoretical research on the stability and convergence of our algorithm for the nonlocal problem.

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References

[1] H. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, A new collection of real world applications of fractional calculus in

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science and engineering, *Commun. Nonlinear Sci.*, **64** (2018), 213–231. `https://doi.org/10.1016/j.cnsns.2018.04.019`

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier (2006).

[3] J. T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, *Commun. Nonlinear Sci.*, **16** (2011), 1140–1153. `https://doi.org/10.1016/j.cnsns.2010.05.027`

[4] I. Podlubny, *Fractional differential equations*, London: Academic Press (1999).

[5] A. A. Alikhanov, Stability and convergence of difference schemes approximating a two-parameter nonlocal boundary value problem for time-fractional diffusion equation, *Comput. Math. Model.*, **26** (2015), 252–272. `https://doi.org/10.1007/s10598-015-9271-4`

[6] Z. Sun, G. Gao, *Fractional differential equations: finite difference methods*, Berlin: De Gruyter (2020). `https://doi.org/10.1515/9783110616064`

[7] N. H. Tuan, N. A. Triet, N. H. Luc, N. D. Phuong, On a time fractional diffusion with nonlocal in time conditions, *Adv. Differ. Equ.*, **2021** (2021), 204. `https://doi.org/10.1186/s13662-021-03365-1`

[8] E. Ozbilge, F. Kanca, E. Özbilge, Inverse problem for a time fractional parabolic equation with nonlocal boundary conditions, *Mathematics*, **10** (2022), 1479. `https://doi.org/10.3390/math10091479`

[9] M. A. Sultanov, Stability of three-layer difference scheme [In Russian], *Sib. Elektron. Mat. Izv.*, **12** (2015), 28–44. `https://doi.org/10.17377/semi.2015.12.004`

Received: July 18, 2025

[10] M. A. Sultanov, E. N. Akimova, V. E. Misilov, Y. Nurlanuly, Parallel direct and iterative methods for solving the time-fractional diffusion equation on multicore processors, *Mathematics*, **10** (2022), 323. `https://doi.org/10.3390/math10030323`

[11] V. A. Morozov, *Methods for solving incorrectly posed problems*, Berlin: Springer (1984).

[12] A. A. Samarskii, P. N. Vabishchevich, *Numerical methods for solving inverse problems of mathematical physics*, Berlin: De Gruyter (2007).

[13] V.V. Vasin, I.I. Eremin, *Operators and Iterative Processes of Fejér Type: Theory and Applications*, Berlin: De Gruyter (2009).

[14] S. I. Kabanikhin, *Inverse and ill-posed problems*, Berlin: De Gruyter (2011).

[15] D. K. Durdiev, *Inverse Problems for Fractional Diffusion Equations*, Singapore: Springer Nature (2025). `https://doi.org/10.1007/978-981-96-5338-6`

[16] D. K. Durdiev, M. A. Sultanov, A. A. Rahmonov, Y. Nurlanuly, Inverse Problems for a Time-Fractional Diffusion Equation with Unknown Right-Hand Side, *Prog. Fract. Differ. Appl.*, **9(4)** (2023), 639–653. `https://doi.org/10.18576/pfda/090408`

[17] E. N. Akimova, V. E. Misilov, M. A. Sultanov, Regularized gradient algorithms for solving the nonlinear gravimetry problem for the Multilayered Medium, *Math. Meth. Appl. Sci.*, **45(15)** (2022), 8760–8768. `https://doi.org/10.1002/mma.7012`

[18] M.A. Sultanov, M.I. Akylbaev, R. Ibragimov, Conditional stability of a solution of a difference scheme for an ill-posed

International Journal of Applied Mathematics
Volume 38 No. 1s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Cauchy problem, *Electron. J. Differ. Equ.,* **2018** (33) (2018), 1–17.

[19] E. N. Akimova, V. E. Misilov, M. A. Sultanov, Solving the Structural Inverse Gravimetry Problem in the Case of Multilayered Medium Using GPU, In: *Proceedings of the 4th Ural Workshop on Parallel, Distributed, and Cloud Computing for Young Scientists*, CEUR-WS (2018) 33–39. `https://ceur-ws.org/Vol-2281/paper-04.pdf`

[20] E. N. Akimova, V. E. Misilov, M. A. Sultanov, Parallel Implementation of the Conjugate Gradient Method for Solving the Inverse Gravimetry Problem on GPU, In: *18th International Conference on Geoinformatics - Theoretical and Applied Aspects*, EAGE Publications, (2019) 1–5. `https://doi.org/10.3997/2214-4609.201902037`

[21] E. N. Akimova, V. E. Misilov, M. A. Sultanov, R. Zh. Turebekov, Memory efficient algorithm for solving the inverse gravimetry problem of finding several boundary surfaces in multilayered medium, In: *Proceedings of Third International Conference of Mathematical Sciences (ICMS 2019)*, AIP Conf. Proc., 2183 (2019) 070006. `https://doi.org/10.1063/1.5136168`

[22] T. Wei, J.G. Wang, A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation, *Appl. Numer. Math.*, **78** (2014), 95–111. `https://doi.org/10.1016/j.apnum.2013.12.002`

[23] F. Yang, Y.P. Ren, X.X. Li, D.G. Li, Landweber iterative method for identifying a space-dependent source for the time-fractional diffusion equation, *Bound. Value Probl.*, **2017** (2017), 163. `https://doi.org/10.1186/s13661-017-0898-2`

[24] H.T. Nguyen, D.L. Le, V.T. Nguyen, Regularized solution of an inverse source problem for a time fractional diffusion

Received: July 18, 2025

equation, *Appl. Math. Model.*, **40** (2016), 8244–8264. `https://doi.org/10.1016/j.apm.2016.04.009`

[25] L.D. Su, V.I. Vasil'ev, T.S. Jiang, G. Wang, Identification of stationary source in the anomalous diffusion equation, *Inverse Probl. Sci. Eng.* **29** (2021), 3406–3422. `https://doi.org/10.1080/17415977.2021.2000609`

[26] E. N. Akimova, M. A. Sultanov, V. E. Misilov, Y. Nurlanuly, Parallel Algorithm for Solving the Inverse Two-Dimensional Fractional Diffusion Problem of Identifying the Source Term *Fractal Fract.*, **7(11)** (2023), 801. `https://doi.org/10.3390/fractalfract7110801`

[27] J. Cui, H. Cheng, Source term identification of time-fractional diffusion equation under Robin boundary condition, *Math Meth Appl Sci.* **46(6)** (2023), 6590–6612. `https://doi.org/10.1002/mma.8927`

[28] A. V. Bitsadze, A. A. Samarskii, Some elementary generalizations of linear elliptic boundary value problems, *Dokl. Akad. Nauk SSSR*, **185** (1969), 739–740.

[29] N. I. Ionkin, E. I. Moiseev, A problem for a heat equation with two-point boundary conditions, *Differ. Uravn.*, **15** (1979), 1284–1295.

[30] N. S. Imanbaev, Distribution of eigenvalues of a third-order differential operator with strongly regular boundary conditions, In: *Proceedings of International Conference on Analysis and Applied Mathematics(ICAAM 2018)*, AIP Conf. Proc., 1997 (2018) 020027. `https://doi.org/10.1063/1.5049021`

[31] M. A. Sadybekov, Initial-boundary value problem for a heat equation with not strongly regular boundary conditions, In: *Functional analysis in interdisciplinary applications*, Cham, Springer (2017) 330–348. `https://doi.org/10.1007/978-3-319-67053-9_32`

Received: July 18, 2025

[32] M. A. Sultanov, V. E. Misilov, M. A. Sadybekov, Numerical method for solving the subdiffusion differential equation with nonlocal boundary conditions, *AIMS Math.*, **9** (2024), 36385–36404. `https://doi.org/10.3934/math.20241726`

[33] A. Samarskii, E. Nikolaev, *Numerical methods for grid equations, volume I: direct methods*, Basel: Birkhäuser (1989). `https://doi.org/10.1007/978-3-0348-9272-8`

[34] Y. Saad, *Iterative methods for sparse linear systems. 2nd ed.*, Philadelphia: SIAM (2003).