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OPTIMIZING GRAPH-BASED COMPUTATION THROUGH PERMUTATION LABELING IN STRONG FACE PLANE GRAPHS

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Abstract

A graph $G = (V, E)$ is defined as a permutation graph if there exist a bijective mapping $f : V(G) \rightarrow \{1, 2, . . . , n\}$ such that edge values acquired by several permutations of a bigger vertex value taken by a smaller vertex value at a time are distinct, then it is called a permutation graph.

The application of permutation labeling within graphs having strong face-planes seek to optimize network routing by reducing conflict and improving resource allocation. It helps in laying out VLSI circuits with better layout effi- ciency and design rule adherence. This is a useful technique in computer science that can improve parallel computing systems' data routing and scheduling algorithms. It also helps to optimize memory data structure layouts, which re- sults in quicker access times. It offers more distinct and lucid representations for graph-based algorithms, which can help with analysis and debugging. It can also enhance the quality of graph visualization with clear & distinguishable layouts for complex networks. In this research, we compare permutation labeling with strong face permutation labeling and examine permutation labeling for graphs with strong face planes using various algorithmic techniques.

Math. Subject Classification: 05C78, 05C85, 68R10. **Key Words and Phrases:** Permutation graphs, strong face planes, permutation labeling, strong face permutation labeling, graph labeling algorithms.

1 Introduction

Graph labeling is a fundamental concept in graph theory that in- volves assigning labels, typically integers, to the vertices or edges of a graph based on specific rules. Because of its uses in a variety of domains, including data structure optimization, circuit design, and communication networks, permutation labeling has attracted the most interest among the

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several kinds of graph labeling. A graph $G = (V, E)$ is a permutation graph if there exists a bijective map- ping $f : V(G)\{1, 2, \ldots, n\}$ such that the edge values derived from the vertex labels are distinct. This distinctness is provided by an induced edge function $g_f : E(G) \rightarrow N$, which ensures that the edge labels are different based on the permutations of the vertex labels. The concept of strong face permutation labeling extends the tra- ditional notion of permutation labeling by incorporating the struc- tural complexities of strong face graphs. For every face in a basic plane graph, a new vertex is added to create a strong face graph (excluding the exterior face) and connecting it to the vertices sur- rounding that face. This additional layer of vertices and edges allows for more sophisticated labeling techniques that can optimize routing paths and reduce conflicts in applications such as VLSI design and parallel computing systems.

The advantages of strong face permutation labeling over stan- dard permutation labeling are significant. While traditional per- mutation labeling is applicable to a wide range of graph types, strong face permutation labeling specifically targets complex struc- tures like fan graphs, wheel graphs, ladder graphs, and friendship graphs. This targeted approach not only enhances the distinctness of edge labels but also improves the overall efficiency of graph-based algorithms. For instance, in satellite communication net- works, strong face labeling can minimize interference by ensuring that non-adjacent nodes utilize distinct frequencies, thereby opti- mizing communication paths and enhancing network reliability.

In the context of the works cited, such as those by J. Baskar Babujee and V. Vishnupriya [3, 4], the foundational principles of permutation labeling are explored in relation to specific graph types. Gallian's survey [5] provides a comprehensive overview of graph la- beling techniques, while Hegde and Shetty [6] delve into combina- torial approaches that further enrich the understanding of labeling. Rosa's early contributions [7] laid the groundwork for these devel- opments, and Seoud and Salim [8] demonstrated the existence of permutation labeling for various graph orders.

This paper aims to investigate the extension of strong face per- mutation labeling to specific graph structures, demonstrating its superiority over traditional permutation labeling. By analyzing the unique properties of strong face graphs, we will illustrate how this advanced labeling technique can lead to more efficient and reliable solutions in both theoretical and practical applications. Through detailed examination and algorithmic approaches, we will highlight the potential of strong face permutation labeling to enhance graph labeling methodologies across various domains.

1.1 Literature Review

Table 1: The Comparative Literature Review of Face La- beling Approaches (2021–2025), highlighting the Evolu- tion Toward Strong Face Permutation Labeling for Vari- ous Graph Classes and Real-World Applications.

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Year	Title	Authors	Contribution	Relation to Strong Face Permutation Labeling
1967	On certain valuations of the vertices of graph	A. Rosa [7]	Introduced foundational concepts in graph labeling, establishing a basis for future research in graph theory.	Provides the theoretical groundwork for permutation labeling, essential for developing strong face permutation labeling.
2008	Permutation labeling for some trees	J. Baskar Babujee, V. Vishnupriya [3]	Investigates permutation labeling specifically for tree structures, demonstrating its applicability in various contexts.	Sets a precedent for applying permutation labeling techniques to more complex structures like strong face graphs.
2011	On permutation labeling	M.A. Seoud, M.A. Salin [8]	Discusses the existence of permutation labeling for specific graph orders, expand- ing the scope of labeling techniques.	Highlights the potential for extending permutation labeling concepts to strong face graphs, particularly in terms of order and structure.
2021	End-to-End Detection-Segmentation System for Face Labeling	Wen, S., Dong, M., Yang, Y., Zhou, P. et al.	AI-based computational system for labeling, focusing on facial segmentation and recognition.	Focuses on labeling but unrelated to permutation label- ing in graph the- ory.

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2022	Face Antimagic Labeling for Double Duplication of Barycentric Graphs	Vasuki B., Shobana L., Ahmed M. A.	Applied antimagic labeling to duplicated barycentric and middle graphs.	Explores face-based graph labeling con- cepts but lacks application to strong face permutation labeling.
2023	On Antimagic Labeling for Some Families of Graphs	Mohammed Ali, Noor Khalil	Expanded antimagic labeling to more graph families.	Discusses antimagic properties, but not directly related to permutation labeling.
2023	Face Antimagic Labeling of Special Graphs	R. Kuppan, L. Shobana	Focused on antimagic labeling for duplications in special graph classes.	Also focused on face-based labeling but does not cover permutation labeling techniques.
2025	Strong Face Permutation Labeling for Various Graphs	Present Analysis	Explored strong face permuta- tion labeling for planar graphs. Broader scope (fan, wheel, friendship, lad- der graphs), real-world appli- cations (satellite communication).	Expands scope and application of strong face permutation labeling to multiple graph structures.

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2 Preliminaries

2.1 Basic Graph Theory

Definition 1 (Graph). [5] An ordered pair (V, E), where V is a collection of vertices and E is a set of edges, is called a graph G joining vertices in pairs.

Definition 2 (Vertex)**.** [5] A vertex is an individual element of the graph, denoted as v; for $i = 1, 2, \ldots, n$.

Definition 3 (Edge)**.** [5] An edge is a connection between two vertices, represented as uv where $u, v \in V$.

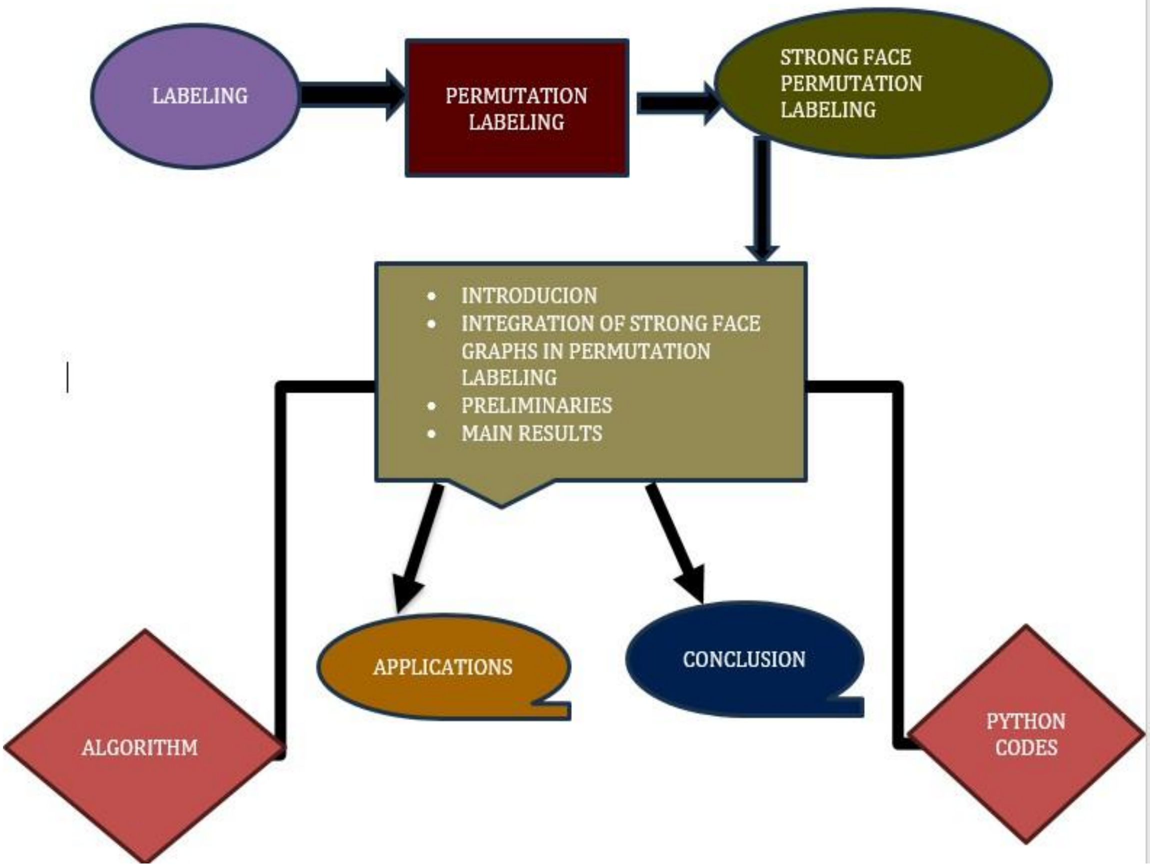

Figure 1: Conceptual Overview of the Paper's Organization − Trac- ing the Progression from General Labeling to Strong Face Permu- tation Labeling, and Concluding with Algorithms, Applications, Conclusions, and Python Implementations.

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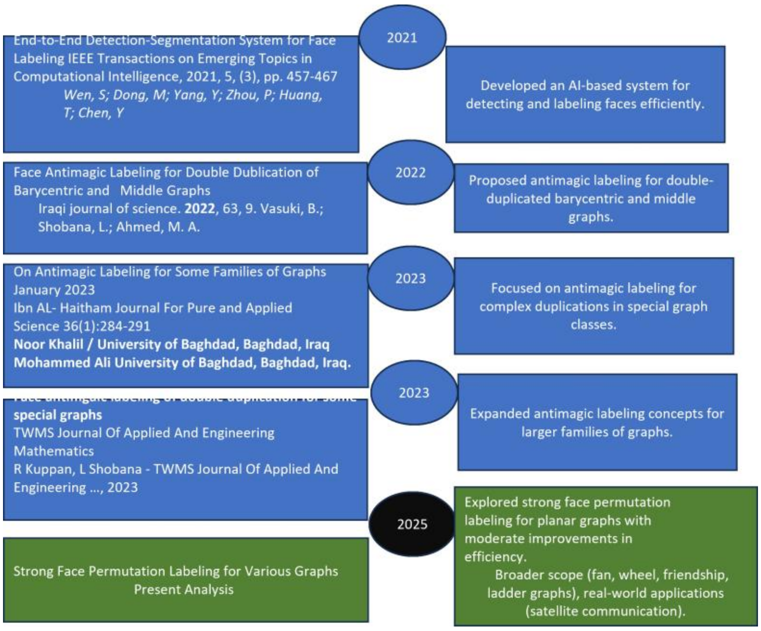
2.2 Permutation Graphs and Labeling

Definition 4 (Permutation Graph)**.** [3, 4, 8] A permutation graph is a graph that can be constructed from a permutation di- agram, where vertices represent elements of the permutation, and edges connect pairs of vertices whose corresponding elements are inverted in the permutation.

Definition 5 (Permutation Labeling)**.** [3, 4, 8] The technique of assigning integers to a graph's vertices or edges in compliance with preset rules while ensuring that the resulting edge values are distinct is known as permutation labeling.

Table 2: Year wise Comparison of Graph Labeling Techniques.



2.3 Strong Face Grahs and Labeling

Definition 6 (Strong Face Graph)**.** [1, 2] By creating a new vertex for each face of a basic linked planar graph G (apart from the exterior face) and attaching it to every vertex surrounding that face, a strong face graph G^* is created. This new graph's faces are all isomorphic to cycle C_3.

Definition 7 (Strong Face Permutation Labeling)**.** [1, 2] A specific type of permutation labeling used on strong face graphs is called "strong face permutation labeling," which entails assigning labels to vertices and edges so that the induced edge.

Definition 8 (Induced Edge Map)**.** [6] An induced map $g : E \to N$ assigns a unique positive integer to each edge based on the vertex labels, which is crucial for establishing the graph as a permutation graph.

2.4 Special Types of Graphs

Definition 9 (Fan Graph)**.** [5] A path graph P_n's vertices are connected to a single vertex, known as the center, to produce F_n.

Definition 10 (Wheel Graph)**.** [5] A wheel graph W_n is cre- ated by joining a cycle graph's isolated central vertex to every ver- tex C_n.

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Definition 11 (Ladder Graph)**.** [5] A ladder graph is a pla- nar graph that resembles a ladder, consisting of two parallel paths (rails) connected by a series of rungs.

Definition 12 (Friendship Graph)**.** [5] A friendship graph F_n, sometimes referred to as an n-fan or Dutch windmill graph, is created by combining n copies of the triangle-shaped cycle graph C_3 at a single central vertex.

2.5 Schematic Representation

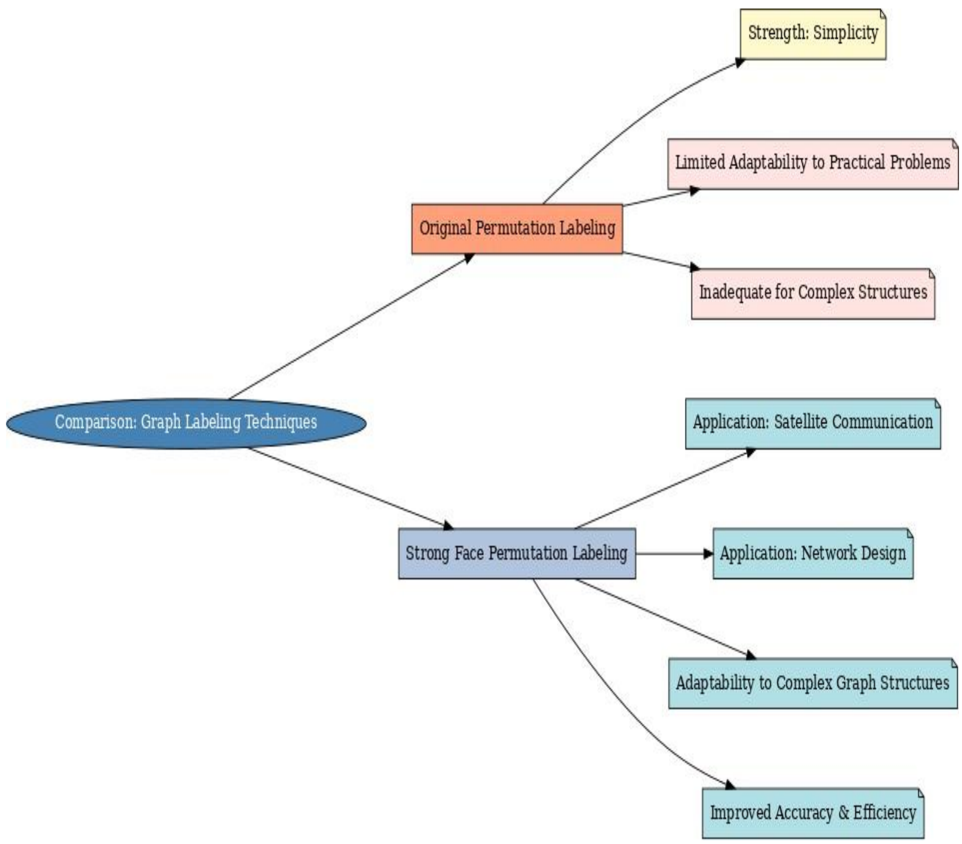

Figure 2: Schematic Representation Comparing Graph Labeling Techniques, Illustrating the Advantages of Strong Face Permuta- tion labeling Over Traditional Permutation Labeling.

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3 Essential Findings

Definition 13. [1] Assume that G is a simple plane graph with links. A sturdy face graph To make G^*, add a new vertex to each of G's faces (excluding the outer face) and bind it to all the other vertex surrounding that face. Everyone on this new network is isomorphic to cycle C_3.

Furthermore, the number of faces doubles two times if the orig- inal graph G's faces are C_3.

Theorem 14. *Fan graphs with a strong face F^*_n is a permutation graph with $n \geq 3$.*

Proof. Let the strong face fan graph's vertex and edge sets be

F^*_n be described as

if $i = 1, 2, \dots, n\} \cup \{u_i$: if $i = 1, 2, \dots, n-1\}$, $V(F^*_n) = \{u, v_i E_1 \cup E_2 \cup E_3 = E(F^*n)$. were, E_1 is equal to $\{uv_i : i = 1, 2, \dots, n\}$. In the case of $i = 1, 2, \dots, n-1$, $E_2 = \{v_iv_{i+1}$; If $i = 1, 2, 3, \dots, n-1$, then $E_3 = \{uu_i, v_iu_i, v_{i+1}u_i\}$ Make $f : V \to \{1, 2, \dots, 2n\}$ a bijective function such that $f(v_i) = 2i$ for $1 < i \leq n$ and $f(u_i) = 2i - 1$ for $1 \leq i \leq n-1$. $f(u) = 2n-1$. The following definition applies to an induced map, $g_f : E \to N$, where N is a positive integer. ${}^{(2n-1)}P_{2i} = gf(uv_i)$: $gf(v_iv_i{+}1) = {}^{2(i+1)}gf(u_iv_i) = {}^{2i}P_{2i-1}$: $1 \leq i \leq n-1$ $g_f(uu_i) = (2n-1) = P_{2i}$: $-1 < i \leq n$ $g_f(u_iv_{i+1}) = {}^{2(i+1)}P_{2i-1}$: $1 \leq i \leq n-1$ $gf(uv_n) = {}^{2n}P_{2n-1}$ is P_{2i-1}.

For $i = 1, 2, \dots, n\} \cup \{u_i$: for $i = 1, 2, \dots, n-1\}$,

$V(F^*_n) = \{u, v_i$

$E_1 \cup E_2 \cup E_3 = E(F^*_n)$.

were,

$E_1 = \{uv_i : i = 1, 2, \dots, n\}$

$E_2 = \{v_iv_{i+1}$; for $i = 1, 2, \dots, n-1\}$

$E_3 = \{uu_i, v_iu_i, v_{i+1}u_i$; for $i = 1, 2, 3, \dots, n-1\}$

Establish a bijective function $f : V \to \{1, 2, \dots, 2n\}$ such that, for $1 < i \leq n$, $f(v_i) = 2i$, and for $1 \leq i \leq n-1$, $f(u_i) = 2i - 1$.

$f(u) = 2n - 1$

An induced map, $g_f : E \to N$, in which N is a positive integer with the following definition

$gf(uv_i) = {}^{(2n-1)}P_{2i}$:

$g_f(v_iv_{i+1}) = {}^{2(i+1)}P_{2i}$: $-1 < i \leq n$ $g_f(u_iv_i) = {}^{2i}P_{2i-1}$: $1 \leq i \leq n-1$ $g_f(uu_i) = {}^{(2n-1)}P_{2i}$: $1 \leq i \leq n-1$

$gf(u_iv_{i+1}) = {}^{2(i+1)}P_{2i-1}$: $1 \leq i \leq n-1$

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$gf(uv_n) = {}^{2n}P_{2n-1}$ is P_{2i-1}.

Every induced labeling is unique, For any two edges $uv_i \in E_1$, and

$v_iv_{i+1} \in E_2$ if $g_f(vu_i) = g_f(v_{i+1}v_i)$

$$(2n-1)_{P_{2i}} = 2(i+1)_{P_{2i}}$$

which is a contradiction.

for any two edges $v_iv_{i+1} \in E_2$ and $uu_i \in E_3$ if

$$g_f(uu_i) \models g_f(v_iv_{i+1})$$

for any two edges in E_3E_1 the labels are distinct.

Consider any two edges $v_iu_i \in E_3$ and $uv_i \in E_1$, if

$$g_f(v_iu_i) \models g_f(uv_i)$$

From all the above possibilities we see that the induced labelling for edges is distinct. Hence, Strong face fan graph F^*_n is a permutation graph. □

Algorithmic Approach for Theorem 14

Begin Algorithm: Theorem 14 −Permutation Graph Proof for F_n

Input

An edge set $E(F_n)$ and the vertex set $V(F_n)$ of a strong face fan graph.

Output

A proof that F_n is a permutation graph Steps:

1. Define the Vertex and Edge Sets:

Vertex set:

The collection of vertex values is $V(F_n) = \{u, v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\}$.

Edge set:

$E(F_n) = E_1 \cup E_2 \cup E_3$ is the edge set, where $E_1 = \{uv_i : i = 1, 2, \ldots, n\}$ E_2 is identical to $\{v_iv_{i+1} : i = 1, 2, \ldots, n-1\}$.

$\{uu_i, v_iu_i, v_{i+1}u_i : i = 1, 2, 3, \ldots, n-1\}$ is equivalent to E_3.

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2. Define the function of bijection: Explain the following f :

   $V \to \{1, 2, \ldots, 2n\}$ for $1 \leq i \leq n$ $f(u_i) = 2i - 1$;

   $f(v_i) = 2i$; $f(u) = 2n - 1$ for $1 < i \leq n - 1$.

3. Explain what the induced Induced Edge Map is, $g_f : E \to N$

   can be defined as follows:

   $g_f(uv_i) = {}^{(2n-1)}P_{2i}$; for $1 \leq i \leq n-1$ $g_f(v_iv_{i+1}) = {}^{2(i+1)}P_{2i}$; for $1 \leq i \leq n-1$ $g_f(u_iv_i) = {}^{2ii}P_{2i-1}$; for $1 \leq i \leq n-1$ $g_f(uu_i) = {}^{(2n-1)}P_{2i-1}$; for $1 \leq i \leq n-1$ $g_f(u_iv_{i+1}) = {}^{2(i+1)}P_{2i-1}$; for $1 \leq i \leq n-1$ $g_f(uv_n) = {}^{2n}P_{2n-1}$

   Note: "P" denotes the specific operation defined in the con-

   text of the labeling.

4. Verify Distinctness of Edge Labels:

   Case 1: For $uv_i \in E_1$ and $v_iv_{i+1} \in E_2$

   If $g_f(uv_i) = g_f(v_{i+1}v_i)$, then ${}^{(2n-1)}P_{2i} = {}^{2(i+1)}P_{2i}$, which is a contradiction.

   Case 2: For $v_iv_{i+1} \in E_2$ and $uu_i \in E_3$

   It follows that $g_f(uu_i) \neq g_f(v_iv_{i+1})$, ensuring distinct labels.

   Case 3: For $uv_i \in E_1$ and $v_{i+1}u_i \in E_3$

   If $g_f(v_{i+1}u_i) \neq g_f(uv_i)$, then ${}^{2(i+1)}P_{2i-1} \neq {}^{(2n-1)}P_{2i}$

   Conclusion:

   Since all induced edge labels are distinct, the labeling confirms that

   F_n is a permutation graph.

End Algorithm

Python Coding and Output

3.1 Strong Face Fan Graph Application in Satel- lite Communication Networks

In a satellite network, signals from the satellite must arrive at the appropriate ground station as soon as possible and free from in- terference. Ensuring that every ground station has an individual and ideal routing path is possible with strong face labeling. Then by ensuring that the signal travels the shortest path and lower- ing the likelihood of signal overlap and interference, this increases communication speed and dependability.

3.2 Reducing Interference

If the Interference could arise nearby ground stations in the series interact on frequencies that overlap. Then by using strong face

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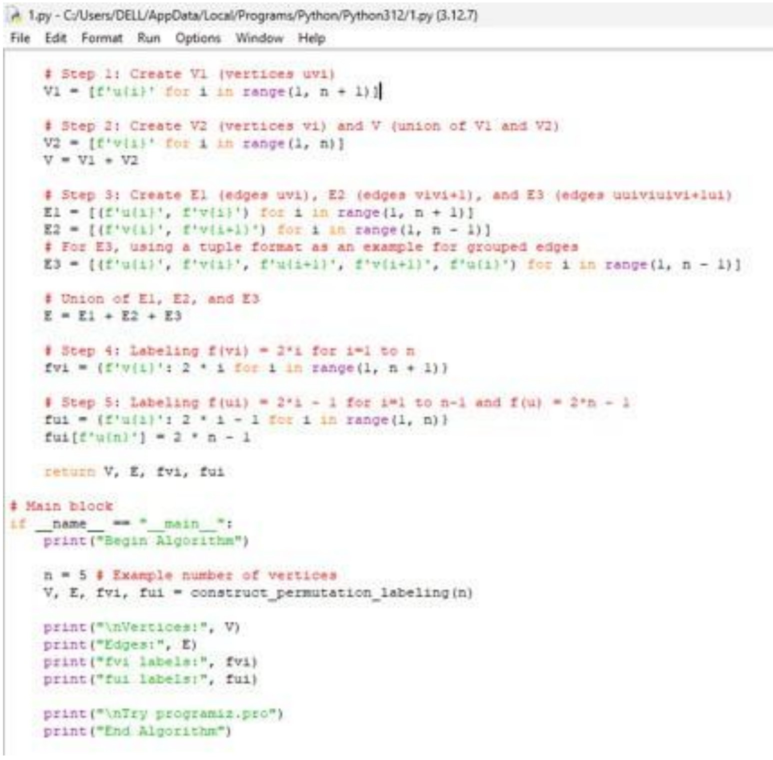

Figure 3: Input Configuration for the Algorithm −Building the Strong Face Fan Graph's Vertex and Edge Sets.

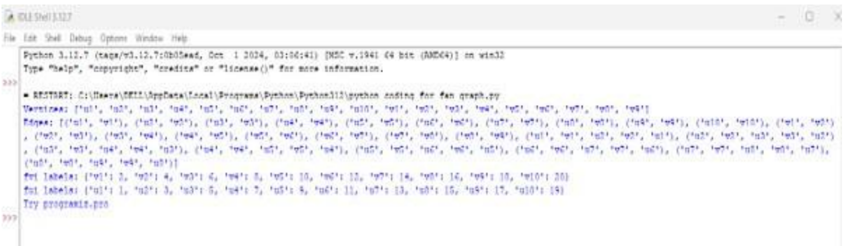

Figure 4: Output of the Algorithm −Permutation Labeling Results with Assigned Vertex Labels and Computed Edge Labels.

labeling, the graph can be labeled to minimize the danger of inter-ference by ensuring that non-adjacent ground stations, which are more likely to interfere, use distinct frequencies or channels.

3.3 Robust Connectivity

The network should be able to communicate effectively even in the event of a failure at one ground station, without having to reroute through it. Then the event of a node failure, robust face labeling in the closed fan graph can guarantee that alternative paths are optimized, offering redundancy and preserving connectivity.

3.4 Strong Face Permutation's Advantages

Labeling Optimal Routing Paths: This reduces delay by en- suring that data packets take the most effective routes.

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Minimized Interference: Minimizes interference from signals by allocating communication frequencies as efficiently as pos- sible.

Strong connectivity: improves the network's ability to with- stand faults by offering backup paths in the event that a node or link fails.

3.5 Practical Example

Consider a satellite network with nine strategically placed ground stations arranged in a consecutive pattern to cover a remote area (such as a coastline or mountain range).

Optimization: Strong face labeling makes sure that every ground station and the satellite have opti- mal communication, and closed loop makes sure that the network can continue effectively route sig- nals through nearby stations even in the event that one station is momentarily unavailable.

Let a satellite network in which nine ground stations (peripheral nodes) are in communication with the satellite (hub).

3.6 Graph Representation

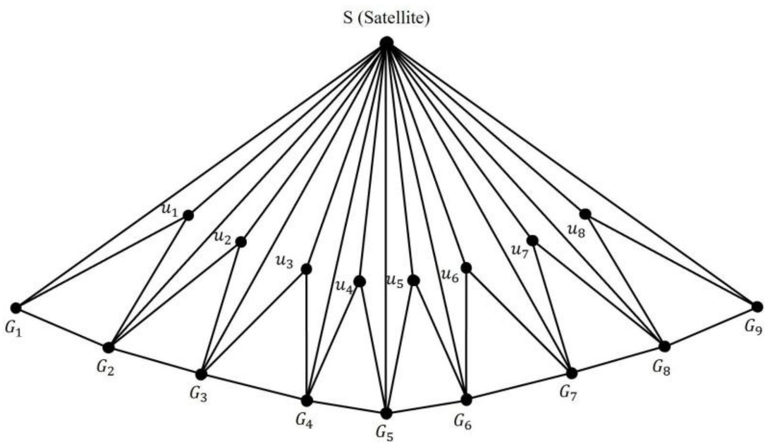

Figure 5: A fan graph representation of the satellite communication network. The central satellite node SSS is connected to ground sta- tions $G_1, G_2, \ldots, G_9 G_1, G_2, \ldots, G_9 G_1, G_2, \ldots, G_9$, illustrating how robust connectivity, interference reduction, and routing optimiza- tion can be facilitated through appropriate face labeling.

1. Nodes:

   S satellite

   $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9$

2. Edges:

   S connected to $G_1 G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9$

   G_1 Connected to G_2

   G_2 Connected to G_3

   G_3 Connected to G_4

   G_4 Connected to G_5

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G_5 Connected to G_2

G_6 Connected to G_2

G_7 Connected to G_2

G_8 Connected to G_9

In the enclosed fan graph model of a satellite communication net- work, robust connectivity, interference reduction, and routing op- timization are all greatly aided by good face labeling. Commu- nication can be made more effective and dependable by carefully labeling the graph. This is crucial in satellite networks, where rout- ing efficiency and signal integrity are critical.

Theorem 15. *Graphs with Strong Face Planes wheel graph W^* is a permutation graph.*

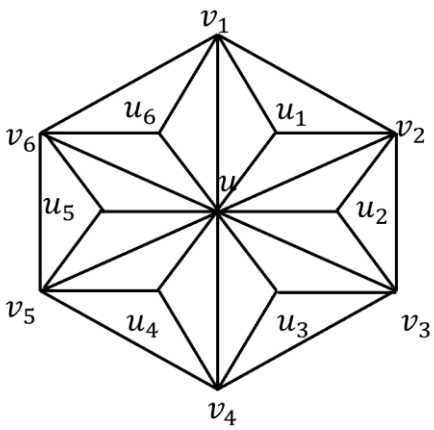

Figure 6: The strong face plane representation of the wheel graph $W^*W^*W^*$. As demonstrated in Theorem 14, this structure ad- mits a permutation labeling, confirming $W^*W^*W^*$ as a permuta- tion graph.

Proof. Let W^*'s vertex set be defined as The formula

$V(W^*) = \{u, v_i; i = 1, 2, \ldots, n\} \cup \{u_i; i = 1, 2, \ldots, n\}$

n

The edge set be

$E = E_1 \cup E_2 \cup E_3 \cup E_4$

$E_1 = \{uv_i : 1 \leq i \leq n\}$

$E_2 = \{v_1 v_n, v_i v_{i+1}; 1 \leq i \leq n-1\}$

$E_3 = \{uu_i, u_i v_i; 1 \leq i \leq n\}$

$E_4 = \{u_n v_1, u_i v_{i+1}; 1 \leq i \leq n-1\}$

Define a bijective function $f : V \to \{1, 2, \ldots, 2n + 1\}$ such that

$f(v_i) = 2i$; for $1 \leq i \leq n$ $f(u) = 2n + 1$

$f(u_i) = 2i - 1$; for $1 \leq i \leq n$

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The induced map $g_f : E \to N$ where N is the set of positive integers is defined as follows:

$$g_f(uv) = \begin{cases} {}^{f(u)}P_{f(v)} & \text{if } f(u) > f(v) \\ {}^{f(v)}P_{f(u)} & \text{if } f(v) > f(u) \end{cases}$$

In E_1,

$g_r(uv_i) = {}^{(2n+1)}P_{2i};\ 1 \le i \le n$

For edges in E_2,

$g_f(v_1v_n) = {}^{(2n)}P_2 = 4n^2 - 2n$ and

$g_f(v_iv_{i+1}) = {}^{2(i+1)}P_{2i};\ 1 \le i \le n$

All induced labeling for edges E_1 and E_2 are distinct. For edges in

E_3,

For edges in E_3,

$g_f(uu_i) = {}^{(2n+1)}P_{2i-1};\ 1 \le i \le n$ $g_f(u_iv_i) = {}^{(2i)}P_{2i-1} = (2i)!;\ 1 \le i \le n$

For edges in E_4 for $g_f(u_nv_1) = {}^{(2n+1)}P_2 = 4n^2 + 2n$

$g_f(u_iv_{i+1}) = {}^{2(i+1)}P_{2i-1};\ 1 \le i \le n-1$ Let, consider any two edges $uv_i \in E_1$ and $v_iv_{i+1} \in E_2$

$g_f(uv_i) = g_f(v_iv_{i+1});\ 1 \le i \le n$

which is a contradiction.

Consider any two edges in $u_iv_i \in E_3$ and $u_iv_{i+1} \in E_4$

$g_f(u_iv_i) = g_f(v_iv_{i+1})$, which is a contradiction.

For any two edges $v_1v_n \in E_2$ and $u_nv_1 \in E_4$

$g_f(v_1v_n) = g_f(u_nv_1)$ $4n^2 - 2n = 4n^2 + 2n$

Which is a contradiction. Hence induced labeling for strong face wheel graph W_n^* is a permutation graph. □

Theorem 16. *The strong face ladder graph L_n^*, for $n \ge 3$ is a permutation graph.*

Proof. Let the vertex set be $V(L_n^*) = V_1 \cup V_2$

where $V_1 = \{v_i, i = 1, 2, \ldots 2n\}$

& $V_2 = \{u_i, i = 1, 2, \ldots, n-1\}$

and the edge set be $E(L_n^*) = E_1 \cup E_2 \cup E_3$ where $E_1 = \{v_iv_{i+1}, v_{n+i}v_{n+i+1};$ for $i = 1, 2, \ldots, n-1\}$ $E_2 = \{v_iv_{n+i};$ for $i = 1, 2, \ldots, n\}$

$E_3 = \{u_iv_i, u_iv_{i+1}, u_iv_{n+i}, u_iv_{n+i+1};$ for $i = 1, 2, \ldots, n-1\}$

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Define a bijective function

$f : V \to \{1, 2, \ldots, 3n-1\}$ such that

$f(v_i) = 2i-1$; for $1 \le i \le n$ $f(v_{n+i}) = 2i$; for $1 \le i \le n$ and $f(u_i) = 2n+i$; for $1 \le i \le n-1$ Define an induced map $g_f : E \to N$ In E_1, for any edges

$$g_f(v_i v_{i+1}) = {}^{(2i+1)}P_{2i-1}$$
$$g_f(v_{n+i} v_{n+i+1}) = {}^{2(i+1)}P_{2i}$$

For edges in E_2,

$g_f(v_i v_{n+1}) = (2i)!$ $1 \le i \le n$ $g_f(u_i v_{n+i}) = {}^{(2n+i)}P_{2i}$

$g_f(u_i v_{n+i+1}) = {}^{(2n+i)}P_{2(i+1)}$ $1 \le i \le n$

For edges in E_3,

$g_f(u_i v_i) = {}^{(2n+i)}P_{2i}$

$g_f(u_i v_{i+1}) = {}^{(2n+i)}P_{2(i+1)-1}$

For any two edges in E_1 and E_2 (or) E_1 and E_3 the labels are distinct.

Consider any two edges $v_i v_{n+i} \in E_2$ and $u_i v_{n+i} \in E_3$

if $g_f(v_i v_{n+i}) = g_f(u_i v_{n+i})$, $(2i)! = {}^{(2n+i)}P_{2i}$

which is a contradiction. Also

$g_f(v_i v_{n+i}) \neq g_f(u_i v_i)$ $g_f(v_i v_{n+i}) \neq g_f(u_i v_{i+1})$ $g_f(v_i v_{n+i}) \neq g_f(u_i v_{n+i+1})$

Hence all edge labels are distinct.

Therefore, L^*_n is a permutation graph. □

Theorem 17. *Graphs with Strong Face Planes friendship graph $n \ge 3$ is a permutation graph.*

The strong face friendship graph $(C^n_3)^*$, can be created by linking n copies of the cycle graph C^n_3 together at a common vertex, which serves as the graph's universal vertex.

Let the vertex set and edge set of strong face friendship graph $(C^n_3)^*$, is defined as

$V((C^n_3)^*) = \{u, u_i, v_i, w_i\}$

$E((C^n_3)^*) = E_1 \cup E_2$ where

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$E_1 = \{uu_i, uv_i, uw_i; 1 \leq i \leq n\}$ $E_2 = \{u_iv_i, u_iw_i, v_iw_i; 1 \leq i \leq n\}$

Define a bijective function $f : V \to \{1, 2, \dots, 3n + 1\}$ and their induced mapping $g_f : E \to N$,

The edges defined for the following two cases.

Case (I) Let n be odd

$f(u) = 1, f(u_i) = i + 1; 1 \leq i \leq n$ $f(v_i) = 2i + n; 1 \leq i \leq n$

$f(w_i) = 2i + n + 1; 1 \leq i \leq n$

For any edges in E_1

$g_f(uu_i) = i + 1; 1 \leq i \leq n$ $g_f(uv_i) = 2i + n; 1 \leq i \leq n$ $g_f(uw_i) = 2i + n + 1; 1 \leq i \leq n$ $g_f(u_iv_i) = {}^{(2i+n)}P_{i+1}; 1 \leq i \leq n$ $gf(u_iw_i) = {}^{(2i+n+1)}P_{i+1}; 1 \leq i \leq n$

$gf(v_iw_i) = {}^{(2i+n+1)}P_{2i+n}; 1 \leq i \leq n$

All induced labeling for edges in E_1 is distinct. For any edge in E_2,

$g_f(u_iv_i) = {}^{(2i+n)}P_{i+1}; 1 \leq i \leq n$ $gf(u_iw_i) = {}^{(2i+n+1)}P_{i+1}; 1 \leq i \leq n$ $gf(v_iw_i) = {}^{(2i+n+1)}P_{2i+n}; 1 \leq i \leq n$

All induced labeling for edges in E_2 is distinct.

Now consider any two edges $uw_i \in E_1$ and $u_iw_i \in E_2$,

$g_f(uw_i) = g_f(u_iw_i)$

$2i + n + 1 = {}^{(2i+n+1)}P_{i+1},$

Which is a contradiction. Hence all induced labeling for edges is distinct.

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Case (II) Let n be even

$f(u) = 3n + 1, f(u_i) = 3i - 1; 1 \leq i \leq n\ f(v_i) = 3i - 2; 1 \leq i \leq n$

$f(w_i) = 3i; 1 \leq i \leq n$

For any edges in E_1

$g_f(uu_i) = {}^{(3n+1)}P_{3i-1}; 1 \leq i \leq n\ g_f(uv_i) = {}^{(3n+1)}P_{3i-2}; 1 \leq i \leq n\ g_f(uw_i) = {}^{(3n+1)}P_{3i}; 1 \leq i \leq n$

All induced labeling for edges in E_1 is distinct. For any edge E_2,

$g_f(u_iv_i) = {}^{(3i-1)}P_{3i-2}; 1 \leq i \leq n\ g_f(u_iw_i) = {}^{(3i)}P_{3i-1}; 1 \leq i \leq n\ g_f(v_iw_i) = {}^{(3i)}P_{3i-2}; 1 \leq i \leq n$

All induced labeling for edges in E_2 is distinct.

Now consider any two edges $uu_i \in E_1$ and $u_iv_i \in E_2$,

$$g_f(uu_i) = g_f(u_iv_i)$$

$$^{(3n+1)}P_{3i-1} = {}^{(3i-1)}P_{3i-2},$$

which is a contradiction. Hence all induced labeling for edges is distinct.

Hence from case I and case II, $(C^n)^*$ is a permutation graph.

Table 3: Comparison of Strong Face Permutation Labeling versus Permutation Labeling

Aspect	**Older Papers on Permutation Labeling**	**This Paper's Contribution**
Graph Types Addressed	Mostly focused on simple graphs (e.g., paths, cycles)	Explores Fan, Wheel, Friendship, and Ladder graphs
Labeling Technique	Original permutation labeling methods	Strong face permutation labeling for enhanced precision
Algorithmic Focus	General algorithms for all graph types	Dedicated algorithm for Fan Graphs tailored to satellite systems
Application Coverage	Limited real-world applications	Satellite Communication and Network Design applications
Structural Adaptability	Minimal adaptability to complex graph structures	High adaptability to complex and dense graph structures

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Efficiency Gains	Moderate performance improvement	Significant improvements in accuracy, reliability, and efficiency
Novel Contribution	Conceptual permutation labeling	Practical algorithm and implementation for enhanced solutions
Relevance to Practical Problems	Theoretical focus with minimal real-world testing	Focus on solving practical communication and network challenges

4 Future Development Opportunities for Strong Face Permutation Labeling

For future development of Strong Face Permutation Labeling, sev- eral avenues can be explored to expand the theoretical and practical contributions of this research. One critical direction is the expan- sion to additional graph structures beyond fan, wheel, friendship, and ladder graphs. Exploring complex graph families such as hy- pergraphs, bipartite graphs, and dynamic graphs will broaden the scope and increase applicability in real-world scenarios like social networks and multi-relationship systems.

Further, the development of multi-dimensional applications could enhance the relevance of this labeling method. Practical applica- tions in Internet of Things (IoT) networks and smart cities can leverage strong face permutation labeling for more efficient commu- nication. Similarly, applying this labeling to social network analy- sis could help identify clusters and improve influence propagation. Biological networks, such as protein-protein interaction networks, could also benefit from pathway optimization using this technique.

Algorithmic optimization is another promising area. Developing adaptive algorithms that adjust dynamically based on graph size and complexity can improve labeling efficiency. Moreover, incor- porating parallel or multi-threaded algorithms could ensure faster processing and scalability, particularly for dense or large graphs.

The paper's future theoretical work could include mathemati- cal generalization of strong face permutation labeling across wider classes of graphs. Studying graph isomorphisms and symmetries may also help identify optimal labeling patterns for more complex structures. Integrating machine learning (ML) could further ad- vance the field by enabling label prediction models and self-learning systems to refine labeling techniques over time.

Lastly, real-time applications in areas such as satellite commu- nication and smart transportation can be explored. These use cases would benefit from real-time data management and optimized com- munication routing, where strong face permutation labeling could enhance efficiency. Conducting comprehensive performance bench- marking across different graph types–focusing on speed, memory usage, accuracy, and scalability–will further establish the superior- ity of this method. These future developments will help bridge the gap between theory and practical applications, solidifying the role of strong face permutation labeling across various domains.

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5 Conclusion

To seeks maximize graph labeling for a range of uses. Offers a more intricate and specialized method that focuses on particular graph architectures and their special characteristics, including algorith- mic methods for edge label distinctness and rigorous proofs. Ex- pand the use of permutation labeling to include additional intricate and advanced graph structures. Examine permutation labeling in non-planar and three-dimensional graphs. Use permutation label- ing in practical applications including data structure organization, network design, and circuit layout optimization. Utilizing permu- tation labeling techniques, assess the performance gains in VLSI design and parallel computing systems. By concentrating on these long-term objectives, the study can make a substantial contribu- tion to the theoretical underpinnings and real-world uses of graph labeling in a variety of domains.

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