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ON THE ITERATED CONVOLUTION OPERATORS $\circledast^k$ AND THEIR DISTRIBUTIONAL FUNDAMENTAL SOLUTIONS IN $\mathbb{R}^n$

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Abstract

We investigate the fundamental solution of a class of higher-order differential operators defined via an iterated convolution structure, denoted by $\circledast^k$, which generalizes certain polyharmonic-type and ultra-hyperbolic operators. Within the framework of distribution theory, we construct explicit formulas for the fundamental solutions corresponding to these operators and analyze their behavior depending on the iteration order k and the dimension n. We also provide a classification of solutions to associated inhomogeneous equations, describing conditions under which they are classical functions, tempered distributions, or purely singular distributions. Our results offer a systematic method for solving a broad class of linear partial differential equations with constant coefficients and possess potential applications in mathematical physics and generalized function theory.

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Key Words and Phrases: fundamental solution, distribution theory, partial differential equations

1 Introduction

Fundamental solutions play a crucial role in distribution theory and have numerous applications in mathematical physics, elasticity, heat transfer, and quantum mechanics. The concept was originally introduced by Ram P. [7] , where a fundamental solution $E(x)$ is defined as a generalized function that satisfies the equation $LE(x) = \delta$, with δ denoting the Dirac delta distribution. This solution is not unique since any solution of the homogeneous equation $LE(x) = 0$ can be added to $E(x)$. Kananthai [1] studied the fundamental solution of the iterated diamond operator equation $\Diamond^k E(x) = \delta$, where the solution has the form

$$E(x) = R^H_{2k}(u) * (-1)^k R^e_{2k}(v),$$

with $*$ denoting convolution. Nozaki [14] defined and investigated properties of the *ultra-hyperbolic kernel* and the *elliptic kernel of Marcel Riesz*.

Later, Kananthai et al. [2] studied the relationships between the operator $\oplus^k$ and both the wave and Laplace operators. They showed that the fundamental solution $K(x)$ of the equation $\oplus^k K(x) = \delta$ can be expressed as

$$K(x) = [R^H_{2k}(u) * (-1)^k R^e_{2k}(v)] * S_{2k}(w) * T_{2k}(z),$$

where the operator $\oplus^k$ is defined by

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$$\begin{aligned}\oplus^k &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k \\ &= \Diamond^k L_1^k L_2^k,\end{aligned} \tag{1.1}$$

with L_1^k and L_2^k given by

$$L_1^k = \left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k, \tag{1.2}$$

$$L_2^k = \left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k. \tag{1.3}$$

It follows that

$$L^k = L_1^k L_2^k = L_2^k L_1^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \tag{1.4}$$

Kananthai [1] also defined the *diamond operator* as

$$\Diamond^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \tag{1.5}$$

This operator can be decomposed as $\Diamond^k = \Box^k \triangle^k = \triangle^k \Box^k$, where

$$\Box^k = \left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k, \tag{1.6}$$

$$\triangle^k = \left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k, \tag{1.7}$$

with $p + q = n$.

By setting $p = k = 1$ and $x_1 = t$ (time), equation (1.6) reduces to the classical wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}. \tag{1.8}$$

Tariboon and Kananthai [4] introduced the operator

$$(\oplus + m^2)^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4 + m^2\right]^k, \tag{1.9}$$

and demonstrated its relationship to the iterated Klein–Gordon and Helmholtz operators. Moreover, they considered the equation

$$\oplus + m2kGx = \delta x, \tag{1.10}$$

where $(\oplus + m^2)^k$ is the k-times iterated operator defined in equation (1.9), δ denotes the Dirac delta distribution, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and k is a non-negative integer. The solution $G(x)$ is given by

$$G(x) = Y_{2k,2k,2k,2k}(u, v, w, z, m),$$

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where

$$Y_{2k,2k,2k,2k}(u,v,w,z,m) = \sum_{r=0}^{\infty} \frac{(-1)^r\, \Gamma(k+r)}{r!\; \Gamma(k)} (m^2)^r K_{2k+2r,2k+2r,2k+2r,2k+2r}(u,v,w,z)$$

serves as the fundamental solution of equation (1.10). Here, m is a non-negative real number, and

$$K_{2k+2r,2k+2r,2k+2r,2k+2r}(u,v,w,z)$$

is defined by

$$K_{\alpha+2r,\beta+2r,\gamma+2r,\nu+2r}(u,v,w,z) = (-1)^{\beta/2+r} R^H_{\alpha+2r}(u) * R^e_{\beta+2r}(v) * S_{\gamma+2r}(w) * T_{\nu+2r}(z),$$

(1.11)

with $\alpha = \beta = \gamma = \nu = 2k$. Satsanit also defined the operator

$$\odot^k = \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k = \left(\frac{\triangle^2 + \Box^2}{2}\right)^k. \qquad (1.12)$$

From this, the operator $\otimes^k$ is given by

$$\begin{aligned} \otimes^k &= \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^4 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k \\ &= \left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\Diamond^2\right)^k. \end{aligned} \qquad (1.13)$$

(1.14)

Trione [11] studied the fundamental solution of the iterated Klein-Gordon operator $(\Box + m^2)^k$, defined as

$$(\Box + m^2)^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right) + m^2\right]^k. \qquad (1.15)$$

From the operators in (1.9) and (1.13), we define the new operator $\circledast^k$ as

$$\begin{aligned} \circledast^k &= \left[\left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^4 + \frac{m^2}{2}\right)^2 - \left(\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4 - \frac{m^2}{2}\right)^2\right]^k \\ &= (\oplus + m^2)^k \otimes^k. \end{aligned} \qquad (1.16)$$

For $m = 0$, this reduces to

$$\circledast^k = \oplus^k \otimes^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^8 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^8\right]^k. \qquad (1.17)$$

In this paper, we study the fundamental solution $u(x)$ of the following equation:

$$\circledast^k u(x) = \sum_{r=0}^{t} c_r \circledast^r \delta(x), \qquad (1.18)$$

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where k and t are non-negative integers, $m \geq 0$, and $\delta(x)$ is the Dirac delta distribution. In particular, we consider the case $m = 0$ and analyze the fundamental solution associated with the operator

$$\left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^8 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^8\right]^k.$$

2 Preliminaries

In this section, we provide several definitions and lemmas regarding fundamental solutions associated with various differential operators. These results, drawn from classical and recent works, will serve as the theoretical foundation for our analysis.

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and define

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.1}$$

where $p + q = n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ denote the interior of the forward cone, and $\overline{\Gamma}_+$ its closure. Following Nozaki [14] , define the ultra-hyperbolic kernel of Marcel Riesz by

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

where α is a complex parameter, and

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{2.3}$$

$R_\alpha^H(u)$ is a locally integrable function if $\operatorname{Re} \alpha \geq n$ and a distribution if $\operatorname{Re} \alpha < n$. Its support is contained in $\overline{\Gamma}_+$.

For the special case $p = 1$, the kernel simplifies to the hyperbolic kernel of Marcel Riesz, denoted by $M_\alpha(u)$, defined as

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

with $u = x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{(n-1)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha-n+2}{2}\right)$.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and define

$$v = x_1^2 + x_2^2 + \cdots + x_p^2 + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2. \tag{2.5}$$

Then the elliptic kernel of Marcel Riesz is given by

$$R_\alpha^e(v) = \frac{v^{\frac{\alpha-n}{2}}}{H_n(\alpha)} \tag{2.6}$$

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where

$$H_n(\alpha) = \frac{\pi^{1/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.7}$$

We now summarize a collection of lemmas regarding fundamental solutions of various iterated differential operators. These results will be used in subsequent sections.

Lemma 2.3 ([1]). *Let $\triangle^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\triangle^k$ denotes the iterated Laplacian operator. Then the fundamental solution is given by*

$$u(x) = (-1)^k R_{2k}^e(v), \tag{2.8}$$

where

$$R_{2k}^e(v) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)} |v|^{2k-n}. \tag{2.9}$$

Lemma 2.4 (11). *Let $\square^k u(x) = \delta$ for $x \in \Gamma_+ \subset \mathbb{R}^n$, where $\square^k$ denotes the iterated ultra-hyperbolic operator. Then the fundamental solution is*

$$u(x) = R_{2k}^H(u), \tag{2.10}$$

where

$$R_{2k}^H(u) = \frac{\left(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2\right)^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)}. \tag{2.11}$$

Lemma 2.5 ([1]). *Let $\diamondsuit^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\diamondsuit^k$ is the iterated diamond operator. Then the fundamental solution is*

$$u(x) = (-1)^k R_{2k}^e(v) * R_{2k}^H(u), \tag{2.12}$$

which is also a tempered distribution.

Lemma 2.6 ([2]). *Let $L_1^k u(x) = \delta$ in $\mathbb{R}^n$, with L_1^k defined by* (1.2). *Then the fundamental solution is*

$$u(x) = (-1)^k (-i)^{q/2} S_{2k}(w), \tag{2.13}$$

where

$$S_{2k}(w) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)} \left[x_1^2 + \cdots + x_p^2 - i\left(x_{p+1}^2 + \cdots + x_{p+q}^2\right)\right]^{\left(\frac{2k-n}{2}\right)}. \tag{2.14}$$

Lemma 2.7 ([2]). *Let $L_2^k u(x) = \delta$ in $\mathbb{R}^n$, with L_2^k defined by* (1.3). *Then the fundamental solution is*

$$u(x) = (-1)^k (i)^{q/2} T_{2k}(z), \tag{2.15}$$

where

$$T_{2k}(z) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)} \left[x_1^2 + \cdots + x_p^2 + i\left(x_{p+1}^2 + \cdots + x_{p+q}^2\right)\right]^{\left(\frac{2k-n}{2}\right)}. \tag{2.16}$$

Lemma 2.8 ([2]). *Let $L^k u(x) = \delta$ in $\mathbb{R}^n$, where L^k is defined by* (1.4). *Then the fundamental solution is*

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$$u(x) = S_{2k}(w) * T_{2k}(z). \quad (2.17)$$

Lemma 2.9 ([6],[9],[12]). *The functions $R^H_{-2k}(u)$ and $(-1)^k R^e_{-2k}(v)$ are the inverse in the convolution algebra of $R^H_{2k}(u)$ and $(-1)^k R^e_{2k}(v)$, respectively. That is,*

$$R^H_{-2k}(u) * R^H_{2k}(u) = R^H_{-2k+2k}(u) = R^H_0(u) = \delta,$$
$$(-1)^k R^e_{-2k}(v) * (-1)^k R^e_{2k}(v) = (-1)^{2k} R^e_{-2k+2k}(v) = R^e_0(v) = \delta.$$

Lemma 2.10 ([13]). *(Convolution of $R^e_\alpha(v)$ and $R^H_\alpha(u)$). Let $R^H_\alpha(u)$ and $R^e_\alpha(v)$ defined by* (2.2) *and* (2.6) *respectively, then we obtain:*

(i) $R^e_\alpha(v) * R^e_\beta(v) = R^e_{\alpha+\beta}(v)$ *where α and β are complex parameters;*

(ii) $R^H_\alpha(u) * R^H_\beta(u) = R^H_{\alpha+\beta}(u)$ *for α and β are both integers and except only the case both α and β are both integers.*

Lemma 2.11 ([3]). *(Convolution of $S_\gamma(w)$ and $T_\gamma(z)$)*

(i) $S_\gamma(w) * S_{\gamma'}(w) = (i)^{q/2} S_{\gamma+\gamma'}(w)$;

(ii) $T_\gamma(z) * T_{\gamma'}(z) = (-i)^{q/2} T_{\gamma+\gamma'}(z)$.

Moreover,

$$S_0(w) = (i)^{q/2}\delta, \quad T_0(z) = (-i)^{q/2}\delta.$$

Lemma 2.12 ([4]). *Let $(\oplus + m^2)^k u(x) = \delta$, with $k \in \mathbb{N}_0$, $m \geq 0$, and $(\oplus + m^2)^k$ defined by* (1.9). *Then the fundamental solution is*

$$u(x) = Y_{2k}(u,v,w,z,m)$$
$$= \sum_{r=0}^{\infty} \binom{-k}{r} m^{2r} R^H_{2k+2r}(u) * (-1)^{k+r} R^e_{2k+2r}(v) * S_{2k+2r}(w) * T_{2k+2r}(z).$$
(2.18)

(2.19)

In particular, when $m = 0$, we recover

$$Y_{2k}(u,v,w,z,0) = R^H_{2k}(u) * (-1)^k R^e_{2k}(v) * S_{2k}(w) * T_{2k}(z), \quad (2.20)$$

which is the fundamental solution of $\oplus^k$.

Lemma 2.13 ([8]). *Let $x \in \mathbb{R}^n$, $m \geq 0$, and $k \in \mathbb{N}_0$. Consider the equation*

$$\circledast^k G(x,m) = (\oplus + m^2)^k \otimes^k G(x,m) = \delta, \quad (2.21)$$

where δ is the Dirac delta distribution, the operators $(\oplus + m^2)^k$ and $\otimes^k$ denote k-times iterated versions of the operators defined in (1.9) *and* (1.13)*, respectively.*

Then, a fundamental solution of the operator $\circledast^k$ is given by

$$G(x,m) = Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right],$$
(2.22)

or equivalently,

$$G(x,m) = Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right],$$
(2.23)

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where $Y_{2k,2k,2k,2k}(u,v,w,z,m)$ is a generalized function, and $*$ denotes convolution in appropriate variables. Furthermore, the following identities hold:

(*i*) *The fundamental solution of the ultra-hyperbolic operator $\square^k$ defined in* (1.6) *is given by*

$$[R^H_{-12k}(u) * (-1)^{7k} R^e_{-14k}(v) * S_{-2k}(w) * T_{-2k}(z)] * \left(H^{*k}(x)\right) * G(x,0) = R^H_{2k}(u). \tag{2.24}$$

(*ii*) *The fundamental solution of the iterated Laplacian $\triangle^k$ defined in* (1.7) *is given by either*

$$[R^H_{-14k}(u) * (-1)^{6k} R^e_{-12k}(v) * S_{-2k}(w) * T_{-2k}(z)] * \left(H^{*k}(x)\right) * G(x,0) = (-1)^k R^e_{2k}(v), \tag{2.25}$$

or

$$[R^H_{-14k}(u) * R^e_{-12k}(v) * S_{-2k}(w) * T_{-2k}(z)] * \left(H^{*k}(x)\right) * G(x,0) = (-1)^k R^e_{2k}(v). \tag{2.26}$$

(*iii*) *The fundamental solution of the operator $L^k = L_1^k L_2^k$, as defined in* (1.4)*, is*

$$[R^H_{-14k}(u) * (-1)^{7k} R^e_{-14k}(v)] * \left(H^{*k}(x)\right) * G(x,0) = S_{2k}(w) * T_{2k}(z). \tag{2.27}$$

Here, the distributions $R^e_{-14k}(v)$, $R^H_{-14k}(u)$, $S_{-2k}(w)$, and $T_{-2k}(z)$ denote the convolution inverses of $R^e_{14k}(v)$, $R^H_{14k}(u)$, $S_{2k}(w)$, and $T_{2k}(z)$, respectively.

3 Main Results

We now establish one of the main results concerning the behavior of the iterated convolution operator $\circledast^k$ acting on the associated fundamental solution.

Theorem 3.1. *Let k and r be positive integers with $0 < r < k$. Then the following identities hold:*

$$\begin{aligned}&\circledast^k \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right)\\ &= Y_{2(k-r),2(k-r),2(k-r),2(k-r)}(u,v,w,z,m) * R^H_{12(k-r)}(u)\\ &\quad * (-1)^{6(k-r)} R^e_{12(k-r)}(v) * \left(H^{*(k-r)}(x)\right)^{*-1}.\end{aligned} \tag{3.1}$$

Furthermore, for all integers $t \geq k$, we have

$$\circledast^t \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right) = \circledast^{t-k} \delta. \tag{3.2}$$

Proof. Let $0 < r < k$. By Lemma 2.13, we have

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$$\circledast^k \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right) = \delta.$$

Applying the decomposition $\circledast^k = \circledast^{k-r} \circledast^r$, we obtain

$$\circledast^{k-r} \left(\circledast^r \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right)\right) = \delta.$$

Equivalently,

$$\circledast^{k-r}\, \delta * \circledast^r \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right) = \delta.$$

Now convolve both sides with the fundamental solution corresponding to $\circledast^{k-r}$, namely $Y_{2(k-r),2(k-r),2(k-r),2(k-r)}(u,v,w,z,m) * R^H_{12(k-r)}(u) * (-1)^{6(k-r)} R^e_{12(k-r)}(v) * \left(H^{*(k-r)}(x)\right)^{*-1}$.

Using the associativity of convolution, the left-hand side becomes

$$\circledast^{k-r} \left(Y_{2(k-r),2(k-r),2(k-r),2(k-r)}(u,v,w,z,m) * R^H_{12(k-r)}(u) * (-1)^{6(k-r)} R^e_{12(k-r)}(v) * \left(H^{*(k-r)}(x)\right)^{*-1}\right)$$
$$* \circledast^r \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right),$$

while the right-hand side becomes

$$Y_{2(k-r),2(k-r),2(k-r),2(k-r)}(u,v,w,z,m) * R^H_{12(k-r)}(u) * (-1)^{6(k-r)} R^e_{12(k-r)}(v) * \left(H^{*(k-r)}(x)\right)^{*-1} * \delta.$$

By Lemma 2.13, convolution with δ leaves the function unchanged. Therefore, equating both sides implies

$$\circledast^r \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right)$$
$$= Y_{2(k-r),2(k-r),2(k-r),2(k-r)}(u,v,w,z,m) * R^H_{12(k-r)}(u) * (-1)^{6(k-r)} R^e_{12(k-r)}(v) * \left(H^{*(k-r)}(x)\right)^{*-1},$$

as required.

For the case $t \geq k$, using again Lemma 2.13, we have

$$\circledast^t \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right)$$
$$= \circledast^{t-k} \left(\circledast^k \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right)\right)$$
$$= \circledast^{t-k}\, \delta.$$

This completes the proof. □

Theorem 3.2. *Consider the linear differential equation*

$$\circledast^k u(x) = \sum_{r=0}^{t} c_r \circledast^r \delta, \tag{3.3}$$

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where $p+q=n$, $x\in\mathbb{R}^n$, *each* $c_r\in\mathbb{R}$ *is a constant,* δ *is the Dirac delta distribution, and we define* $\circledast^0\,\delta=\delta$. *Then, the solution* $u(x)$ *depends on the relationship between* k *and* t, *as follows*:

(1) ***Case*** $t=0<k$***:*** *The solution is given by*

$$u(x)=c_0\,Y_{2k,2k,2k,2k}(u,v,w,z,m)*R^H_{12k}(u)*(-1)^{6k}R^e_{12k}(v)*\left(H^{*k}(x)\right)^{*-1}, \tag{3.4}$$

which represents the fundamental solution to the operator $\circledast^k$ *in Lemma 2.13. This solution is an ordinary function when* $2k\geq n$ *and* $12k\geq n$, *and it is a tempered distribution when* $2k<n$ *and* $12k<n$.

(2) ***Case*** $t=0<k$ ***with*** $m=0$***:*** *Then the solution is*

$$u(x)=c_0\,(-1)^{7k}R^e_{14k}(v)*R^H_{14k}(u)*S_{2k}(w)*T_{2k}(z)*\left(H^{*k}(x)\right)^{*-1},$$

which corresponds to the fundamental solution of the operator defined by equation (1.17). *This is an ordinary function when* $2k\geq n$ *and* $14k\geq n$, *and a tempered distribution when* $2k<n$ *and* $14k<n$.

(3) ***Case*** $0<t<k$***:*** *The solution takes the form*

$$u(x)=\sum_{r=0}^{t}c_r\;Y_{2(k-r),2(k-r),2(k-r),2(k-r)}(u,v,w,z,m)*R^H_{12(k-r)}(u)*$$
$$(-1)^{6(k-r)}R^e_{12(k-r)}(v)*\left(H^{*(k-r)}(x)\right)^{*-1},$$

which is an ordinary function when $2(k-r)\geq n$ *and* $12(k-r)\geq n$ *for all* r, *and a tempered distribution when* $2(k-r)<n$ *and* $12(k-r)<n$.

(4) ***Case*** $t\geq k$***:*** *Then the solution is given by* $u(x)=\sum_{r=k}^{t}c_r\circledast^{r-k}\delta$, *which is a purely singular distribution and cannot be expressed as an ordinary or tempered function.*

Proof.

(1) For $t=0$, the equation reduces to

$$\circledast^k\,u(x)=c_0\delta.$$

By Lemma 2.13, the solution is

$$u(x)=c_0\,Y_{2k,2k,2k,2k}(u,v,w,z,m)*R^H_{12k}(u)*(-1)^{6k}R^e_{12k}(v)*\left(H^{*k}(x)\right)^{*-1}. \tag{3.5}$$

When $2k\geq n$ and $12k\geq n$, each of the functions

$$Y_{2k,2k,2k,2k}(u,v,w,z,m),\quad R^H_{12k}(u),\quad R^e_{12k}(v),\quad \left(H^{*k}(x)\right)^{*-1}$$

is analytic, and their convolution in (3.5) exists as an analytic (ordinary) function by (2.22). On the other hand, for $2k<n$ and $12k<n$, by Lemmas 2.5 and 2.13, the solution is a tempered distribution.

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(2) For the case $t = m = 0$, similarly,

$$\circledast^k u(x) = c_0\delta,$$

and by Lemmas 2.10, 2.13 and equation (2.20), we have

$$\begin{aligned} u(x) &= c_0\, Y_{2k,2k,2k,2k}(u,v,w,z,0) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1} \\ &= c_0(-1)^{7k} R^e_{14k}(v) * R^H_{14k}(u) * S_{2k}(w) * T_{2k}(z) * \left(H^{*k}(x)\right)^{*-1}. \end{aligned}$$

When $2k \geq n$ and $14k \geq n$, all components are analytic and their convolution defines an ordinary function by (2.22). For $2k < n$ and $14k < n$, the solution is a tempered distribution by Lemmas 2.5 and 2.13.

(3) For $0 < t < k$, the equation is

$$\circledast^k u(x) = \sum_{r=1}^{t} c_r \circledast^r \delta.$$

Convolve both sides by the fundamental solution

$$Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1},$$

to obtain

$$\begin{aligned} &\circledast^k \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right) * u(x) \\ &= \sum_{r=1}^{t} c_r \circledast^r \left(Y_{2k,2k,2k,2k}(u,v,w,z,m) * R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * \left(H^{*k}(x)\right)^{*-1}\right). \end{aligned}$$

By Lemma 2.13 and Theorem 3.1, this implies

$$\begin{aligned} u(x) &= \sum_{r=1}^{t} c_r\; Y_{2(k-r),2(k-r),2(k-r),2(k-r)}(u,v,w,z,m) * R^H_{12(k-r)}(u) \\ &\quad * (-1)^{6(k-r)} R^e_{12(k-r)}(v) * \left(H^{*(k-r)}(x)\right)^{*-1}. \end{aligned}$$

As in case (1), $u(x)$ is an ordinary function if $2(k-r) \geq n$ and $12(k-r) \geq n$; otherwise, it is a tempered distribution.

(4) For $t \geq k$ with $k \leq t \leq M$, the equation is

$$\circledast^k u(x) = \sum_{r=k}^{M} c_r \circledast^r \delta.$$

Convolving both sides by the fundamental solution as before, and using Lemma 2.13 and Theorem 3.1, yields

$$u(x) = \sum_{r=k}^{M} c_r \circledast^{r-k} \delta.$$

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Since $\circledast^{r-k}\,\delta$ are singular distributions, $u(x)$ is a singular distribution in this case. This completes the proof. □

Theorem 3.3. *Consider the equation*

$$\circledast^k\, h(x) = f(x), \tag{3.6}$$

where $f(x)$ is a given generalized function and $h(x)$ is the unknown function. Then

$$h(x) = G(x,m) * f(x) \tag{3.7}$$

is a solution to equation (3.6)*, where $G(x,m)$ is the fundamental solution of the operator $\circledast^k$ given in Lemma 2.13.*

Proof. Convolve both sides of (3.6) with $G(x,m)$, the fundamental solution for $\circledast^k$ from Lemma 2.13, to obtain

$$G(x,m) * \circledast^k\, h(x) = G(x,m) * f(x).$$

Since convolution is associative and commutative in this context, this can be rewritten as

$$\circledast^k\, G(x,m) * h(x) = G(x,m) * f(x).$$

By Lemma 2.13, we have $\circledast^k\, G(x,m) = \delta$, the Dirac delta distribution, hence

$$\delta * h(x) = G(x,m) * f(x).$$

Since convolution with δ leaves the function unchanged, it follows that

$$h(x) = G(x,m) * f(x),$$

which completes the proof. □

Corollary 3.4. *If $t < k$ and $t = m = 0$, then equation* (3.3) *has the solution*

$$u(x) = c_0(-1)^{7k} R^e_{14k}(v) * R^H_{14k}(u) * S_{2k}(w) * T_{2k}(z) * \left(H^{*k}(x)\right)^{*-1}$$

which is the fundamental solution corresponding to the operator

$$\left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^8 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^8\right]^k u(x)$$

$$= \sum_{r=0}^{t} c_r \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^8 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^8\right]^r \delta,$$

with $c_0 = 1$.

4 Applications

The results obtained in this paper have potential applications in several areas of mathematical analysis and mathematical physics. In particular, the explicit construction of fundamental solutions to the iterated operator $\circledast^k$ enables the following:

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(1) The operator $\circledast^k$ generalizes various elliptic and hyperbolic differential operators. The fundamental solution $G(x, m)$ can be used to solve inhomogeneous partial differential equations (PDEs) of the form

$$\circledast^k h(x) = f(x),$$

where $f(x)$ is a generalized function. This includes problems involving anisotropic media or directional diffusion processes.

(2) The convolution representation

$$h(x) = G(x, m) * f(x)$$

can be interpreted as a Green's function method for linear PDEs with constant coefficients, particularly in higher-dimensional settings. This facilitates both analytical and numerical approaches for boundary value problems.

(3) When the operator $\circledast^k$ is constructed to resemble the difference between powers of Laplacians on complementary subspaces, the solution represents directional wave propagation. This is applicable in problems in elasticity, seismology, and the modeling of anisotropic wave fields.

(4) The presence of delta distributions on the right-hand side of the equation

$$\circledast^k u(x) = \sum_{r=0}^{t} c_r \circledast^r \delta$$

models physical systems with concentrated sources, such as point charges, heat sources, or impulsive forces. The solution form derived in Theorem 4 provides insight into how such sources propagate under the operator.

(5) The distinction between ordinary functions and tempered distributions (depending on the relation between k, n, etc.) allows these solutions to be used in generalized function spaces, such as $\mathcal{S}'(\mathbb{R}^n)$, which are commonly used in Fourier analysis and quantum field theory.

These applications show that the operator-theoretic framework and the convolutional approach presented here are not only theoretically significant but also practically valuable in the analysis of complex systems governed by differential operators.

5 Conclusion

In this paper, we investigated the fundamental solution associated with the iterated operator $\circledast^k$, which generalizes a class of high-order partial differential operators defined on $\mathbb{R}^n$. By employing the framework of distribution theory and convolution techniques, we derived explicit forms of solutions under various assumptions on the source terms. In particular, we provided analytic representations of the fundamental solutions for the cases where the operator acts on Dirac delta distributions and their iterates.

We classified the nature of the solutions based on the relative sizes of the iteration order k, the spatial dimension n, and the order of the delta terms involved. It was shown that under certain conditions, the fundamental solutions are ordinary analytic functions, while in other settings, they exist only in the sense of tempered distributions or singular distributions.

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Moreover, we established general convolution formulas that allow the solution of the equation $\circledast^k h(x) = f(x)$ for arbitrary generalized source terms $f(x)$. These results not only provide insight into the structure of high-order differential equations but also serve as a foundation for further theoretical analysis and applications in mathematical physics, wave propagation, and generalized function theory.

Future work may extend these results to include variable-coefficient operators, non-Euclidean geometries, or nonlinear extensions of the $\circledast^k$ operator.

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