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COMPARATIVE ERROR ANALYSIS OF THE α-CUT METHOD AND STANDARD APPROXIMATION METHOD FOR DIVISION OF TRAPEZOIDAL AND TRIANGULAR FUZZY NUMBERS

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Abstract:

One of the fundamental operations in fuzzy arithmetic (FA) is fuzzy division, where the results can be either exact or approximate. This paper compares two methods for dividing trapezoidal fuzzy numbers (TrFN) and triangular fuzzy numbers (TFN). The first method is an exact approach that uses interval arithmetic via the α-cut technique, while the second is an approximation known as the standard approximation method (SAM). This study presents numerical examples, graphical representations, and tabulated error analyses to illustrate the differences between the exact and approximate results. The findings show that the α-cut method preserves the theoretical structure of fuzzy numbers (FNs), while SAM, although computationally simpler, introduces a measurable approximation error.

Keywords: Error analysis, α-cut method, Standard approximation method, Trapezoidal fuzzy number, Triangular fuzzy number.

1. Introduction

We may face various instances of uncertainty in contemporary society. Nonetheless, this uncertainty cannot be resolved through conventional set theory. Therefore, comprehending fuzzy set theory (FST) and fuzzy logic is crucial. (Zadeh, 1965) was the inaugural proponent of fuzzy sets (FSs). His concept has introduced a new dimension. We are adept at employing the concept of ambiguity. The notion of ambiguity has been thoroughly explored and applied across numerous fields, including artificial intelligence, psychology, physics, chemistry, engineering, operations research, computer science, robotics, and medical science. Based on the research of numerous scholars, we are progressively acquainted with various types of FSs, including type-2 FS (Zadeh, 1975), interval-valued FS (Grattan-Guinness, 1976), intuitionistic FS (Atanassov, 1986), neutrosophic FS (Smarandache, 1999) and spherical FS (Gündoğdu and Kahraman, 2019).

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FA is essential in the domain of FST. The framework and principles of FA have advanced significantly, with several academics contributing across various domains, including decision-making (Gazi et al., 2023), optimization (Alzahrani et al., 2023), medicine (Momena et al., 2023), game theory (Liu et al., 2022), complex fuzzy set (Ali, 2024), topological spaces (Abdullah et al., 2023), linear programing (Hassan et al., 2022), fuzzy neural network (Abduladhem Abdulkareem and Ghassaq, 2021), among others.

FNs are extensions of typical FSs that aid in addressing many complex decision-making situations. Consequently, numerous FNs have been produced by the application of various FSs. The FN, a unique variant of the FS proposed by (Zadeh, 1965) is essential for adeptly managing confusing values through a pragmatic method. Numerous previously published articles indicate that FNs have been utilised in diverse applications across various fields, necessitating FA for estimation and approximation calculations. FA is a fundamental component of FST. (Dubois and Prade, 1978) introduced certain arithmetic operations on fuzzy integers. Recently, many scholars have expressed interest in fuzzy mathematics, and the literature on FNs has expanded regarding contributions to FA operations.

Operations using FNs are conducted using several strategies, including the α-cut method (essentially it is interval arithmetic, (Moore, 1979)), the extension principle , and the vertex method, among others. The configuration of the membership functions (MFs) for FNs considerably influences the results of our calculations when utilising them. For example, examine the interval arithmetic operations on TrFNs. The division of two ambiguous FNs cannot yield a consistent outcome; they do not preserve their form. Often the problem is resolved using the standard approximate method (SAM) division of FNs. (Mondal and Mandal, 2017) have demonstrated a strong interest in investigating the arithmetic operations of pentagonal FNs and have effectively utilized this understanding to solve fuzzy differential equations (FDEs). (Singh et al., 2023) employ FNs to develop a malaria model and solution employing FDEs. Alongside the above-stated academics, (Ranjbar et al., 2022), (Thiripurasundari et al., 2019) and (Jana, 2016) have made substantial contributions to the domain of FN arithmetic by investigating diverse approaches and evaluating the results.

In this paper, we examine the results of our divisions with some specific examples, and we want to conduct an accurate comparison between two distinct division methods for TrFNs and TFNs and determine the error between the two methods. We also want to show which method provides us more accurate and better results.

This paper is structured as follows: Section 1 presents fundamental definitions pertaining to FSs and FNs. Section 2 discusses the α-cut method and SAM. Section 3 discusses the error analysis and section 4 presents numerical examples and graphical results for various cases.

1.1. Preliminaries

This section presents the fundamental concepts, useful results and essential notation used throughout the paper.

Definition 1. (Zadeh, 1965). An FS $\widetilde{\mathcal{M}}$ on a space (the universe) $X \in \mathbb{R}^n$ of elements is a set of ordered pairs,

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$$\widetilde{\mathcal{M}} = \{(\kappa, \widetilde{\mathcal{M}}(\kappa)) \mid \kappa \in X, \widetilde{\mathcal{M}}(\kappa) \in [0,1]\},$$

where $\widetilde{\mathcal{M}}: X \to [0,1]$ and $\widetilde{\mathcal{M}}(x)$ is the MF that maps each element κ to its membership value (MV).

Example 1. Suppose $X = \{1, 2, 3, 4, 5, 6, 7\}$, Consider the FS $\widetilde{\mathcal{M}}$ as the set of numbers closer to 5. Now, take the MVs as $\widetilde{\mathcal{M}}(1) = 0, \widetilde{\mathcal{M}}(2) = 0.2, \widetilde{\mathcal{M}}(3) = 0.5, \widetilde{\mathcal{M}}(4) = 0.8, \widetilde{\mathcal{M}}(5) = 1, \widetilde{\mathcal{M}}(6) = 0.8$ and $\widetilde{\mathcal{M}}(7) = 0.5$; then we write the FS $\widetilde{\mathcal{M}}$ as

$$\widetilde{\mathcal{M}} = \{(1, 0), (2, 0.2), (3, 0.5), (4, 0.8), (5, 1), (6, 0.8), (7, 0.5)\}.$$

Definition 2. (Dutta et al., 2025). The support of an FS $\widetilde{\mathcal{M}}$ is defined as the set of all elements κ in the universe for which $\widetilde{\mathcal{M}}(\kappa)$ is greater than 0. Equation (1) illustrates the support of FS $\widetilde{\mathcal{M}}$.

$$Support(\widetilde{\mathcal{M}}) = \{\kappa \mid \kappa \in X, \widetilde{\mathcal{M}}(\kappa) > 0\}. \tag{1}$$

Definition 3. (Dutta et al., 2025). The core of an FS $\widetilde{\mathcal{M}}$ consists of all elements κ in the universe that have $\widetilde{\mathcal{M}}(\kappa) = 1$. Equation (2) illustrates the core of FS $\widetilde{\mathcal{M}}$.

$$Core(\widetilde{\mathcal{M}}) = \{\kappa \mid \kappa \in X, \widetilde{\mathcal{M}}(\kappa) = 1\}. \tag{2}$$

Definition 4. (Gadjiev et al., 2023). An α-cut set of the FS $\widetilde{\mathcal{M}}$, referred to as $\widetilde{\mathcal{M}}_\alpha$, is a set of all elements $\kappa \in X$ that have the MV greater than or equal to a specified value of α, where $\alpha \in [0, 1]$. Equation (3) provides the α-cut of the FS $\widetilde{\mathcal{M}}$.

$$\widetilde{\mathcal{M}}_\alpha = \{\kappa \mid \kappa \in X, \widetilde{\mathcal{M}}(\kappa) \geqslant \alpha\}. \tag{3}$$

Example 2. Examine example 1. The support of FS $\widetilde{\mathcal{M}}$ is $Support(\widetilde{\mathcal{M}}) = \{2, 3, 4, 5, 6, 7\}$. The core of FS $\widetilde{\mathcal{M}}$ is $Core(\widetilde{\mathcal{M}}) = \{5\}$. The α-cuts of FS $\widetilde{\mathcal{M}}$ can be found as follows: $\widetilde{\mathcal{M}}_{0.1} = \{2, 3, 4, 5, 6, 7\}, \widetilde{\mathcal{M}}_{0.3} = \{3, 4, 5, 6, 7\}, \widetilde{\mathcal{M}}_{0.6} = \{4, 5, 6\}$ and so on.

An FN is a specific type of FS, characterised by a unimodal and normal MF. (Zadeh, 1965) presented FNs as a pragmatic method for effectively managing indefinite numerical values.

Definition 5. (Dubois and Prade, 1978), (Font et al., 2025). An FN is a FS $\widetilde{\mathcal{M}}$ on $\mathbb{R}$ that satisfies the following conditions:

1) $\widetilde{\mathcal{M}}$ is normal FS. i.e., $\exists m \in \mathbb{R}$ for which $\widetilde{\mathcal{M}}(m) = 1$.
2) $\widetilde{\mathcal{M}}$ is convex FS. i.e., $\widetilde{\mathcal{M}}(\lambda m + (1 - \lambda)n) \geqslant \min(\widetilde{\mathcal{M}}(m), \widetilde{\mathcal{M}}(n))$ for any $m, n \in \mathbb{R}$ and $\lambda \in [0,1]$.
3) $\widetilde{\mathcal{M}}$ is upper semi-continuous on $\mathbb{R}$.
4) $\widetilde{\mathcal{M}}_0$ is compact.

Remark 1. The family of trapezoidal and triangular FNs is the most important subset of FSs (for other types of FNs see (Ban et al., 2015) and the references in (Mukherjee et al., 2023)). It is caused by the simplicity of representation which makes easier all transformations and calculations made on trapezoidal and triangular FNs, simplifies computer applications and usually gives more intuitive and more natural interpretation. This is also the reason why the

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trapezoidal and triangular approximation of FNs is a matter of great importance. Therefore, we only work with these two types of FNs.

Definition 6. (Pujaru et al., 2024). A TrFN denoted by $\widetilde{\mathcal{M}} = (t, u, v, w)$, where $t < u < v < w$ and t, u, v, w are real numbers and its MF $\widetilde{\mathcal{M}}(x)$ is denoted as (shown in Figure 1)

$$\widetilde{\mathcal{M}}(x) = \begin{cases} (x-t)/(u-t), & if \quad t \le x \le u \\ 1, & if \quad u \le x \le v \\ (w-x)/(w-v), & if \quad v \le x \le w \\ 0, & otherwise \end{cases}$$

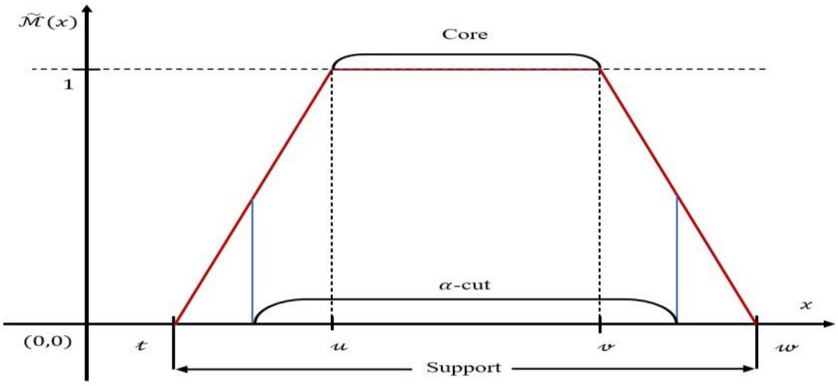

__Figure 1:__ The graph of MF of TrFN $\widetilde{\mathcal{M}} = (t, u, v, w)$ with their α-cut, support and core.

Definition 7. (Mukherjee et al., 2023). A TFN denoted by $\widetilde{\mathcal{M}} = (t, u, w)$, where $t < u < w$ and t, u, w are real numbers and its MF $\widetilde{\mathcal{M}}(x)$ is denoted as (shown in Figure 2)

$$\widetilde{\mathcal{M}}(x) = \begin{cases} (x-t)/(u-t) & if \quad t \le x \le u \\ (w-x)/(w-u) & if \quad u \le x \le w \\ 0 & otherwise \end{cases}$$

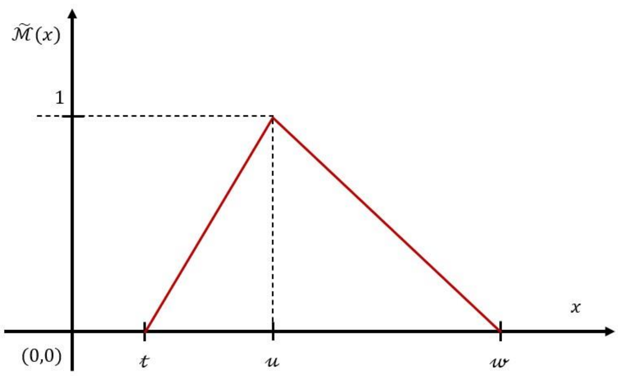

__Figure 2:__ The graph of MF of TFN $\widetilde{\mathcal{M}} = (t, u, w)$.

Remark 2. A TrFN is a TFN if $u = v$. Therefore, It is evident that the MFs and any other formula of TrFN can be applied to TFN when $\widetilde{\mathcal{M}} = (t, u, w) = (t, u, u, w)$.

For every $\alpha \in [0,1]$, the α-cut of any TrFN $\widetilde{\mathcal{M}} = (t, u, v, w)$ constitute a nonempty closed and bounded (compact) interval of the form

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$$\widetilde{\mathcal{M}}_\alpha = [t + \alpha(u - t), w - \alpha(w - v)] = \left[\underline{m}_{(\alpha)}, \overline{m}_{(\alpha)}\right] \quad (4)$$

Remark 3. We have several types of TrFNs (TFNs), listed below:

1) $\widetilde{\mathcal{M}} = (t, u, v, w)$ is a positive TrFN (PTrFN) if $t > 0$.
2) $\widetilde{\mathcal{M}} = (t, u, v, w)$ is a negative TrFN (NTrFN) if $w < 0$.
3) A TrFN $\widetilde{\mathcal{M}} = (t, u, v, w)$ is partial negative TrFN (PNTrFN), where at least one or two or three components of $\widetilde{\mathcal{M}}$ are negative and not all.

2. α-cut Method and SAM for TrFNs and TFNs

The exact result of division can be determined using the α-cut method, and the result is a trapezoidal-shaped or triangular-shaped MF; the result is approximated to be a trapezoidal or triangular MF.

2. 1. α-cut method

The exact result is determined by reformulating the MF to establish a collection of closed intervals as indicated in equation (4), (Mukherjee et al., 2023).

Assuming that $\widetilde{\mathcal{M}} = (t, u, v, w)$ and $\widetilde{\mathcal{N}} = (p, q, r, s)$ are both PTrFNs or both NTrFNs with α-cuts $\widetilde{\mathcal{M}}_\alpha = \left[\underline{m}_{(\alpha)}, \overline{m}_{(\alpha)}\right]$ and $\widetilde{\mathcal{N}}_\alpha = \left[\underline{n}_{(\alpha)}, \overline{n}_{(\alpha)}\right]$. Then for all $\alpha \in [0,1]$ the division operation can be computed as follows

$$\widetilde{\mathcal{M}}_\alpha / \widetilde{\mathcal{N}}_\alpha = [\min Q, \max Q], \quad (5)$$

where $Q = \left\{\underline{m}_{(\alpha)} / \underline{n}_{(\alpha)}, \underline{m}_{(\alpha)} / \overline{n}_{(\alpha)}, \overline{m}_{(\alpha)} / \underline{n}_{(\alpha)}, \overline{m}_{(\alpha)} / \overline{n}_{(\alpha)}\right\}$, with $0 \notin \widetilde{\mathcal{N}}_\alpha$.

When $\widetilde{\mathcal{M}}$ is a PNTrFN and $\widetilde{\mathcal{N}}$ is PTrFN or NTrFN, then the division operation $\widetilde{\mathcal{M}}_\alpha / \widetilde{\mathcal{N}}_\alpha$ is different from that of equation (5) because the values of $\underline{m}_{(\alpha)}$ or $\overline{m}_{(\alpha)}$ will vary within the interval from 0 to 1. Thereby, the interval of [0,1] will be partitioned into two segments.

When $\widetilde{\mathcal{M}}$ is negative and $\widetilde{\mathcal{N}}$ is a positive FN, or vice versa, the division of $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ can then be found as in expression (5).

2. 2. SAM

The SAM for division of $\widetilde{\mathcal{M}} = (t, u, v, w)$ and $\widetilde{\mathcal{N}} = (p, q, r, s)$ is defined as follows:

$$\widetilde{\mathcal{M}} / \widetilde{\mathcal{N}} = (\min \mathcal{U}, \min \mathcal{V}, \max \mathcal{V}, \max \mathcal{U}), \quad (6)$$

where $\mathcal{U} = \{t/p, t/s, w/p, w/s\}$, $\mathcal{V} = \{u/q, u/r, v/q, v/r\}$ and $\widetilde{\mathcal{N}}$ is positive or negative.

The expression (5) is known as analytical method for FA operation. The approximation and exact results are shown in Figures 3, 4 and 5. In exact division the lines connecting the ends points are parabolic and in SAM lines connecting the ends points are triangular or trapezoidal form.

3. Error analysis

The absolute error is the difference, at a specified α-cut, among the approximated MF and the exact findings in equations (5) and (6). Each TrFN or TFN can be partitioned into left and right portions. The exact division in equation (5) will yield the value x at a specified α denoted as $\mathcal{T}_l$ for the left portion, and $\mathcal{T}_r$ for the right portion. The SAM in equation (6) will have a value x, at a specified α defined as the left portion $\mathcal{A}_l$ and right portion $\mathcal{A}_r$.

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(Butt, 2021). Following that, we will define the absolute error for the left and right portions in this manner:
$e_l = |\mathcal{A}_l - \mathcal{T}_l|$ is the left and $e_r = |\mathcal{A}_r - \mathcal{T}_r|$ is the right.

This is represented graphically as the horizontal distance between the two curves, as illustrated in Figures 3, 4, and 5. The numerical error is presented in Tables 1, 2, and 3.

4. Numerical study

We provide below some numerical examples using tables and graphs that illustrate the difference between the two division methods.

Case study 1: If both FNs are positive or negative.

Example 3. Let $\widetilde{\mathcal{M}} = (2,4,6,8)$ and $\widetilde{\mathcal{N}} = (1,2,3,4)$ be two PTrFNs, the α-cut of $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{M}}_\alpha = [2+2\alpha, 8-2\alpha]$ and the α-cut of $\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{N}}_\alpha = [1+\alpha, 4-\alpha]$. Therefore, for any $\alpha \in [0,1]$ we have that $\widetilde{\mathcal{M}}_\alpha > 0$ and $\widetilde{\mathcal{N}}_\alpha > 0$. Then by equation (5) we have

$$\widetilde{\mathcal{M}}_\alpha / \widetilde{\mathcal{N}}_\alpha = [(2+2\alpha)/(4-\alpha), (8-2\alpha)/(1+\alpha)].$$

Let $x = (2+2\alpha)/(4-\alpha)$ then $\alpha = (4x-2)/(2+x)$ is an increasing function for $1/2 \leq x \leq 4/3$. Let $x = (8-2\alpha)/(1+\alpha)$ then $\alpha = (8-x)/(2+x)$ is a decreasing function for $3 \leq x \leq 8$. Hence, the MF of $\widetilde{\mathcal{M}}_\alpha / \widetilde{\mathcal{N}}_\alpha$ can be expressed as follows

$$\left(\widetilde{\mathcal{M}}_\alpha / \widetilde{\mathcal{N}}_\alpha\right)(x) = \begin{cases} (4x-2)/(2+x) & if \quad 1/2 \leq x \leq 4/3 \\ 1 & if \quad 4/3 \leq x \leq 3 \\ (8-x)/(2+x) & if \quad 3 \leq x \leq 8 \\ 0 & otherwise \end{cases}$$

In SAM, $\widetilde{\mathcal{M}} / \widetilde{\mathcal{N}} = (1/2, 4/3, 3, 8)$ and the MF is

$$\left(\widetilde{\mathcal{M}} / \widetilde{\mathcal{N}}\right)(x) = \begin{cases} (6x-3)/5 & if \quad 1/2 \leq x \leq 4/3 \\ 1 & if \quad 4/3 \leq x \leq 3 \\ (8-x)/5 & if \quad 3 \leq x \leq 8 \\ 0 & otherwise \end{cases}$$

Table 1: *Comparison of both division methods of example 3.*

α	Exact division		Standard approximation		Absolute error	
	$\mathcal{T}_l$	$\mathcal{T}_r$	$\mathcal{A}_l$	$\mathcal{A}_r$	e_l	e_r
0	0.500	8.000	0.500	8.000	0.000	0.000
0.1	0.564	7.091	0.583	7.500	0.019	0.409
0.2	0.632	6.333	0.667	7.000	0.035	0.667
0.3	0.703	5.692	0.750	6.500	0.047	0.808

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0.4	0.778	5.143	0.833	6.000	0.055	0.857
0.5	0.857	4.667	0.917	5.500	0.060	0.833
0.6	0.941	4.250	1.000	5.000	0.059	0.750
0.7	1.030	3.882	1.083	4.500	0.053	0.618
0.8	1.125	3.556	1.167	4.000	0.042	0.444
0.9	1.226	3.263	1.250	3.500	0.024	0.234
1	1.333	3.000	1.333	3.000	0.000	0.000

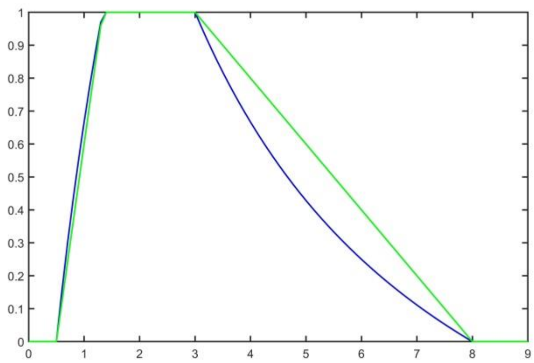

Figure 3: *The graph of the MF* $\left(\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha\right)(x)$ *(blue) and the MF* $\left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}\right)(x)$ *(green) of example 3.*

Case study 2: If $\widetilde{\mathcal{M}}$ is partial negative and $\widetilde{\mathcal{N}}$ is negative or positive.

Example 4. Let $\widetilde{\mathcal{M}} = (-7,-5,-4,6)$ is a PNTrFN and $\widetilde{\mathcal{N}} = (-5,-4,-2,-1)$ is a NTrFN, the α-cut of $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{M}}_\alpha = [-7+2\alpha, 6-10\alpha]$ and the α-cut of $\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{N}}_\alpha = [-5+\alpha, -1-\alpha]$. Therefore, for any $\alpha \in [0,1]$ implise $\underline{m}_{(\alpha)} < 0$ and $\widetilde{\mathcal{N}}_\alpha < 0$ but $\overline{m}_{(\alpha)}$ will not exhibit the same sign throughout the interval [0,1]. Here, we partition the interval [0,1] into two sub-intervals as $[0, 3/5]$ and $[3/5, 1]$ where $6-10\alpha = 0 \Longrightarrow \alpha = 3/5$. So, for $\alpha \in [0, 3/5]$ implise $\overline{m}_{(\alpha)} > 0$. Again for $\alpha \in [3/5, 1]$ implise $\overline{m}_{(\alpha)} < 0$. Then by equation (5), we have

$$\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha = \begin{cases} [(10\alpha-6)/(1+\alpha), (7-2\alpha)/(1+\alpha)], \forall\alpha \in [0,3/5], \\ [(6-10\alpha)/(\alpha-5), (7-2\alpha)/(1+\alpha)], \forall\alpha \in [3/5,1]. \end{cases}$$

Let $x = (10\alpha-6)/(1+\alpha)$ then $\alpha = (6+x)/(10-x)$ is an increasing function for $-6 \leq x \leq 0$. Let $x = (7-2\alpha)/(1+\alpha)$ then $\alpha = (7-x)/(2+x)$ is a decreasing function for $5/2 \leq x \leq 7$. Let $x = (6-10\alpha)/(\alpha-5)$ then and $\alpha = (6+5x)/(x+10)$ is an increasing function for $0 \leq x \leq 1$. Hence, the MF of $\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha$ can be expressed as follows

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$$(\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha)(x) = \begin{cases} (6+x)/(10-x) & if \quad -6 \le x \le 0 \\ (6+5x)/(x+10) & if \quad 0 \le x \le 1 \\ 1 & if \quad 1 \le x \le 5/2 \\ (7-x)/(2+x) & if \quad 5/2 \le x \le 7 \\ 0 & otherwise \end{cases}$$

In SAM, $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}} = (-6, 1, 5/2, 7)$ and the MF is

$$(\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}})(x) = \begin{cases} (6+x)/7 & if \quad -6 \le x \le 1 \\ 1 & if \quad 1 \le x \le 5/2 \\ (14-2x)/9 & if \quad 5/2 \le x \le 7 \\ 0 & otherwise \end{cases}$$

Table 2: *Comparison of both division methods of example 4.*

α	Exact division		Standard approximation		Absolute error	
	$\mathcal{T}_l$	$\mathcal{T}_r$	$\mathcal{A}_l$	$\mathcal{A}_r$	e_l	e_r
0	−6.000	7.000	−6.000	7.000	0.000	0.000
0.1	−4.546	6.182	−5.300	6.550	0.754	0.368
0.2	−3.333	5.500	−4.600	6.100	1.267	0.600
0.3	−2.308	4.923	−3.900	5.650	1.592	0.727
0.4	−1.429	4.429	−3.200	5.200	1.771	0.771
0.5	−0.668	4.000	−2.500	4.750	1.832	0.750
0.6	0.000	3.625	−1.800	4.300	1.800	0.675
0.7	0.233	3.294	−1.100	3.850	1.333	0.556
0.8	0.476	3.000	−0.400	3.400	0.876	0.400
0.9	0.732	2.737	0.300	2.950	0.432	0.213
1	1.000	2.500	1.000	2.500	0.000	0.000

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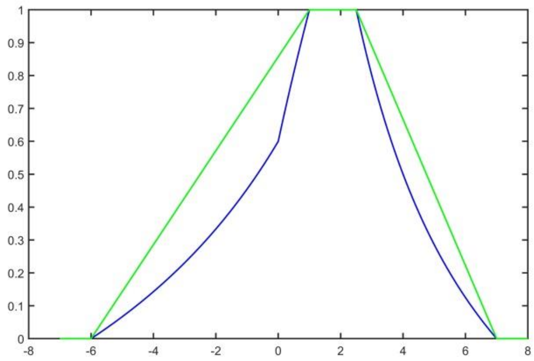

Figure 4: *The graph of the MF $(\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha)(x)$ (blue) and the MF $(\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}})(x)$ (green) of example 4.*

Case study 3: When $\widetilde{\mathcal{M}}$ is negative and $\widetilde{\mathcal{N}}$ is a positive or vice versa.

Example 5. Let $\widetilde{\mathcal{M}} = (4, 6, 9)$ is PTFN and $\widetilde{\mathcal{N}} = (-3, -2, -1)$ is NTFN, the α-cut of $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{M}}_\alpha = [4 + 2\alpha, 9 - 3\alpha]$ and the α-cut of $\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{N}}_\alpha = [\alpha - 3, -1 - \alpha]$. Therefore, for any $\alpha \in [0,1]$, $\widetilde{\mathcal{M}}_\alpha > 0$ and $\widetilde{\mathcal{N}}_\alpha < 0$. Then by equation (5) we have

$$\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha = [(9 - 3\alpha)/(-1 - \alpha), (4 + 2\alpha)/(\alpha - 3)].$$

Let $x = (9 - 3\alpha)/(-1 - \alpha)$ then $\alpha = (-9 - x)/(x - 3)$ is an increasing function for $-9 \le x \le -3$. Let $x = (4 + 2\alpha)/(\alpha - 3)$ then $\alpha = (-4 - 3x)/(2 - x)$ is a decreasing function for $-3 \le x \le -4/3$. Hence, the MF of $\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha$ can be expressed as follows

$$(\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha)(x) = \begin{cases} (9 + x)/(3 - x) & if \ -9 \le x \le -3 \\ (4 + 3x)/(x - 2) & if \ -3 \le x \le -4/3 \\ 0 & otherwise \end{cases}$$

In SAM, $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}} = (-9, -3, -4/3)$ and the MF is

$$(\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}})(x) = \begin{cases} (x + 9)/6 & if \ -9 \le x \le -3 \\ (-4 - 3x)/5 & if \ -3 \le x \le -4/3 \\ 0 & otherwise \end{cases}$$

Table 3: *Comparison of both division methods of example 5.*

α	Exact division		Standard approximation		Absolute error	
	$\mathcal{T}_l$	$\mathcal{T}_r$	$\mathcal{A}_l$	$\mathcal{A}_r$	e_l	e_r
0	-9.000	-1.333	-9.000	-1.333	0.000	0.000
0.1	-7.909	-1.448	-8.400	-1.500	0.491	0.052

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0.2	−7.000	−1.571	−7.800	−1.667	0.800	0.096
0.3	−6.231	−1.704	−7.200	−1.833	0.969	0.129
0.4	−5.571	−1.846	−6.600	−2.000	1.029	0.154
0.5	−5.000	−2.000	−6.000	−2.167	1.000	0.167
0.6	−4.500	−2.167	−5.400	−2.333	0.900	0.166
0.7	−4.059	−2.348	−4.800	−2.500	0.741	0.152
0.8	−3.667	−2.546	−4.200	−2.667	0.533	0.121
0.9	−3.316	−2.762	−3.600	−2.833	0.284	0.071
1	−3.000	−3.000	−3.000	−3.000	0.000	0.000

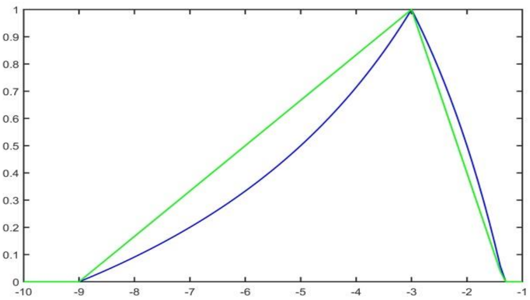

Figure 5: *The graph of the MF* $(\widetilde{\mathcal{M}}_\alpha/\widetilde{\mathcal{N}}_\alpha)(x)$ *(blue) and the MF* $(\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}})(x)$ *(green) of example 5.*

We have conducted comparisons here only for these few cases; we can use the same method for all other cases.

5. Conclusion

Based on the comparative analysis presented in this study, the α-cut method is an exact approach for dividing trapezoidal and triangular FNs that preserves the theoretical integrity of the FN structure. In contrast, the standard approximation method (SAM), while computationally efficient and simpler to apply, introduces a measurable approximation error, as demonstrated in various cases involving positive, negative, and partially negative FNs. Graphical and tabulated error analyses emphasize the inherent trade-off between accuracy and practicality and suggest that the α-cut method is preferred in areas requiring high accuracy, such as theoretical developments and precise fuzzy modelling. Conversely, SAM may be appropriately employed in applications where computational ease outweighs the need for accurate results, such as simulation or optimization projects.

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