

**A NUMERICAL STUDY ON THE ACCURACY AND SPEED OF  
GEOMETRIC SERIES AND RIEMANN INTEGRAL METHODS FOR  
LOGARITHMIC CALCULATIONS**

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**Abstract**

This study investigates the approximation of the natural logarithm function  $\ln(x)$  for integer values  $x = 2, 3, \dots, 100$  using two approaches: an infinite geometric series and a Riemann integral-based method developed in this research. The study first proves the recursive formula of  $\ln(x)$  through the Riemann integral using mathematical induction, establishing a theoretical foundation for the method. Numerical evaluations are then carried out, with the Riemann integral computed both analytically and numerically using partition values  $n = 10, 100, 1000, 10000, 100000, 1000000$ . The two approaches are compared in terms of accuracy and computational efficiency, with accuracy measured using the Mean Absolute Percentage Error (MAPE). The results show that while the geometric series provides higher accuracy, the proposed Riemann integral method demonstrates superior execution speed and serves as a fundamental basis for developing more advanced numerical integration techniques.

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**Key Words and Phrases:** natural logarithm approximation, geometric series, Riemann integral, Mean Absolute Percentage Error, quadrature methods

## 1. Introduction

Education is a deliberate effort aimed at enhancing human quality through a goal-oriented process. It operates continuously across all types and levels, forming an integral and unified system (Farida et al. [5]). Mathematics Education, in particular, is structured within three key frameworks identified by Thanheiser [13]: 1) abstract knowledge and methods, 2) contextual understanding of the world, and 3) an expression of human identity. The focus is on fostering problem-solving knowledge and strategies within these frameworks to develop students' abilities effectively.

Problem-solving skills refer to strategic competence in understanding problems and selecting appropriate strategies to find solutions. One key goal of mathematics education is developing problem-solving abilities, including understanding problems, formulating solution models, solving these models, and providing appropriate solutions (Oktaviyanti [10]). These skills are essential not only for individuals delving deeper into mathematics but also for those applying mathematical concepts in other fields and everyday life (Sofyan et al. [12]).

Solving mathematical problems can be done analytically or numerically. The solution of the numerical method is an overlay, while the analytical method is a true solution. This study performs numerical solutions in determining the value of natural logarithms. The natural logarithm function ( $\ln$ ) is a logarithm with base  $e = 2.718281828\dots$ , i.e.  $\ln(e^x) = x, x > 0$ , where  $e^x$  is an exponential function. Natural logarithm functions are defined by  $\ln(x) = \int_1^x \frac{1}{t} dt$ , for  $x > 0$ . The derivative of the natural logarithm is  $\frac{d}{dx} \ln(x) = \frac{1}{x}$ .

Natural logarithms are one of the most useful functions in mathematics, with many applications in the fields of econometrics, physics, chemistry, biology, and others. For example, their use in econometrics includes analyzing the impact of pandemic indices on tourist arrivals (Karabulut et al. [8]). Similarly, natural logarithms are used to measure the daily growth of cases and deaths due to COVID-19 in panel regression analysis (Al-Awadhi et al. [1]). Natural logarithms follow the same rules of mathematical operations as general logarithms (logarithms with base 10, usually written as  $\log$ ), namely:  $\ln(ab) = \ln(a) + \ln(b)$ ;  $\ln(a/b) = \ln(a) - \ln(b)$ ; and  $\ln(a^b) = b \ln(a)$ .

In this study, we propose a novel recursive formulation of  $\ln(x)$  for  $x = 2, 3, 4, \dots$ , derived via the Riemann integral approach and proven using mathematical induction. This formulation allows us to establish a systematic method for approximating  $\ln(x)$ , which, to the best of our knowledge, has not been explicitly presented in previous literature. Furthermore, we compare this method with the classical infinite geometric series in terms of accuracy and computational efficiency, highlighting the trade-off between precision and speed.

## 2. Research Design

In this study, the formula of  $\ln(x), x = 2, 3, 4, \dots$ , was first determined by using Riemann integrals and mathematical induction. The results obtained were compared with the actual values for  $x = 2, 3, 4, \dots, 100$  and  $n = 10, 100, 1000, 10000, 100000$  and  $1000000$ .

Similarly,  $\ln(x)$  for  $x = 2, 3, 4, \dots, 100$ , obtained from the development of the infinite geometric series with  $n = 10, 100, 1000, 10000, 100000, \text{ and } 1000000$  terms, was compared with the actual values, and the MAPE value was also calculated for both approaches. The tool used in this study is the MATLAB software to find the approximation value of  $\ln(x), x = 2, 3, 4, \dots, 100$ .

The work steps mentioned above are summarized in several stages of the algorithm of the method of this research:

1. Identify the problem.

Research refers to  $\int_1^x \frac{1}{t} dt$ , both analytically and numerically by using Riemann integrals.

2. Problem solutions

Calculation of the result of  $\int_1^x \frac{1}{t} dt$ , both analytically and numerically using the Riemann integral performed several times for different integrant values, further proved by mathematical induction, for  $x = 2, 3, 4, \dots$

3. Simulations of the methods

In this stage, the results of  $\ln(x), x = 2, 3, 4, \dots, 100$  are simulated from the development of infinite geometric series as well as from the development of the Riemann integral for  $n = 10, 100, 1000, 10000, 100000$  and  $1000000$ , where  $n$  is the number of partitions.

4. Retraction of conclusions

From the results and comparisons of  $\ln(x), x = 2, 3, 4, \dots, 100$  the values of the approximation obtained from the development of infinite geometric series as well as from the development of the Riemann integral with the values of  $\ln(x)$  contained in MATLAB which are considered as actual values.

### **3. Riemann Integral**

Integrals are one of the most important and fundamental mathematical concepts and have many applications in physics and engineering. Integrals must be calculated in several practical problems, among them calculating the flat plane area under a curve. The bounded integral  $\int_a^b f(x)dx$  can be determined after the antiderivative  $F(x)$  of  $f(x)$  is known. The Newton-Leibniz formula gives a certain integral  $f(x)$  at the interval  $[a, b]$ , as follows  $\int_a^b f(x)dx = F(b) - F(a)$  (Zhao and Li [2]). Integral formulation  $\int_a^b f(x)dx$  is also the area of the curve bounded by  $y = f(x)$ , line  $x = a$ , line  $x = b$  and  $x$ -axis, as shown in Figure 1.

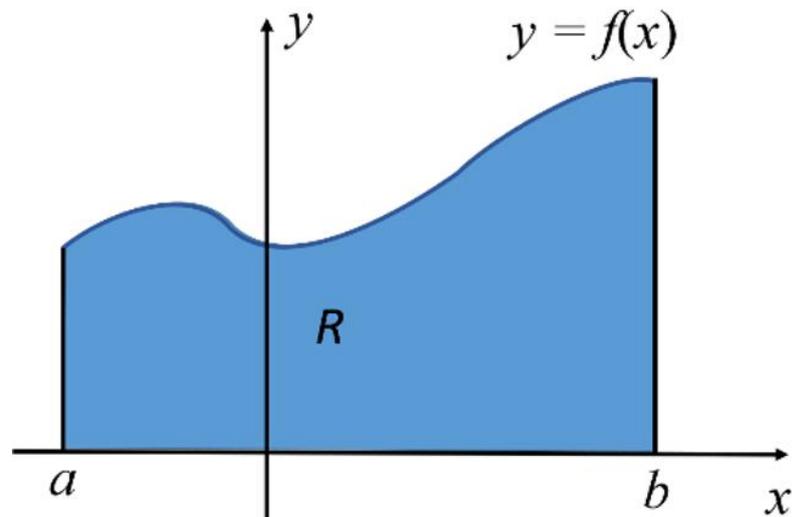


Figure 1. The area of the area bounded by the curve  $y = f(x)$ , line  $x = a$ , line  $x = b$  and  $x$ -axis

To understand the conceptual meaning of integration, we rely on the definition of the definite integral which is the limit of the Riemann sum. We can use the fundamental theorem of calculus in calculating definite integrals. The fundamental theorem of calculus can solve integrals necessarily by using processes to convert integrals into anti-derivative ones (Oh [9]). However, the anti-derivative  $F(x)$  is not easy to obtain. In addition, some  $f(x)$  values are only available at certain points. It often happens that values are derived from experimental data, but the need demands to calculate a certain integral. For many functions and applications, the Riemann integral can be evaluated with the fundamental theorems of calculus or approximated by numerical integration.

Many textbooks use rectangles of the same width to calculate the limit of the number of Riemann partitions but then point out a more general definition of an integral. Here the subintervals do not have to be of equal length, followed by the statement that when functions can be integrated then the sum of the Riemann limits corresponding to partitions of the same length gives an integral value (Dawson [4] and Thomson [14]). In the usual Riemann integral arrangement, the Riemann norm is adopted for Riemann quantities (Chen [3]), whereas in real analysis, Riemann integration is the most commonly used form to estimate areas under curves. The interval domain in  $x$ -axis is divided into segments. A rectangle is built on top of each segment to fit below (or above) the curve, and its area is estimated as the sum of the rectangular areas (Duffy et al. [5]). This can be seen from Figure 2.

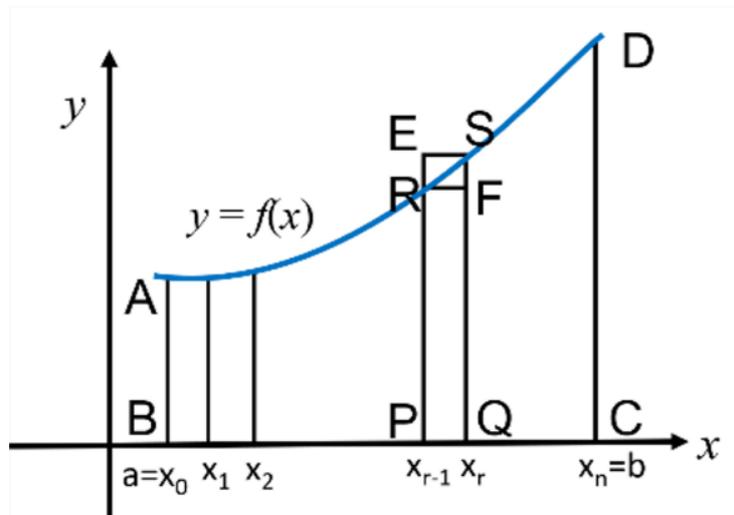


Figure 2. The approximation of the area delimited by the curve  $y = f(x)$ , line  $x = a$ , line  $x = b$  and  $x$ -axis

Notation  $\int_a^b f(x)dx$  is the area bounded by the curve  $y = f(x)$ . This region is represented by the  $ABCD$  region as shown in Figure 2. The entire region lying between  $[a, b]$  is divided into  $n$  equal subintervals given by  $[x_0, x_1], [x_1, x_2], \dots, [x_{r-1}, x_r], \dots, [x_{n-1}, x_n]$ . Think of the width of each subinterval as  $dx, x_0 = a, x_1 = a + dx, x_2 = a + 2dx, \dots, x_r = a + rdx, x_n = b = a + ndx$  and  $n = (b - a)/dx$ . From Figure 2 it can be seen that the area of the  $PQFR$  rectangle  $<$  the area of the  $PQSR$   $<$  the area of the  $PQSE$  rectangle.

Define the variables  $s_n$  and  $S_n$  that express the sum of Equations (1) and (2):

$$s_n = [f(x_0) + f(x_1) + \dots + f(x_{n-1})]dx = dx \sum_{r=0}^{n-1} f(x_r) \quad (1)$$

$$S_n = [f(x_1) + f(x_2) + \dots + f(x_n)]dx = dx \sum_{r=1}^n f(x_r) \quad (2)$$

Based on Equations (1) and (2) by considering any arbitrary sub interval  $[x_{r-1}, x_r]$  with  $r = 1, 2, 3, \dots, n$ , it can be said that:

$$s_n < \text{area } ABCD < S_n$$

Because of  $n \rightarrow \infty$ , then  $dx \rightarrow 0$ , therefore  $x_r - x_{r-1} \rightarrow 0$  so the rectangular line is very narrow. It can be assumed that the limit value of  $s_n$  and  $S_n$  are the same, and the shared boundary values give us the area under the curve, which is:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} S_n = \text{area } ABCD = \int_a^b f(x)dx \quad (3)$$

Furthermore, to measure the accuracy of an approach value with the actual value there are several ways to do, one of which is the Mean Absolute Percentage Error (MAPE). MAPE

is one of the most widely used and recommended measures of forecast accuracy in most textbooks. Suppose  $A_t$  and  $F_t$  show the actual and approximate values at the respective data point  $t$ . MAPE is defined as:

$$\text{MAPE} = \frac{1}{N} \sum_{t=1}^N \left| \frac{A_t - F_t}{A_t} \right| * 100\% \quad (4)$$

where  $N$  is the amount of data (Kim and Kim [7]). The criteria of MAPE are as follows: < 10% excellent approach model capabilities; 10% - 20% good approach model capabilities; 20% - 50% feasible approach model capabilities; > 50% poor approach model capabilities (Cheng [15]).

#### 4. Geometric Series Development

The current numerical approximation of  $\ln(x)$  is based on the development of infinite geometric series. Salas and Hille [11] explain that the value of  $\ln(x)$  be approximated using a formula that requires solving for a specific value of  $x-1 < x < 1$  implicitly in the left hand side of the equation, but the use in the right hand side is only a substitution of the solution to the left one.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots, -1 < x < 1 \quad (5)$$

By integrating both sides of Equation (5) we get Equation (6):

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \dots, \quad (6)$$

$$-1 < x < 1.$$

Further the infinite geometric series is determined in the same way for Equation (7):

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots, -1 < x < 1 \quad (7)$$

By integrating both sides of Equation (7) we get Equation (8):

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots, \quad (8)$$

$$-1 < x < 1.$$

Then by adding Equations (6) and (8) we get:

$$\begin{aligned} \ln(1+x) - \ln(1-x) &= \ln \frac{1+x}{1-x} \\ &= 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right) - 1 < x < 1 \end{aligned} \quad (9)$$

From Equation (9) we could determine the value of  $\ln$  argument larger than 1. For example, we calculate the value of  $\ln(2)$ , meaning we use the value  $x = \frac{1}{3}$ , and for  $\ln(3)$  we use the value  $x = \frac{1}{2}$ .

**5. The Riemann Integral Method for Natural Logarithm Function Determination**

In this section the proof of Riemann Integral method in the determination of a formula for  $\ln(m)$  for a natural number  $m$  is derived. The value of  $\int_a^b f(x)dx$  is developed based on Equation (2), namely  $\sum_{i=1}^n f(x_i)dx$ ,  $x_i = a + idx, i = 1, 2, 3, \dots, n$ ,  $dx = \frac{b-a}{n}$ , with  $n$  large numbers or  $n \rightarrow \infty$ . The area bounded by the curve  $y = \frac{1}{x}$ , the horizontal axis  $y = 0$ , and the lines  $x = a, x = b$  is the bounded integral  $L = \int_a^b \frac{1}{x} dx$ . Using the Riemann sum, then it can be written as  $\sum_{i=1}^n \frac{1}{x_i} dx, x_i = a + idx, i = 1, 2, 3, \dots, n$ . Here it will be proven that  $\ln(m) = \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{mn} \right), m = 2, 3, 4, \dots$

*Proof:*

Define  $L_k = \int_k^{k+1} \frac{1}{x} dx, k = 1, 2, 3, \dots$ . By using the mathematical induction, first it was done for  $k = 1$ , or for  $m = 2$  in the equation of the problem.  $L_1 = \int_1^2 \frac{1}{x} dx = \ln(2) - \ln(1) = \ln(2)$ .

$$L_1 = \sum_{i=1}^n f(x_i)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{x_i} dx = \lim_{n \rightarrow \infty} \left( \frac{1}{1+\frac{1}{n}} \frac{1}{n} + \frac{1}{1+\frac{2}{n}} \frac{1}{n} + \frac{1}{1+\frac{3}{n}} \frac{1}{n} + \dots + \frac{1}{1+\frac{n}{n}} \frac{1}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n} \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} \right).$$

So,  $\ln(2) = \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} \right)$ . The same was done for  $m = 3, L_2 = \int_2^3 \frac{1}{x} dx = \ln(3) - \ln(2)$ .

$$L_2 = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{x_i} dx = \lim_{n \rightarrow \infty} \left( \frac{1}{2+\frac{1}{n}} \frac{1}{n} + \frac{1}{2+\frac{2}{n}} \frac{1}{n} + \frac{1}{2+\frac{3}{n}} \frac{1}{n} + \dots + \frac{1}{2+\frac{n}{n}} \frac{1}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1}{2n+1} + \frac{1}{2n+2} + \frac{1}{2n+3} + \dots + \frac{1}{2n+n} \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{2n+1} + \frac{1}{2n+2} + \frac{1}{2n+3} + \dots + \frac{1}{3n} \right).$$

It can be written that  $\ln(3) = \ln(3) - \ln(2) + \ln(2) = \ln(2) + L_2$ . So,  $\ln(3) = \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} \right) + \lim_{n \rightarrow \infty} \left( \frac{1}{2n+1} + \frac{1}{2n+2} + \frac{1}{2n+3} + \dots + \frac{1}{3n} \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{3n} \right)$ .

Assume that is true for  $m = k$  with  $k \geq 2$ ,  $k$  integer so  $\ln(k) = \lim_{n \rightarrow \infty} (\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{kn})$ . It will be shown that the statement is also true for  $m = k + 1$ , in other words it will be pointed out that  $\ln(k + 1) = \lim_{n \rightarrow \infty} (\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{(k+1)n})$ .

Since  $L_k = \int_k^{k+1} \frac{1}{x} dx = \ln(k + 1) - \ln(k)$ , and

$$L_k = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{x_i} dx = \lim_{n \rightarrow \infty} (\frac{1}{k+\frac{1}{n}} + \frac{1}{k+\frac{2}{n}} + \frac{1}{k+\frac{3}{n}} + \dots + \frac{1}{k+\frac{n}{n}})$$

$$= \lim_{n \rightarrow \infty} (\frac{1}{kn+1} + \frac{1}{kn+2} + \frac{1}{kn+3} + \dots + \frac{1}{(k+1)n}).$$

Again,  $\ln(k + 1) = \ln(k + 1) - \ln(k) + \ln(k) = \ln(k) + L_k$ , so that

$$\ln(k + 1) = \lim_{n \rightarrow \infty} (\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{kn}) + \lim_{n \rightarrow \infty} (\frac{1}{kn+1} + \frac{1}{kn+2} + \frac{1}{kn+3} + \dots + \frac{1}{(k+1)n}) =$$

$$\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{(k+1)n}).$$

Therefore, it was proven that in general the formula of  $\ln(m)$  is

$$\ln(m) = \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{mn} \right), \quad (10)$$

$m = 2, 3, 4 \dots$

### 3. Riemann Integral

In this study, the formula of  $\ln(x)$ ,  $x = 2, 3, 4, \dots$ , was first determined by using Riemann integrals and mathematical induction. The results obtained were compared with the actual values for  $x = 2, 3, 4, \dots, 100$  and  $n = 10, 100, 1000, 10000, 100000$  and  $1000000$ . Similarly,  $\ln(x)$  for  $x = 2, 3, 4, \dots, 100$ , obtained from the development of the infinite geometric series with  $n = 10, 100, 1000, 10000, 100000$ , and  $1000000$  terms, was compared with the actual values, and the MAPE value was also calculated for both approaches.

### 6. Comparison of the Results Calculated by Riemann Integral Expansion and Geometric Series

The MAPE value is calculated using the Riemann Integral and Geometric Series, and compared with the calculation results from MATLAB. Meanwhile, for execution times, the *tic* and *toc* functions available in MATLAB are used. The *tic* function records the current time, and the *toc* function uses the recorded value to calculate the elapsed time.

Table 1. MAPE values and execution times of the  $\ln(x)$  approach using the Riemann integral and the Geometric series

Value s of $n$	Integral Riemann MAPE (%)	Execution times for Integral Riemann (s)	Geometric Series MAPE (%)	Execution times for Geometric Series (s)
$10^1$	1.372686663	0.00089690	7.63575482 1	0.010236700
$10^2$	0.139492509	0.00172900	0.01357085 5	0.004579000
$10^3$	0.013971514	0.01364210	8.70422E- 14	0.027610200
$10^4$	0.001397374	0.16512990	8.70422E- 14	0.228449400
$10^5$	0.000139740	1.21839370	8.70422E- 14	1.528378600
$10^6$	0.000013974	11.55178340	8.70422E- 14	14.073831100

Table 1 shows a comparison of the  $\ln$  values obtained from the development of the Riemann Integral and the Geometric Series in terms of their MAPE, using the actual  $\ln$  values calculated using MATLAB, along with their execution times. It is seen that for the development of the Riemann integral with a variation of  $n$  from 10 to  $10^6$  the MAPE value is less than 10%, so the approximation of the value of  $\ln$  using the Riemann integral is very good. It can also be seen that the greater the value of  $n$  then the MAPE value is getting smaller. The same can be seen that the MAPE value for  $n$  is from 10 to  $10^6$  using the expansion of the geometric series everything is below 10%. The  $\ln$  value approach using geometric series development is very good compared to the Riemann integral development, with  $n$  values of  $n = 10^3$  until  $n = 10^6$  MAPE values show convergence and stable i.e. 8.70422E-14.

From these results, it can be seen that the solution using the geometric series converges faster compared to the solution from the Riemann integral. This can be observed from Equation (9) for the geometric series. For example, if we calculate  $\ln(50)$  using the geometric series with  $x = \frac{49}{51}$  and choose  $n = 10$ , the result is:

$$\ln(50) = 2 \left( \frac{49}{51} + \frac{\left(\frac{49}{51}\right)^3}{3} + \frac{\left(\frac{49}{51}\right)^5}{5} + \dots + \frac{\left(\frac{49}{51}\right)^{19}}{19} \right) = 3.6018226066047.$$





39	3.66356165	3.66323690	0.00886509	3.66356165	0.00000000000000848525224690714
40	3.68887945	3.68855449	0.00881004	3.68887945	0.0000000000000722315622466009
41	3.71357207	3.71324690	0.00875692	3.71357207	0.0000000000000478341824931034
42	3.73766962	3.73734426	0.00870564	3.73766962	0.0000000000000594072316715386
43	3.76120012	3.76087457	0.00865608	3.76120012	0.0000000000000354213439479415
44	3.78418963	3.78386391	0.00860815	3.78418963	0.0000000000000821476926284923
45	3.80666249	3.80633660	0.00856175	3.80666249	0.0000000000000349983123833648
46	3.82864140	3.82831535	0.00851680	3.82864140	0.0000000000000463965322275726
47	3.85014760	3.84982140	0.00847323	3.85014760	0.0000000000000461373698663209
48	3.87120101	3.87087466	0.00843096	3.87120101	0.0000000000000688296797710197
49	3.89182030	3.89149380	0.00838993	3.89182030	0.0000000000000114108354403120
50	3.91202301	3.91169638	0.00835008	3.91202301	0.0000000000000227038138187769
51	3.93182563	3.93149887	0.00831134	3.93182563	0.0000000000000338841991990136
52	3.95124372	3.95091683	0.00827368	3.95124372	0.0000000000000561961298111851
53	3.97029191	3.96996491	0.00823702	3.97029191	0.0000000000001006677336497090
54	3.98898405	3.98865692	0.00820134	3.98898405	0.0000000000001781262415356350
55	4.00733319	4.00700595	0.00816659	4.00733319	0.0000000000000886553105165476
56	4.02535169	4.02502435	0.00813272	4.02535169	0.0000000000001323876999484610
57	4.04305127	4.04272382	0.00809970	4.04305127	0.0000000000001757441814831760
58	4.06044301	4.06011546	0.00806748	4.06044301	0.0000000000003062350054785890
59	4.07753744	4.07720980	0.00803605	4.07753744	0.0000000000001524755817311540
60	4.09434456	4.09401682	0.00800536	4.09434456	0.0000000000001735424864619780
61	4.11087386	4.11054603	0.00797538	4.11087386	0.0000000000001080279338464620
62	4.12713439	4.12680647	0.00794609	4.12713439	0.0000000000001291227767506420
63	4.14313473	4.14280672	0.00791747	4.14313473	0.0000000000002786855899846140
64	4.15888308	4.15855500	0.00788947	4.15888308	0.00000000000000000000000000
65	4.17438727	4.17405910	0.00786209	4.17438727	0.0000000000002553222867764250
66	4.18965474	4.18932650	0.00783529	4.18965474	0.0000000000000211993224833256
67	4.20469262	4.20436430	0.00780906	4.20469262	0.0000000000000633705124320437
68	4.21950771	4.21917931	0.00778337	4.21950771	0.0000000000000420986750947552
69	4.23410650	4.23377804	0.00775821	4.23410650	0.0000000000002097676141910390

70	4.24849524	4.24816671	0.00773356	4.24849524	0.0000000000000836228706022222
71	4.26267988	4.26235128	0.00770939	4.26267988	0.0000000000000833446043634504
72	4.27666612	4.27633745	0.00768570	4.27666612	0.0000000000001869120841993680
73	4.29045944	4.29013071	0.00766247	4.29045944	0.0000000000002484149117966900
74	4.30406509	4.30373630	0.00763968	4.30406509	0.00000000000000000000000000
75	4.31748811	4.31715926	0.00761732	4.31748811	0.0000000000000822865885296112
76	4.33073334	4.33040443	0.00759537	4.33073334	0.0000000000000615261908257830
77	4.34380542	4.34347645	0.00757382	4.34380542	0.0000000000001431290846183370
78	4.35670883	4.35637980	0.00755267	4.35670883	0.0000000000000407729070283084
79	4.36944785	4.36911878	0.00753188	4.36944785	0.000000000000203270172728724
80	4.38202663	4.38169751	0.00751146	4.38202663	0.0000000000001621493420733120
81	4.39444915	4.39411997	0.00749140	4.39444915	0.0000000000003435933060881590
82	4.40671925	4.40639002	0.00747168	4.40671925	0.0000000000002620162258639410
83	4.41884061	4.41851133	0.00745229	4.41884061	0.0000000000000803992267232290
84	4.43081680	4.43048747	0.00743322	4.43081680	0.0000000000000400909565898544
85	4.44265126	4.44232188	0.00741447	4.44265126	0.0000000000000399841611876501
86	4.45434730	4.45401788	0.00739603	4.45434730	0.00000000000003589125519702730
87	4.46590812	4.46557865	0.00737788	4.46590812	0.0000000000001591037515510190
88	4.47733681	4.47700731	0.00736002	4.47733681	0.00000000000003570696647816400
89	4.48863637	4.48830682	0.00734243	4.48863637	0.0000000000002968089904840510
90	4.49980967	4.49948008	0.00732513	4.49980967	0.0000000000000394762660988254
91	4.51085951	4.51052987	0.00730808	4.51085951	0.0000000000002953467355002880
92	4.52178858	4.52145890	0.00729130	4.52178858	0.0000000000001374953479571640
93	4.53259949	4.53226978	0.00727477	4.53259949	0.0000000000001959534304854800
94	4.54329478	4.54296503	0.00725848	4.54329478	0.0000000000001368444980097570
95	4.55387689	4.55354710	0.00724243	4.55387689	0.0000000000000780151454105699
96	4.56434819	4.56401837	0.00722662	4.56434819	0.0000000000001362132921743970
97	4.57471098	4.57438112	0.00721103	4.57471098	0.0000000000001359047373072490
98	4.58496748	4.58463758	0.00719566	4.58496748	0.0000000000002130868465731020
99	4.59511985	4.59478992	0.00718051	4.59511985	0.0000000000000966436620444309
100	4.60517019	4.60484022	0.00716557	4.60517019	0.0000000000002892982399659870

<b>Total</b>	<b>0.92226818</b>		<b>0.0000000000096192321420575400</b>
<b>MAPE</b>	<b>0.00931584</b>		<b>0.000000000000971639610308843</b>

Table 2 shows the results for  $\ln(2)$  to  $\ln(100)$  using MATLAB, the Riemann integral, and the geometric series for  $n = 1500$ , along with the MAPE values. The MAPE value for the Riemann integral is 0.00931584, while the MAPE value for the geometric series is 0.000000000000971639610308843. The MAPE value for the Riemann integral is larger compared to that of the geometric series. Next, the comparison of logarithmic calculations  $\ln(x)$  using MATLAB, the Riemann integral, and geometric series methods is shown in Figure 3.

In an effort to reduce errors in the Riemann integral method, various numerical quadrature methods can be used to improve the accuracy of calculations. For a single partition  $[x_0, x_1]$ , the integral can be computed using both the Riemann integral method and the Trapezoidal method. In the Riemann integral method, the integral result is calculated using the inner polygon with the formula  $hf_0$ , where  $h = dx$ . On the other hand, the outer polygon uses the formula  $hf_1$ . In contrast, the Trapezoidal method uses the formula  $\left(\frac{h}{2}\right)(f_0 + f_1)$ , which provides a better approximation by utilizing the average function value at both endpoints.

When the interval is divided into two partitions,  $[x_0, x_1]$  and  $[x_1, x_2]$ , Simpson's 1/3 rule can be applied. The formula for this is  $\left(\frac{h}{3}\right)(f_0 + 4f_1 + f_2)$ . For three partitions  $[x_0, x_1]$ ,  $[x_1, x_2]$ , and  $[x_2, x_3]$ , Simpson's 3/8 rule is used, with the formula  $\left(\frac{3h}{8}\right)(f_0 + 3f_1 + 3f_2 + f_3)$ . These formulas significantly improve the accuracy of the integral compared to the simple Riemann integral method.

In general, the relationship between the number of partitions  $n$ , the weights  $a_i$ , and the normalization scale  $m$  in quadrature methods can be expressed as:

$$h \frac{\sum_{i=0}^n a_i}{m} = nh \tag{11}$$

For example, in Simpson's 1/3 rule,  $n = 2$  with weights  $a_0 = 1, a_1 = 4, a_2 = 1$ , and  $m = 3$ . Using Equation (11), the result is:

$$\frac{h(1 + 4 + 1)}{3} = 2h$$

Meanwhile, for Simpson's 3/8 rule, with  $n = 3$ , weights  $a_0 = 1, a_1 = 3, a_2 = 3, a_3 = 1$ , and  $m = 8/3$ , the result is:

$$\frac{h(1 + 3 + 3 + 1)}{\frac{8}{3}} = 3h$$

This approach allows the development of new combinations of formulas to solve numerical integrals. For instance, for three partitions, a combination like  $\left(\frac{3h}{7}\right) (f_0 + 2f_1 + 3f_2 + f_3)$  can be used, with weights  $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 1, n = 3,$  and  $m = 7/3.$  The result from this formula is:

$$\frac{h(1 + 2 + 3 + 1)}{\frac{7}{3}} = 3h.$$

This approach can also be extended to four partitions, such as using the formula  $\left(\frac{4h}{9}\right) (f_0 + 2f_1 + 3f_2 + 2f_3 + f_4),$  which yields a result of  $4h.$

Table 3. The MAPE values of the Riemann Integral method and other quadrature methods

Method	MAPE
Trapezium $\left(\frac{h}{2}\right) (f_0 + f_1)$	2.198857028134900e-06
Simpson 1/3, $\left(\frac{h}{3}\right) (f_0 + 4f_1 + f_2)$	3.672766177836382e-13
$\left(\frac{h}{3}\right) (2f_0 + 2f_1 + 2f_2)$	2.198857028134900e-06
Simpson 3/8, $\left(\frac{3h}{8}\right) (f_0 + 3f_1 + 3f_2 + f_3)$	1.056721818837513e-12
$\left(\frac{3h}{7}\right) (f_0 + 2f_1 + 3f_2 + f_3)$	0.001330403697326
$\left(\frac{4h}{9}\right) (f_0 + 2f_1 + 3f_2 + 2f_3 + f_4)$	1.277715750104592e-12

From Table 3, it can be seen that the MAPE values of the other quadrature methods are smaller than the MAPE value of the Riemann integral. The development of these methods demonstrates the flexibility of numerical integration in enhancing calculation accuracy. By determining appropriate weights and normalization scales, various combinations of formulas can be designed to optimize the integral results, especially in the computation of complex functions with larger partitions.

### 7. Conclusions

In this study, we have successfully derived the function  $\ln(m) = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{mn}\right), m = 2, 3, 4 \dots$  which was obtained recursively and proven using mathematical induction through the Riemann integral approach. Based on the MAPE analysis, the geometric series expansion yields a more accurate solution compared to the Riemann integral

approximation for computing the natural logarithm ( $\ln$ ) function. The geometric series solution demonstrates higher precision, as it avoids extensive summation of terms. Conversely, the Riemann integral approximation requires summation over a greater number of terms, yet the computational process remains more straightforward compared to the geometric series. From the perspective of computational efficiency, the Riemann integral method performs faster than the geometric series expansion.

Various numerical quadrature methods can be used to improve the accuracy of Riemann integral calculations. The results show that the MAPE values of other quadrature methods are smaller than those of the Riemann integral. However, the Riemann integral remains essential as a foundation for developing other quadrature methods. The development of these methods demonstrates the flexibility of numerical integration in enhancing calculation accuracy.

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