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ON A PROBLEM OF MULTIDIMENSIONAL CROSS-DIFFUSION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract

Recently, the cross-diffusion problem has gained importance in solving problems of mathematical modelling of complex processes such as population dynamics, diffusion in multiphase media and reaction-diffusion systems and therefore has attracted considerable attention. Studies have shown that cross-diffusion elements in models can significantly change the qualitative and quantitative properties of solutions. However, many aspects of solutions, especially nonlinear boundary value problems, have not been sufficiently studied, which requires further and more in-depth study of these issues and creates the need for a more in-depth theoretical analysis. Based on the above considerations, the objective of this study is to formulate and analyse the scientific problem associated with the dynamic behaviour of the cross-diffusion problem based on the conditions of existence and non-existence of a global solution and the influence of boundary and parametric conditions on solutions based on self-similar analysis. The research methodology is based on the use of a self-similar approach, which allows simplifying the system of differential equations by introducing variables that are invariant with respect to changes in measurements. The cross-diffusion problem with nonlinear boundary conditions is considered, resulting in a qualitative analysis based on the conditions for the existence and non-existence of solutions. In addition, the methods of the theory of

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ordinary and partial differential equations, analytical and numerical methods are used to study the behaviour of solutions over large time intervals. The analysis shows that, depending on the choice of system parameters and boundary conditions, both the existence of global solutions and their non-existence are possible. The conditions under which the solution of the system retains its regularity over a finite time interval are established, as well as the one under which the emergence or failure of the solution in a finite time is observed. In addition, the self-similar approach allows us to determine the key parameters responsible for the system's critical behavior. The main conclusion of the work is that the use of self-similar variables not only significantly simplifies the study of complex cross-diffusion models but also allows us to obtain important information about the stability limits of solutions. The results obtained can be useful for research related to nonlinear diffusion problems, as well as for developing more accurate mathematical models in practical problems.

Keywords cross-diffusion, self-similar analysis nonlinear parabolic system, diffusion, critical exponents of the Fujita type, blow-up, existence, non-existence

1. Introduction

Cross-diffusion is a phenomenon in which the components of a mixture interact with each other through interdependent spatial distributions. The interaction is especially important in multiphase systems such as chemical solutions, biological cell populations, or ecosystem processes. Many areas of the natural sciences, including biology, ecology, chemistry, and physics, use cross-diffusion models. These models better describe complex interactions in multicomponent systems, where the existence and behavior of one component affect the mobility of another. A brief description of cross-diffusion models and their use in several areas of natural science:

- Interactions of molecules whose ions can influence each other through concentration gradients, creating new chemical patterns, are described by cross-diffusion models in chemistry and reaction physics. The Schneckenberg-Turing model is popular and uses cross-diffusion to explain periodic patterns in reaction-diffusion systems. This classification is for reactions in which one reactant causes the other to move, forming self-sustaining patterns. It shows how cross-diffusion can create processes that keep repeating themselves and patterns related to chemical waves and Belousov-Zhabotinsky reactions.;

- In biology and ecology, cross-diffusion models are used to model populations in the presence of other species and how cells migrate in response to chemical signals. For example, the Keller-Segel model of chemotaxis describes how bacteria and other cells move to areas where chemical attractants produced by other cells are present in high concentrations. This model explains the formation and maintenance of aggregation patterns in biological populations, from bacterial colonies to animal tissues. Ecology uses cross-diffusion models to explain the interaction of species in spatial ecosystems Cross-diffusion models of competition and cooperation can explain spatial patterns like stripes or patches in plant or animal communities. Cross-diffusion models provide a powerful tool for describing and predicting the

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behavior of complex systems where components are interdependent.

2. Statement of the problem

This research looks at the features of solutions to a complex system where different dimensions interact and are affected by nonlinear boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla\left(\upsilon^{m_1-1}\nabla u\right), \\ \dfrac{\partial \upsilon}{\partial t} = \nabla\left(u^{m_2-1}\nabla \upsilon\right), \quad x \in R_+^N, \ t > 0, \end{cases} \tag{1}$$

$$\begin{cases} -\upsilon^{m_1-1}\dfrac{\partial u}{\partial x_1}(0,t) = u^{q_1}(0,t), \quad x_1 = 0, \, t > 0, \\ -u^{m_2-1}\dfrac{\partial \upsilon}{\partial x_1}(0,t) = \upsilon^{q_2}(0,t), \quad x_1 = 0, \, t > 0, \end{cases} \tag{2}$$

$$u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad x \in R_+^N, \tag{3}$$

where $R_+^N = \left\{(x_1, x') \mid x' \in R^{N-1}, \, x_1 > 0\right\}$, $m_i > 1$, $q_i > 0 \, (i = 1,2)$, $u_0(x)$ and $\upsilon_0(x)$ -non-negative continuous functions with compact support in R_+^N .

Various fields of natural science encounter cross-diffusion models. The work [1-15] is devoted to the study of nonlinear equations, including an important class of parabolic equations describing reaction-diffusion processes and widely used in nature. These equations serve as mathematical models for physical problems in various fields, such as phase transitions, filtration, biochemistry, and ecology. Often, the equations are degenerate or singular. The existence of degeneracy or singularity complicates and hinders the study. Many new ideas and methods have come up that improve the theory of partial differential equations to tackle the specific problems caused by degeneracy and singularity.

The requirements for the global existence of solutions and the conditions resulting in a blow-up regime have been thoroughly examined recently (see [4-15]). A calculation for the solution near the blow-up time of a nonlocal diffusion problem was made, and the conditions for whether a solution can exist globally or not over time were studied in [8, 9].

$$u_t = u_{xx}, \quad \upsilon_t = \upsilon_{xx}, \quad x > 0, \ 0 < T < 0, \tag{4}$$

$$-u_x(0,t) = u^{\alpha}\upsilon^{p}, \quad -\upsilon_x(0,t) = u^{q}\upsilon^{\beta}, \quad 0 < t < T, \tag{5}$$

$$u(x,0) = u_0(x), \, \upsilon(x,0) = \upsilon_0(x), \, x > 0. \tag{6}$$

It has been established that any solution to issues (4)-(6) is global if $pq \le (1-\alpha)(1-\beta)$.

In [13], they studied properties of solutions of the cross-diffusion system with nonlinear boundary conditions

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$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\upsilon^{m_1-1}\frac{\partial u}{\partial x}\right), \\ \frac{\partial \upsilon}{\partial t} = \frac{\partial}{\partial x}\left(u^{m_2-1}\frac{\partial \upsilon}{\partial x}\right), \quad x \in R_+, \ t > 0, \end{cases} \tag{7}$$

$$\begin{cases} -\upsilon^{m_1-1}\frac{\partial u}{\partial x}(0,t) = u^{q_1}(0,t), \\ -u^{m_2-1}\frac{\partial \upsilon}{\partial x}(0,t) = \upsilon^{q_2}(0,t), \ t > 0, \end{cases} \tag{8}$$

$$u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad x \in R_+, \tag{9}$$

It is proved that under $q_1 \le 1, q_2 \le 1$ conditions the solution of problem (7)-(9) is global in time and under, $q_1 > 1$, $q_2 > 1$, conditions then the solution of problem (7)-(9) is unbounded given sufficiently large initial data and under, $q_1 > m_2 + 1$, $q_2 > m_1 + 1$ conditions and the initial data is sufficiently small, then every solution of problem (7)-(9) is global.

In the paper [14] studied the qualitative properties of solutions of the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\upsilon^{m_1-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right), \\ \frac{\partial \upsilon}{\partial t} = \frac{\partial}{\partial x}\left(u^{m_2-1}\left|\frac{\partial \upsilon}{\partial x}\right|^{p-2}\frac{\partial \upsilon}{\partial x}\right), \quad x \in R_+, \ t > 0, \end{cases} \tag{10}$$

$$\begin{cases} -\upsilon^{m_1-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}(0,t) = u^{q_1}(0,t), \\ -u^{m_2-1}\left|\frac{\partial \upsilon}{\partial x}\right|^{p-2}\frac{\partial \upsilon}{\partial x}(0,t) = \upsilon^{q_2}(0,t), \ t > 0, \end{cases} \tag{11}$$

$$u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad x \in R_+, \tag{12}$$

It is shown that, if $\min\{l_1,l_2\} > 0$, where $l_1 = p(q_1-1)(q_2-1)-(p-2)(q_1-1)-(m_2-1)(q_2-1)$, $l_2 = p(q_1-1)(q_2-1)-(p-2)(q_2-1)-(m_1-1)(q_1-1)$ the solution of problem (10)-(12) is unbounded with sufficiently large initial data, and if $\max\{\alpha_1-\beta,\alpha_2-\beta\} < 0$ and the initial data are sufficiently small, the solution of problem (10)-(12) is global, and If $q_1 \le 1, q_2 \le 1$, the solution of problem (10)-(12) is global.

3. Method of solution

The objective of this study is to determine the conditions for the existence and non-existence of solutions to problems (1)-(3) over time through self-similar analysis. Various self-similar solutions to problems (1)-(3) are developed, estimates and critical exponents of Fujita type, as well as critical exponents for the global existence of a solution, are established.

Theorem 1. If $q_1 \le 1, q_2 \le 1$, are satisfied, then any solution to problems (1)-(3) is global.

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Proof. We demonstrate sufficient conditions for the general time solution of issues (1)–(3) by using bounded upper solutions. We seek constrained upper solutions of the problem in the subsequent self-similar format

$$\begin{cases} \breve{u}(x,t)=e^{L_1 t}\left(K+e^{-M_1\xi_1}\right), \ \xi_1 = x_1 e^{-J_1 t}, \ \ x_i=0, \, i=\overline{2,N}, \, t\ge 0, \\ \breve{\upsilon}(x,t)=e^{L_2 t}\left(K+e^{-M_2\xi_2}\right), \ \xi_2 = x_1 e^{-J_2 t}, \ \ x_i=0, \, i=\overline{2,N}, \, t\ge 0, \end{cases} \tag{13}$$

Where $K\ge \max\left\{\|u_0\|_\infty, \|\upsilon_0\|_\infty\right\}$, $M_1=(K+1)^{\frac{q_1}{m_1-1}}$, $M_2=(K+1)^{\frac{q_2}{m_2-1}}$, $L_1=\dfrac{M_1(K+1)^{m_1-2}\left(M_1K+M_2(m_1-1)\right)}{K+1}$,

$L_2=\dfrac{M_2(K+1)^{m_2-2}\left(M_2K+M_1(m_2-1)\right)}{K+1}$, $\quad J_1=L_2(m_1-1)$, $J_2=L_1(m_2-1)$.

We shall show that the produced functions (11) are higher solutions to problem (1)-(3). The concept of solution comparison requires the fulfillment of the following system of inequalities [1, 4].

$$\begin{cases} \dfrac{\partial \breve{u}}{\partial t}\ge \nabla\left(\breve{\upsilon}^{m_1-1}\nabla \breve{u}\right), \\ \dfrac{\partial \breve{\upsilon}}{\partial t}\ge \nabla\left(\breve{u}^{m_2-1}\nabla \breve{\upsilon}\right), \ \ x\in R_+^N, \ t>0, \end{cases} \tag{14}$$

$$\begin{cases} -\upsilon^{m_1-1}\dfrac{\partial u}{\partial x_1}(0,t)\ge u^{q_1}(0,t), \ \ x_1=0, \, t>0, \\ -u^{m_2-1}\dfrac{\partial \upsilon}{\partial x_1}(0,t)\ge \upsilon^{q_2}(0,t), \ \ x_1=0, \, t>0, \end{cases} \tag{15}$$

After the following calculations

$$\breve{u}_t=L_1e^{L_1t}\left(K+e^{-M_1\xi_1}\right)+M_1J_1xe^{(L_1-J_1)t}\ge L_1e^{L_1t}(K+1), \quad \breve{\upsilon}^{m_1-1}\breve{u}_x=M_1e^{\left(L_2(m_1-1)+L_1-J_1\right)t}\left(K+e^{-M_2\xi_2}\right)^{m_1-1}e^{-M_1\xi_1}$$

$$\left(\breve{\upsilon}^{m_1-1}\breve{u}_x\right)_x=M_1e^{\left(L_2(m_1-1)+L_1-J_1\right)t}\left(K+e^{-M_2\xi_2}\right)^{m_1-2}\left(M_1e^{-(M_1\xi_1+J_1t)}\left(K+e^{-M_2\xi_2}\right)+(m_1-1)M_2e^{-J_2t}\right)\le$$
$$M_1e^{\left(L_2(m_1-1)+L_1-J_1\right)t}(K+1)^{m_1-2}\left(M_1(K+1)+(m_1-1)M_2\right)$$

$$\breve{\upsilon}_t=L_2e^{L_2t}\left(K+e^{-M_2\xi_2}\right)+M_2J_2xe^{(L_2-J_2)t}\ge L_2e^{L_2t}(K+1), \quad \breve{u}^{m_2-1}\breve{\upsilon}_x=M_2e^{\left(L_1(m_2-1)+L_2-J_2\right)t}\left(K+e^{-M_1\xi_1}\right)^{m_2-1}e^{-M_2\xi_2}$$

$$\left(\breve{u}^{m_2-1}\breve{\upsilon}_x\right)_x=M_2e^{\left(L_1(m_2-1)+L_2-J_2\right)t}\left(K+e^{-M_1\xi_1}\right)^{m_2-2}\left(M_2e^{-(M_2\xi_2+J_2t)}\left(K+e^{-M_1\xi_1}\right)+(m_2-1)M_1e^{-J_1t}\right)\le$$
$$M_2e^{\left(L_1(m_2-1)+L_2-J_2\right)t}(K+1)^{m_2-2}\left(M_2(K+1)+(m_2-1)M_1\right)$$

it is clear that the systems of inequalities (14) and (15) will always be valid under the conditions $q_1\le 1, q_2\le 1$ and definition $M_i, J_i, L_i \ (i=1,2), K$.

Theorem 2. Let $q_1>\dfrac{m_2+1}{2}, q_2>\dfrac{m_1+1}{2}$, then any solution to problem (1)-(3) is unbounded for sufficiently large initial data.

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Proof. In proving Theorem 2, we seek a solution to problem (1)-(3) in the following self-similar form:

$$\begin{cases} \underline{u}(x,t) = (T-t)^{-\alpha_1}\varphi(\xi), \\ \underline{\upsilon}(x,t) = (T-t)^{-\alpha_2}\phi(\xi), \end{cases} \tag{16}$$

where $T>0$, $\xi = |\zeta|$, $\zeta_1 = (x_1+h)(T-t)^{-\beta_1}$, $\zeta_i = x_i(T-t)^{-\beta}$, $(i = 2,\ldots,N)$, $0<h<a$, $T>0$, $\beta = \frac{q_1-m_2}{2q_1-m_2-1} = \frac{q_2-m_1}{2q_2-m_1-1}$, $\alpha_1 = \frac{1}{2q_1-m_2-1}$, $\alpha_2 = \frac{1}{2q_2-m_1-1}$, and the functions $(\varphi(\xi),\phi(\xi))$ solution to the following problems

$$\begin{cases} \xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\phi^{m_1-1}\dfrac{d\varphi}{d\xi}\right) - \beta\xi\dfrac{d\varphi}{d\xi} - \alpha_1\varphi = 0, \\ \xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\varphi^{m_2-1}\dfrac{d\phi}{d\xi}\right) - \beta\xi\dfrac{d\phi}{d\xi} - \alpha_2\phi = 0, \end{cases} \tag{17}$$

$$\begin{cases} -\phi^{m_1-1}\dfrac{d\varphi}{d\xi}(h) = \varphi^{q_1}(h), \\ -\varphi^{m_2-1}\dfrac{d\phi}{d\xi}(h) = \phi^{q_2}(h), \end{cases} \tag{18}$$

which arise from substituting (16) into (1)-(3) and performing specific simplifications. Let us define the conditions under which (16) constitutes an unbounded lower solution to problem (1)-(3). We designate the following functions for comparison:

$$\begin{cases} \bar{\varphi}(\xi) = A_1(a-\xi)^{\frac{1}{m_2-1}}, \\ \bar{\phi}(\xi) = A_2(a-\xi)^{\frac{1}{m_1-1}}, \end{cases} \tag{19}$$

where $A_i > 0\ (i=1,2)$. Then, to use the comparison theorem, it is necessary to fulfill

$$\begin{cases} \xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\bar{\phi}^{m_1-1}\dfrac{d\bar{\varphi}}{d\xi}\right) - \beta\xi\dfrac{d\bar{\varphi}}{d\xi} - \alpha_1\bar{\varphi} \ge 0, \\ \xi^{1-N}\dfrac{d}{d\xi}\left(\xi^{N-1}\bar{\varphi}^{m_2-1}\dfrac{d\bar{\phi}}{d\xi}\right) - \beta\xi\dfrac{d\bar{\phi}}{d\xi} - \alpha_2\bar{\phi} \ge 0, \end{cases}$$

$$\begin{cases} -\phi^{m_1-1}\dfrac{d\varphi}{d\xi}(h) \le \varphi^{q_1}(h), \\ -\varphi^{m_2-1}\dfrac{d\phi}{d\xi}(h) \le \phi^{q_2}(h), \end{cases}$$

this corresponds to the realization of the subsequent inequalities

$$\begin{cases} F_1(\xi) = \mu_1\xi^2 + \mu_2\xi - \mu_3 \ge 0 \\ F_2(\xi) = \lambda_1\xi^2 + \lambda_2\xi - \mu_3 \ge 0 \end{cases} \tag{20}$$

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$$\begin{cases} \dfrac{A_1 A_2^{m_1-1}}{m_2-1}(a-h)^{\frac{1}{m_2-1}} \le A_1^{q_1}(a-h)^{\frac{q_1}{m_2-1}} \\ \dfrac{A_2 A_1^{m_2-1}}{m_1-1}(a-h)^{\frac{1}{m_1-1}} \le A_2^{q_2}(a-h)^{\frac{q_2}{m_1-1}} \end{cases} \tag{21}$$

where

$$\mu_1 = \frac{1}{A_2^{m_1-1}}\left(\beta+\alpha_1(m_2-1)\right),\ \lambda_1 = \frac{1}{A_1^{m_2-1}}\left(\beta+\alpha_2(m_1-1)\right),\ \mu_2 = (N-1)+\frac{1}{m_2-1}-\frac{m_2-1}{A_2^{m_1-1}}\alpha_1 a,$$

$$\lambda_2 = (N-1)+\frac{1}{m_1-1}-\frac{m_1-1}{A_1^{m_2-1}}\alpha_2 a,\ \ \mu_3 = (N-1)a.$$

Since the functions $\left(F_1(\xi),F_2(\xi)\right)$ are quadratic, the validity of (20) in $\xi\in(h,+\infty)$ follow from the following conditions:

(i) If $\min(\mu_1,\lambda_1)>0$, then from (20) it follows that $q_1>\dfrac{m_2+1}{2},\ q_2>\dfrac{m_1+1}{2}$;

(ii) If $\mu_2\ge 0$ and $\lambda_2\ge 0$, then from (20) it follows that $(N-1)+\dfrac{1}{m_2-1}\ge\dfrac{m_2-1}{A_2^{m_1-1}}\alpha_1 a$,

$(N-1)+\dfrac{1}{m_1-1}\ge\dfrac{m_1-1}{A_1^{m_2-1}}\alpha_2 a$,

$q_1>\dfrac{m_2+1}{2},\ q_2>\dfrac{m_1+1}{2}$;

(iii) Because $\left(F_1(\xi),F_2(\xi)\right)$ are increasing, to perform $F_1(\xi)\ge 0,\ F_2(\xi)\ge 0$ it is enough to implement them $\xi=h$. Then, from (20) we have

$$\begin{cases} \mu_1 h^2+\mu_2 h-\mu_3\ge 0, \\ \lambda_1 h^2+\lambda_2 h-\mu_3\ge 0. \end{cases}$$

the following restrictions follow from (21):

$$a-h\ge\max\left\{\begin{matrix} A_1^{m_2-1}A_2^{\frac{(m_1-1)(m_2-1)}{q_1-1}}(m_2-1)^{-\frac{m_2-1}{q_1-1}}, \\ A_2^{m_1-1}A_1^{\frac{(m_1-1)(m_2-1)}{q_2-1}}(m_1-1)^{-\frac{m_1-1}{q_2-1}} \end{matrix}\right\} \tag{22}$$

it is easy to check that for any

$$q_1>\frac{m_2+1}{2},\ q_2>\frac{m_1+1}{2}.$$

there will be permanent ones h A_i, a_i, $(i=1,2)$, satisfying inequalities (i)-(iii), (22). Drawing on the concept of the comparison theorem for initial data solutions, we possess

$$\begin{cases} u_0(x)\ge T^{-\alpha_1}\bar{\phi}(\xi), \\ \upsilon_0(x)\ge T^{-\alpha_2}\bar{\phi}(\xi). \end{cases}$$

Thus, the solution to problem (1)-(3) is unlimited.

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$$\begin{cases} u(x,t) \ge (T-t)^{-\alpha_1} \bar{f}(0) \to \infty, \ t \to \infty, \\ \upsilon(x,t) \ge (T-t)^{-\alpha_2} \bar{g}(0) \to \infty, \ t \to \infty. \end{cases}$$

at $q_1 > \frac{m_2+1}{2}$, $q_2 > \frac{m_1+1}{2}$. The proof of the theorem is complete.

Theorem 3. Let $q_1 > m_2 + \frac{1}{N}$, $q_2 > m_1 + \frac{1}{N}$ and the initial data be small enough, then any solution to problem (1)-(3) is global.

Proof. Developing bounded upper solutions will facilitate the establishment of conditions for the total time solvability of the problem (1)-(3). We seek them in the subsequent self-similar configuration:

$$\begin{cases} u_+(x,t) = (T+t)^{-\alpha_1} f(\xi), \\ \upsilon_+(x,t) = (T+t)^{-\alpha_2} g(\xi), \end{cases} \tag{23}$$

where $T > 0$, $\xi = |\zeta|$, $\zeta_1 = (x_1 + h)(T+t)^{-\beta_1}$, $\zeta_i = x_i (T+t)^{-\beta}$, $(i = 2,\ldots,N)$, $0 < h < a$, $\beta = \frac{q_1 - m_2}{2q_1 - m_2 - 1} = \frac{q_2 - m_1}{2q_2 - m_1 - 1}$, $\alpha_1 = \frac{1}{2q_1 - m_2 - 1}$, $\alpha_2 = \frac{1}{2q_2 - m_1 - 1}$, and the functions $(f(\xi), g(\xi))$ taking into account the comparison theorem, the solutions must satisfy the systems of inequalities

$$\begin{cases} \xi^{1-N} \frac{d}{d\xi}\left(\xi^{N-1} g^{m_1-1} \frac{df}{d\xi}\right) + \beta\xi \frac{df}{d\xi} + \alpha_1 f \le 0, \\ \xi^{1-N} \frac{d}{d\xi}\left(\xi^{N-1} f^{m_2-1} \frac{dg}{d\xi}\right) + \beta\xi \frac{dg}{d\xi} + \alpha_2 g \le 0, \end{cases} \tag{24}$$

$$\begin{cases} -g^{m_1-1} \left.\frac{df}{d\xi}\right|_{\xi=h} \ge f^{q_1}(\xi), \\ -f^{m_2-1} \left.\frac{dg}{d\xi}\right|_{\xi=h} \ge g^{q_2}(\xi), \end{cases} \tag{25}$$

let's consider the following functions

$$\begin{cases} \bar{f}(\xi) = A_1 \left(a - \xi^2\right)^{\frac{1}{m_2-1}}, \\ \bar{g}(\xi) = A_2 \left(a - \xi^2\right)^{\frac{1}{m_1-1}}, \end{cases} \tag{26}$$

where $a > 0$, $A_1 = \left(\frac{\beta(m_1-1)}{2}\right)^{\frac{1}{m_2-1}}$, $A_2 = \left(\frac{\beta(m_2-1)}{2}\right)^{\frac{1}{m_1-1}}$. Let us show that the systems of inequalities (24) and (25) are solvable with respect to the unknowns a, h at $q_1 > m_2 + \frac{1}{N}$, $q_2 > m_1 + \frac{1}{N}$. Then, comparing functions (26) in (24) and (25) we obtain

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$$\begin{cases} (\alpha_1 - N\beta)(a - \xi^2) \le 0, \\ (\alpha_2 - N\beta)(a - \xi^2) \le 0, \end{cases}$$

where does the need for restrictions come from $q_1 > m_2 + \frac{1}{N}$, $q_2 > m_1 + \frac{1}{N}$ and the following conditions for numerical parameters $a, h.$

$$a \le \min \begin{Bmatrix} h^2 + \left(A_2^{m_1 - 1} A_1^{1 - q_1} \dfrac{2h}{m_2 - 1} \right)^{\frac{1}{q_1 + m_2 - 2}}, \\ h^2 + \left(A_1^{m_2 - 1} A_2^{1 - q_2} \dfrac{2h}{m_1 - 1} \right)^{\frac{1}{q_2 + m_1 - 2}} \end{Bmatrix} \tag{27}$$

So, if $q_1 > m_2 + \frac{1}{N}$, $q_2 > m_1 + \frac{1}{N}$ and initial functions $u_0(x), \upsilon_0(x)$ satisfies

$$\begin{cases} u_0(x) \le T^{-\alpha_1} \bar{f}(\xi), \\ \upsilon_0(x) \le T^{-\alpha_2} \bar{g}(\xi), \end{cases}$$

where a, h are selected from condition (27), then the solution to problem (1)-(3) is global.

Note 1: Theorem 1 demonstrates that the crucial exponents of the globally existing solution are $q_{10} = \frac{m_2 + 1}{2}$ and $q_{20} = \frac{m_1 + 1}{2}$.

Notes 2. Theorem 3 demonstrates that the crucial exponents of Fujita type are $q_{1c} = m_2 + \frac{1}{N}$ and $q_{2c} = m_1 + \frac{1}{N}$.

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