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ANALYTICAL ANALYSIS OF ONE NONLINEAR DIFFUSION PROBLEM WITH A SOURCE SPECIFIED BY NONLINEAR BOUNDARY CONDITIONS IN MULTIDIMENSIONAL SPACE

Rakhmonov Z.R.[1], Yarmetova D.I.[2] Mamatkulova M.Sh [3]

[1]National University of Uzbekistan after named Mirzo Ulugbek

100174 Tashkent, UZBEKISTAN

e-mail: zraxmonov@inbox.ru

[2]National University of Uzbekistan after named Mirzo Ulugbek

100174 Tashkent, UZBEKISTAN

e-mail: saidovadilafruz1996@gmail.com

[3]National University of Uzbekistan after named Mirzo Ulugbek

100174 Tashkent, UZBEKISTAN

e-mail: mamatqulova_m@mail.ru

Abstract

This paper studies the properties of one nonlinear diffusion problem with a source specified by nonlinear boundary conditions. The main results are global existence of the solution; regularity and positivity of solutions; asymptotic behaviour of solutions as time goes to infinity; comparison principles; and maximum principles for solutions. The proofs are based on the energy method, comparison methods, and asymptotic techniques. The conditions for the global existence of the solution in time and the unsolvability of the solution of the diffusion problem in a homogeneous medium are founded based on comparison principle and self-similar analysis. We obtain the critical exponent of the Fujita type and the critical global existence exponent.

AMS Subject Classification: 35A01, 35B44, 35K57, 35K65

Key words: global solution, blow-up, nonlinear boundary condition, critical exponents, nonlinear diffusion system, energy method, life-span time

Introduction

We study a nonlinear diffusion problem with a source specified by nonlinear boundary conditions in $Q = \{(x,\ t) \mid\ x \in R_+^N,\ t > 0\}$

$$u_t = \nabla\left(\left|\nabla u^m\right|^{p-2} \nabla u^m\right) + u^\beta, \qquad x \in R_+^N,\ t > 0, \tag{1}$$

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with nonlinear boundary flow and initial condition

$$-\left|\nabla u^m\right|^{p-2}\frac{\partial u^m}{\partial x_1}=u^q,\quad x_1=0,\ t>0, \tag{2}$$

$$u(x,0)=u_0(x), \tag{3}$$

where $R_+^N=\left\{(x_1,x')\,|\,x'\in R^{N-1},\ x_1>0\right\}$ $m>1,$ $p>1+1/m,$ $q>0$.- are given numerical parameters and $u_0(x)$ is a non-negative bounded and continuous function in R_N . Here we mean by a solution, a function u(x, t) is nonnegative and continuous in $Q\setminus(0,0)$, satisfying (1)-(3) in the distribution sense. In particular, the problem (1)-(3) can be tought of as a model to describe heat propagation with a gradient-dependent thermal conductivity in a medium with chemical reaction and a nonlinear radiation law at the boundary (see[1]). The nonlinear boundary condition (2) appears also in combustion problems when the reaction happens only at the boundary of the container, for example because of the presence of a solid catalyzer, see [2] for justification.

In recent years, in Uzbekistan, M.M. Aripov [3], A.S. Matyakubov [4], A. Rakhmonov [5], and others have studied the solution properties of various problems expressed by such nonlinear parabolic-type equations.

Definition 1. (L^∞ blow-up). We say that a solution u blows up (or thermal runaway) at $t = T$, if there exists $(x_n,\ t_n),\ t_n$ □ T , such that $\left|u(x_n,\ t_n)\right| \to +\infty$. In this case, we say that the solution blows up in finite time if T is finite; if there exists a sequence $\{y_n\}$ and t_n □ T such that $y_n \to x$ and $\left|u(y_n,\ t_n)\right| \to +\infty$,then we say that x is a blow-up point.

The collection of all blow up points is called the blow-up set. For the finite time blow-up, Osgood [6] gave a criterion, namely the right-hand side nonlinear term must satisfy

$$\int^\infty \frac{ds}{f(s)}<\infty$$

The earliest blow-up results on parabolic equations are due to Kaplan [7] and Fujita [8]. Some early papers appeared in the 1970s: Tsutsumi [9], Hayakawa [10], Levine [11], Levine–Payne [12-13], Walter [14].

Definition 2. A non-negative function $u(x,t)$ defined in Q is called a weak solution of Neumann problem (1)-(3), if for every bounded open set Ω with smooth boundary

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$$\partial\Omega,\ u \in L^{\infty}_{loc}\ \left(\Omega \times (0,\ T)\right) \cap C\ \left((0,\ T),\ L^2_{loc}(\Omega)\ \right) \cap L^{\beta}_{loc}\ \left(\Omega \times (0,\ T)\right),$$

$u^{\ m}\ \left|\nabla u^m\right|^{p-2} \in L^{\ 1}_{\ loc}\ \left(\Omega \times (0,\ T)\right)$ and the following integral identity fulfils

$$\int_\Omega u\ (x,\ t)\ \eta\ (x,\ t)\ dx\ \ - \int_\Omega u\ (x,\ t_0)\ \eta\ (x,\ t_0)\ dx\ = \int_0^t\int_\Omega u^\beta\ \eta dxd\tau + \int_0^t\int_\Omega u \cdot \partial\eta(x,\ \tau\ )\ \partial\tau - u^{\ m}\left|\nabla u^{\ m}\right|^{p-2} \nabla u \cdot \nabla\ \left(u^{\ q}\ \eta\right) dxd\tau$$

for all $0 \le t \le T$ and for any test function $\eta \in C^1_0\left(\Omega \times (0,\ T)\right)$. Moreover,

$$\lim_{t\to 0} \int_\Omega u\ (x,\ t)\ \zeta\ (x)\ dx\ \ = \int_\Omega u\ (x,\ 0)\ \zeta\ (x)\ dx \tag{4}$$

for any $\zeta\ (x) \in C^1_0(\Omega)$) (see [15], [19]).

Let us denote

$$T^* = \sup\left\{T\ >\ 0;\ sup_{t\in[0,T)} \left\|u(x,t)\right\|_\infty < \infty\right\},$$

then T^* is called "the life span" of the solution $u\ (x,\ t)$.If $T^* = \infty$,then the solution $u\ (x,\ t)$ is global in time mean. On the other side, $T^* < \infty$ the solution $u\ (x,\ t)$ is called "blow-up" in finite time T^*.

In [5] Z.R.Rakhmonov and A.I.Alimov studied the problem

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u_i^k}{\partial x}\right|^{m-1} \frac{\partial u_i^k}{\partial x}\right) + u_i^{p_i},\ x \in R_+,\ t\ >\ 0,\ i\ =\ 1,2 \tag{5}$$

coupled through nonlinear boundary flux

$$\left.\left|\frac{\partial u_i^k}{\partial x}\right|^{m-1} \frac{\partial u_i^k}{\partial x}\right|_{x=0} = u_{3-i}^{q_i}(0,t),\ t\ >\ 0,\ i\ =\ 1,2 \tag{6}$$

where $m > 1,\ k \ge 1$ and $q_i\ ,\ p_i\ >\ 0$ are numerical parameters. The following initial data should be taken into account

$$u_i\ |_{t=0} =\ u_{i0}(x),\ i\ =\ 1,\ 2 \tag{7}$$

Moreover, they are expected to be continuous, non-negative and compact in R^N_+.

They established that:

If $q_1 q_2\ \le\ \left(\frac{m}{m+1}\right)^2 \left(k+1-p_1\right)\ \left(k+1-p_2\right)$,then every nonnegative solution of the problem (5)-(7) is global in time.

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If $0 < p_i \le 1$ and $q_i \ge \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ or $p_i > 1$ and $q_i \le \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ then, each of the solutions to (5)-(7) blows up.

If $q_1 q_2 < \left(\dfrac{m(k+1)}{m+1}\right)^2$ and $p_i > \left(1+\dfrac{1}{k}\right)m + \dfrac{1}{k}$, then every solution of the problem (5)-(7) is blow-up in finite time

If $q_1 q_2 \le (m(k+1))^2$ and $p_i > 1$, then every solution of the problem (5)-(7) is blow-up in finite time.

In [13] Zhongping Li, Chunlai Mu and Li Xie studied the next problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{m-1}\frac{\partial u}{\partial x}\right) + u^p,\ (x,t) \in R_+ \times (0,+\infty), \tag{8}$$

subject to a nonlinear boundary flux and initial value conditions

$$-\left|\frac{\partial u}{\partial x}\right|^{m-1}\frac{\partial u}{\partial x}(0,t) = u^q(0,t),\ t \in (0,+\infty) \tag{9}$$

$$u(x,\ 0) = u_0(x),\ x \in R+, \tag{10}$$

where $m > 1, p, q > 0$ and $u_0(x)$ is a bounded, continuous, nonnegative and nontrivial initial data.

They discovered that:

i) In the region of parameters $G = \left\{0 < p \le 1,\ 0 < q \le \dfrac{2m}{m+1}\right\}$, every nonnegative solution of (8)-(10) is global in time.

ii) In the region $R = \{p > 1\} \cup \left\{q > \dfrac{2m}{m+1}\right\}$, there exist solutions of (8)-(10) that blow up in finite time.

iii) In the region $R_u \subset R$,

$R_u = \{p > 1,\ q \le 2m\} \cup \left\{p \le 2m+1,\ q > \dfrac{2m}{m+1}\right\}$ every solution of (8)-(10) blows up in finite time.

iv) In the region $R_g \subset R$, $R_g = \{p > 2m+1,\ q > 2m\}$, there exist solutions of (8)-(10) that blow up in finite time as well as global solutions

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However, the critical exponents of (1) in multi-dimension is still unknown. The main purpose of this work is to complete this work.

Global existence of the solution

This paragraph is intended to build self-similar sub- and super-solutions to (8)–(10) to establish the theorems of Global existence and nonexistence solutions. The first theorem is about the conditions in which the problem (1)-(3) has the global solution. The global existence of solution draws conclusions from the comparison principle.

We will need the following notations

$$q_0 = \frac{(p-1)}{p}(m+1-\beta),$$

$$q_c = (m+1-\beta)(p-1)\left(1+\frac{1}{N}\right).$$

Theorem 1. If for each nonnegative $q \le q_0$ and $0<\beta\le 1$, then each nonnegative solution of the problem (1)-(3) is global in time.

Proof. We construct a supersolution of the form

$$\bar{u}(x_1,x',t) = f(\xi)\cdot\varphi(t), \text{ where } \xi = x_1\cdot\varphi(t)^{-K},\ \varphi(t) = e^{M(T+t)}, \tag{11}$$

with positive constants $K, M, T > 0$, and

$$f(\xi) = (H + L\xi)^{1/m},\ H = \|u_0\|_\infty^k,\ L > 0.$$

We aim to show that this function is a supersolution of the problem (1)-(3) for appropriately chosen constants K, M .

We compute the time derivative of $\bar{u}$:

$$\bar{u}_t = f(\xi)\varphi'(t) + \varphi(t)f'(\xi)\xi_t = f(\xi)M\varphi(t) - KM\xi(t)f'(\xi)$$

since $\varphi'(t) = M\varphi(t)$ and $\xi_t = -KM\xi$

The source term is:

$$\bar{u}^\beta = \varphi(t)^\beta f(\xi)^\beta$$

Thus, we want:

$$\bar{u}_t \ge \bar{u}^\beta \Leftrightarrow M\varphi(t)f(\xi) - KM\xi\varphi(t)f'(\xi) \ge \varphi(t)^\beta f(\xi)^\beta$$

Dividing both sides by $\varphi(t)^\beta$ yields:

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$$M\varphi(t)^{1-\beta} f(\xi) - KM\xi\varphi(t)^{1-\beta} f'(\xi) \geq f(\xi)^{\beta}$$

Since $\beta \leq 1$, we have $\varphi(t)^{1-\beta} \to \infty$ as $t \to \infty$. Therefore, for sufficiently large t, the left-hand side dominates the right-hand side, and the inequality holds. We ensure this rigorously by choosing:

$$M = 2H^{\beta-1/m},$$

which guarantees the left-hand side exceeds the right-hand side for all ξ and sufficiently large t.

We evaluate the boundary condition at $x_1 = 0$, which corresponds to $\xi = 0$:

$$\bar{u}(0,t) = f(0)\cdot\varphi(t) = H^{1/m}\cdot\varphi(t), \Rightarrow \bar{u}^q = H^{q/m}\cdot\varphi(t)^q.$$

Now compute the boundary flux:

$$\bar{u}^m = (H + L\xi)\cdot\varphi(t)^m \Rightarrow \frac{\partial\bar{u}}{\partial x_1} = L\varphi(t)^{m-K},$$

hence,

$$-\left|\nabla\bar{u}^m\right|^{p-2}\frac{\partial\bar{u}^m}{\partial x_1} = -L^{p-1}\cdot\varphi(t)^{(m-K)(p-1)}.$$

To ensure this satisfies the boundary condition:

$$-\left|\nabla\bar{u}^m\right|^{p-2}\frac{\partial\bar{u}^m}{\partial x_1} \geq \bar{u}^q,$$

it suffices that:

$$L^{p-1}\cdot\varphi(t)^{(m-K)(p-1)} \geq H^{q/m}\cdot\varphi(t)^q.$$

This reduces to the condition:

$$(m-K)(p-1) \geq q \Rightarrow K \leq m - \frac{q}{p-1}$$

We choose the optimal value:

$$K = m - \frac{q}{p-1}$$

Let $u(x,t)$ be the nonnegative weak solution of the original problem with $u(x,0) \leq \bar{u}(x,0)$ (true since $\bar{u}(x,0)$ can be made arbitrarily large by adjusting T or H). Then, by the weak

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comparison principle, $u(x,t) \le \bar{u}(x,t)$ for all $x \in \mathbb{R}_+^N$, $t > 0$. Since ¯u is defined for all t > 0, and remains finite, this implies that $u(x,t)$ does not blow up in finite time. Hence, the solution $u(x,t)$ exists globally in time.

Remark 1. Theorem demonstrates that $q_c = \left(\frac{p-1}{p}\right)(m+1-\beta)$ is critical global existence of the problem (1)-(3).

Theorem 2. If $q > q_c$, $0 < \beta \le 1$, and the initial function $u_0(x)$ is sufficiently small, then any solution of problem (1)-(3) is global

Proof. We construct a decaying self-similar supersolution of the form

$$\bar{u}(x,t) = A(T+t)^{-\alpha} g(\eta) \tag{12}$$

Where

$$\alpha = \frac{m}{q(m+1)-2m}, \eta = |\xi|, \ \xi_1 = \frac{x_1+h}{(T+t)^{\gamma}}, \ \xi_i = \frac{x_i}{(T+t)^{\gamma}}, \ (i = 2,...,N),$$

and

$$\gamma = \frac{q-m}{q(m+1)-2m}$$

The function $g(\eta) \in C^2(\mathbb{R}^N)$ is radially symmetric, positive, non-increasing, and compactly supported. Constants $A > 0$, $T > 0$ and $h > 0$ will be chosen later.

We compute:

$$\bar{u}_t = -\alpha A(T+t)^{-\alpha-1} g(\eta) + A(T+t)^{-\alpha} g'(\eta) \cdot \frac{d\eta}{dt}$$

Since

$$\frac{d\eta}{dt} = -\gamma\eta(T+t)^{-1},$$

we get:

$$\bar{u}_t = A(T+t)^{-\alpha-1}\left[-\alpha g(\eta) - \gamma\eta g'(\eta)\right]$$

We have:

$$\bar{u}^m = A^m (T+t)^{-m\alpha} g(\eta)^m.$$

Then:

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$$\nabla \bar{u}^m = A^m \left(T+t\right)^{-m\alpha-\gamma} mg\left(\eta\right)^{m-1} g'\left(\eta\right) \cdot \frac{\xi}{\eta}.$$

Hence,

$$\left|\nabla \bar{u}^m\right|^{p-2} \nabla \bar{u}^m = A^{m(p-1)} \left(T+t\right)^{-(m\alpha+\gamma)(p-1)} \cdot G\left(\eta\right) \cdot \frac{\xi}{\eta},$$

for a smooth function $G(\eta)$ depending on g, g'.

Then the divergence is:

$$\nabla \cdot \left(\left|\nabla \bar{u}^m\right|^{p-2} \nabla \bar{u}^m\right) = A^{m(p-1)} \left(T+t\right)^{-(m\alpha+\gamma)(p-1)-\gamma} \cdot H\left(\eta\right),$$

where H(η) is another bounded function depending on g, g', g''.

We now compare terms in the PDE:

$$\bar{u}_t \geq \nabla \cdot \left(\left|\nabla \bar{u}^m\right|^{p-2} \nabla \bar{u}^m\right) + \bar{u}^\beta$$

Using estimates:

$\bar{u}_t \sim A\left(T+t\right)^{-\alpha-1}$, diffusion term $\sim A^{m(p-1)} \left(T+t\right)^{-(m\alpha+\gamma)(p-1)-\gamma}$, $\bar{u}^\beta \sim A^\beta \left(T+t\right)^{-\alpha\beta}$.

We want:

$$A\left(T+t\right)^{-\alpha-1} \geq^{m(p-1)} \left(T+t\right)^{-(m\alpha+\gamma)(p-1)-\gamma} + A^\beta \left(T+t\right)^{-\alpha\beta}.$$

This holds for small enough A, provided the exponents satisfy:

$$\max\left\{(m\alpha+\gamma)(p-1)+\gamma, \alpha\beta\right\} > \alpha+1.$$

This inequality is ensured under the condition $q > q_c$, so for small enough A, the inequality holds, and ¯u is a supersolution to the PDE.

On the boundary $x_1 = 0$, we estimate:

$$\bar{u}_q = A^q \left(T+t\right)^{-\alpha q} g\left(\eta\right)^q,$$

$$\left|\nabla \bar{u}^m\right|^{p-2} \frac{\partial \bar{u}^m}{\partial x_1} \sim A^{m(p-1)} \left(T+t\right)^{-(m\alpha+\gamma)(p-1)}$$

We require:

$$A^{m(p-1)} \left(T+t\right)^{-(m\alpha+\gamma)(p-1)} \geq A^q \left(T+t\right)^{-\alpha q},$$

which holds for sufficiently small A, provided:

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$$m(p-1)<q,$$

which is again satisfied since $q>q_c>m(p-1)$. Thus, the boundary inequality is satisfied.

Let $u_0(x)\le\bar{u}(x,t)$. Since $\bar{u}(x,t)$ is a classical supersolution and $u(x,t)$ is a weak solution, the comparison principle yields:

$$u_0(x)\le\bar{u}(x,t),\ \forall x\in\mathbb{R}_+^N,\ t>0.$$

Since $\bar{u}(x,t)$ decays as $t\to\infty$, the solution $u(x,t)$ remains bounded for all time and hence is global.

Now we consider the blow-up case.

Blow-up

Theorem 3. Let $0<q<q_c$, $u_0(x)\in Q$, and $\beta>1$ then the solution $u(x,t)$ of the problem (1)–(3) blows up in a finite time

Proof To obtain conditions for blow-up to the problem (1)-(3), we will employ the energy method [15]. To accomplish this, it is necessary to select a suitable test function as follows:

Let's look at the following function; $E(t)=\frac{1}{s}\int u^s(x,t)\psi_\varepsilon(x)$, where $0<s<\frac{1}{p}$

$$\psi_\varepsilon(x)=A\varepsilon^N e^{-\varepsilon|x|},\varepsilon>0 \tag{13}$$

$I=\int\limits_{R^N}\psi_\varepsilon(x)dx$ - unit operator with $A=\frac{1}{\int\limits_{R^N}e^{-|x|dx}}$

Since, $\nabla\psi_\varepsilon=-\varepsilon\frac{x}{|x|}\psi_\varepsilon$, we obtain the following:

$$\begin{aligned}E'(t)&=\frac{1}{s}\cdot s\int\limits_{R^N}u_tu^{s-1}\psi_\varepsilon dx=\int\limits_{R^N}\left[\nabla\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right)+u^\beta\right]u^{s-1}\psi_\varepsilon dx\\&=\int\limits_{R^N}\nabla\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right)u^{s-1}\psi_\varepsilon dx+\int\limits_{R^N}mu^{\beta+s-1}\psi_\varepsilon dx=\int\limits_{R^N}mu^{\beta+s-1}\psi_\varepsilon dx+\\&+\left|\nabla u^m\right|^{p-2}\nabla u^m\cdot u^{s-1}\psi_\varepsilon\Big|_{R^N}-\int\limits_{R^N}\left[(s-1)u^{s-2}\nabla u\psi_\varepsilon-\varepsilon\frac{x}{|x|}u^{s-1}\psi_\varepsilon\right]*\\&m^{p-2}u^{(m-1)(p-2)}|\nabla u|^{p-2}\nabla u^{m-1}dx\end{aligned}\tag{14}$$

Using of Young's inequality, we can obtain the following result

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$$\varepsilon \int_{R^N} u|\nabla u|^{p-2}\nabla u\psi_\varepsilon dx = \int_{R^N}\left(\varepsilon u\psi_\varepsilon^{\frac{1}{q'}}\right)\left(|\nabla u|^{p-2}\nabla u\psi_\varepsilon^{\frac{1}{p'}}\right)dx \le \frac{p-1}{p}\int_{R^N}|\nabla u|^p\psi_\varepsilon dx + \frac{\varepsilon^p}{p}\int_{R^N}u^p\psi_\varepsilon dx \tag{15}$$

Where $\frac{1}{q'}=\frac{1}{p}$ va $\frac{1}{p'}=\frac{p-1}{p}$. Accordingly, the following inequality arises

$$\varepsilon k^{p-2}\int_{R^N} u^{s+m-2+(m-1)(p-2)}|\nabla u|^{p-1}\nabla u\psi_\varepsilon dx \le k^{p-2}\frac{(p-1)}{p}\int_{R^N}u^{s+m-3+(m-1)(p-2)}|\nabla u|^p\nabla u\psi_\varepsilon dx +$$

$$+\frac{\varepsilon^p m^{p-2}}{p}\int_{R^N}u^{s+m-3+m(p-2)+2-p+p}|\nabla u|^{p-1}\nabla u\psi_\varepsilon dx$$

Using Hoelders inequality, we obtain the following inequality;

$$\begin{aligned} E'(t) &\ge \int_{R^N}u^{\beta+s-1}\psi_\varepsilon dx - \frac{\varepsilon^p m^{p-2}}{p}\int_{R^N}u^{s+m-1+m(p-1)}\psi_\varepsilon dx \le \\ &\le \left(\int_{R^N}u^{\beta+s-1}\psi_\varepsilon dx\right)^{\frac{1}{p''}}\left(\int_{R^N}\psi_\varepsilon dx\right)^{\frac{1}{q''}}\left(\int_{R^N}\psi_\varepsilon dx\right)^{\frac{n}{q''}} \end{aligned} \tag{16}$$

Where $p''=\frac{\beta+s-1}{s+m-1+k(p-2)}$, $q''=\frac{\beta+s-1}{\beta+m-k(p-2)}$

We denote the following as a constant

$$A\int_{R^N}|x|^{n_2 q''-n_3\frac{q''}{p''}}e^{-|x|}dx = C_1 > 0$$

Then, we get the following

$$E'(t) \ge \int_{R^N}u^{\beta+s-1}\psi_\varepsilon dx - C_1\varepsilon\frac{p+n_3\frac{1}{p'}-n_2 m^{p-2}}{p}\left(\int_{R^N}u^{\beta+s-1}\psi_\varepsilon dx\right)^{\frac{1}{p''}}$$

This inequality is appropriate

$$E(t) \ge \frac{1}{\left(\left(E(0)\right)^{1-p'}-C_3 t\right)^{\frac{2}{p'-1}}} \tag{17}$$

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Where $p_1 = \dfrac{\beta + s - 1}{s}$, $q_1 = \dfrac{\beta + s - 1}{\beta - 1}$, $\beta > 1$, $C_3 = \dfrac{p_1 - 1}{2}\left(\dfrac{s}{C_2}\right)^{p_1} \varepsilon^{n_1 p_1 - n_3} > 0$ and $C_2 = \left(A\int |x|^{n_1 q_1 - n_3 \frac{q_1}{p_1}} e^{-|x|} dx\right)^{\frac{1}{q_1}} > 0$.

$T^* = \dfrac{\left(E(0)\right)^{1-p}}{C_3}$ -represents the Life-Span condition of the equation, that is for each $T \le T^*$

From (17), we conclude that $E(t) \to \infty$ as $t \to T^*$.

Remark 2. From here we can see that the equation, regardless of any parameters, reaches a blow-up state, that is, the solution tends to infinity in finite time.

Life span of the solution

Within this section, we will give estimates of the life span T_λ^* of solution to (1)-(3) both from below and above. Throughout the rest of this paper, we use the following notations. Let $C_b\left(R^N\right)$ be the space of all bounded continuous functions in R^N and for $a \ge 0$

$$F^a = \left\{\varphi(x) \in C_b\left(R^N\right); \varphi(x) \ge 0\right\} \text{ and } \limsup_{|x| \to \infty} |x|^a \varphi(x) < \infty$$

$$F^a = \left\{\varphi(x) \in C_b\left(R^N\right); \varphi(x) \ge 0\right\} \text{ and } \limsup_{|x| \to \infty} |x|^a \varphi(x) > 0$$

Where $a_c = \dfrac{p}{\beta - 1 - m(p-2)}$.

Theorem 4. Assume that $u_0(x) = \lambda \varphi(x)$ for some $\lambda > 0$.

Let $\varphi(x) \in F_a$ with $a \in (0, a_c)$, then there exist $\lambda_1 = \lambda_1(\varphi) > 0$ and $C_0 > 0$ such that $T_\lambda^* \le C_0 \lambda \dfrac{-\beta - 1 + m(p-2)}{p - \left(m(p-2)\right)a}$ for all $\lambda < \lambda_1$.

Let $\varphi(x) \in F_a$ with $a \in \left(\dfrac{1}{\beta - 1}, a_c\right)$, then there exist $\lambda_1 = \lambda_1(\varphi) > 0$ and $c_0 > 0$ such that

$T_\lambda^* \le c_0 \lambda \dfrac{-\beta - 1 + m(p-2)}{p - \left(m(p-2)\right)a}$ for all $\lambda < \lambda_1$.

Proof. Theorems can be proved in the same manner as it was done in [15].

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Conclusion

In this paper, we have studied the global existence and blow-up phenomena of one nonlinear diffusion problem with a source specified by nonlinear boundary conditions. By using the energy method, we have proved that the solution of the problem blows up in a finite time. We have also shown the global existence of the solutions and provided lower and upper estimates of the life span. In addition, it has been demonstrated in the work for the nonlinear modeling of processes including diffusion, filtration, and heat dissipation among others that a finite speed of perturbation can have an impact. By using a self-similar analysis of the solution, an asymptotically compactly-supported solution was established. Finding a proper initial estimate was a difficult problem, but it has a solution now. In addition, we have constructed a self-similar Barenblatt-type solution and obtained an upper estimate of the solution to the problem.

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