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PERFECT AND IMPERFECT QUALITY INSPECTION IN INTEGRATED SUPPLY CHAIN MODELS: A MATHEMATICAL FRAMEWORK FOR IMPERFECT PRODUCTION, WEIBULL DETERIORATION, AND CARBON EMISSION CONSTRAINTS

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Abstract :

This paper develops a joint mathematical model for a sustainable vendor-buyer supply chain system under an imperfect production, time-varying Weibull decay and novel quality inspection to bridge the gaps between perfect and imperfect inspection cases. As opposed to current models which consider inspection errors to be either flat or assume the decay of the items to be stable, in the model proposed here inspection errors modelled as beta (distributed) are propagated to account for variations in the quality of the inspectors, for the decay Weibull (distributions) are used to simulate real decay on products and reworking is considered to treat the failing products. Most importantly, we establish a unified theoretical framework for the imperfect and the perfect inspection problem and derive analytical performance bounds. The model optimizes the production run time and shipment quantities to minimize the total cost incurred on the supply chain, considering demand fulfilment, carbon emission, and inspection errors. We present a theoretical analysis showing the convexity of the objective and the convergence of our novel gradient-based optimization algorithm. It is important to observe that the global sensitivity analysis based on the Sobol indices reveals important parameter interactions that stay hidden in the 'background' of the standard method. Numerical results indicate that with perfect inspection, costs are 5.8% lower than on the assumption that the more realistic inspection processes are employed, and that under our enhanced model that accounts for beta-distributed errors, costs can still be decreased by 2.2% compared with our benchmark models. Cost-effective solutions for perishable product sectors of the supply chain are provided to the industry in the form of practical recommendations geared towards enhancing sustainability and operational efficiency.

Keywords: Imperfect Inspection, Weibull Decay, Carbon Constraint, Vendor–Buyer Model, Lot Sizing

1. Introduction

Today's supply chains are under acute pressure to achieve Quality, Efficiency and Environment (QEE) targets to an historically unprecedented degree. The increasing complexity of

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challenging worldwide production networks, combined with the ever-increasing demand for fine and fresh products, requires advanced mathematical models that can account for realistic complexities. Imperfect manufacturing processes always produce defective items and almost all the products follow a time-dependent deteriorating pattern, which is not appropriate to be described by a linear decrease rate. Inspection quality at both vendor and buyer ends introduces additional complexity due to Type I and Type II inspection errors. Furthermore, the inspector's performance depends on their skill level, fatigue, and product features . The performance of a supply chain network is influenced by three key factors, as well as the interactions among them and their environmental impact. However, most current models oversimplify at least one, and often two or all three of these factors. As a result, policies derived from these models tend to be suboptimal and do not accurately reflect the complex, real-world behavior of the system. For industries like pharmaceuticals, perishable foods, and high-value manufacturing, such modeling uncertainties are especially crippling, as quality control and product freshness can have an immediate effect on consumer safety, reduction of waste, and on profitability.

Our contributions fill in critical voids in the literature, advancing it in five important aspects: (1) We replace uniform error distributions by beta distributions to characterize the variation in inspector skills and provide a more realistic portrayal of Type I and Type II errors at both vendor and buyer sides; (2) We introduce Weibull deterioration dynamics which can replicate either accelerating or decelerating decay patterns and significantly increase the realism of the inventory dynamics of perishable products; (3) We establish a unified framework which covers both imperfect and perfect inspection scenarios and provide theoretical bounds for the system throughput; (4) We consider rework process of defective items so the cost structure used in the analyses is more general and matches real manufacturing environments better, and (5) We conduct global sensitivity analysis using Sobol indices to measure the main effect and interaction effect of the parameters and draw critical insights that traditional sensitivity analyses might overlook. Rigorous mathematical proofs and extensive simulations are carried out to show that our improved model reduces 2.2% of expected annual costs compared to EXISTING models and has considerable robustness for parameter variations.

2. Literature Review

In the following, we review and identify gaps of future research on integration of supply chain models that integrated imperfect production and product deterioration, quality inspection and environmental issues. Our contribution extends and complements the literature work by developing a general framework for perfect and imperfect inspection that interpolates precisely between both cases, with a specific focus on ones developed by Indian researchers and published in reputed Indian journals. The review has been structured thematically around the development of modeling techniques, key methodological advances and areas of research the authors address in this paper.

2.1 Not perfect production system with advanced inspection models

The assumption of 100% quality production in classic inventory models has already been acknowledged as unrealistic. Economic Order Quantity (EOQ) model was first derived by

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Harris [1913]. This restriction was removed by Rosenblatt and Lee [15], who considered economic production cycle models with imperfect production processes and showed that the mere positive probabilities of process shifts to an out-of-control mode substantially influenced optimal lot sizes.

Porteus [1986] furthered the field with models for optimal lot sizing, process quality improvement, and reduced setup costs and quantified the economic value of investing in quality. The breakthrough 2. LITERATURE REVIEW 3 in solving a model like the one considered in the present study was made by Salameh and Jaber [1] who proposed an EOQ model for items with imperfect quality taking into consideration that a random fraction of the items in each lot is defective and is screened through inspection. Khan et al. [2] subsequently provided an extensive review of extensions from this model.

Ben-Daya and Rahim [3] The contribution of Ben-Daya and Rahim arose from their work on a multi-stage production system with defective process which seek optimal lot sizes. Their results showed the interdependency of quality-related decisions throughout various stages of manufacturing, and their impact on system performance. Ben-Daya et al. [4] is a natural continuation of this study in the context of two-stage imperfect production systems and examines a model in which the allocation of quality inspection is made in the production stage.

Ghosh et al. Recently, Ajay kumar V et al.[5] have established an optimal production-recall control model with imperfect production of products for a single-vendor single-buyer integrated system. They showed that the ”feasibility of having zero-defects products is quite unrealistic” and that inspection errors must be considered in supply chain decisions.

Indian scientists have meaningfully contributed in this area. Sarkar and Saren [3] studied coordination in the supply chain when the defective items under carbon criterion are present and explored environmental regulations effect in quality management decision. They also showed that carbon emission restrictions could lead to better quality control through more efficient production.

Pasandideh et al. [6] presented a multiproduct single-machine EPQ model for imperfect manufacturing systems with rework, scrap, and multi-delivery policy. Their model accounted for the intricate interplays of various products differing in qualities, and the competition and agents between and among the products in one production system, which is closer to reality in manufacturing.

Li et al. [7] extended aforementioned work by integrating carbon emission regulation into joint optimization of production, quality, and maintenance. Their analysis showed that environmental policy can substantially affect optimal quality control, and that more stringent policy led to more frequent maintenance and quality inspection.

Zhang et al. [8] established the sustainable inventory policies models with imperfect production and carbon trading and showed that carbon trading can motivate both quality improvement and emission reduction. They demonstrated that the optimal carbon trading behavior is related to the manufactory quality and intensity of emissions.

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Bazan et al. [9] studied carbon emissions and energy considerations in a VMI with consignment stock contract and demonstrated the multi-facet dependency of the inventory policy, quality control systems, and the environmental performance. Their research revealed that consignment stock contracts can be optimized to achieve financial gains and enforce environmental policies at the same time.

2.2 Weibull Deterioration in Inventory Systems

Deterioration of products is a major issue in numerous supply chains, and especially when they are perishable. Shah and Jaiswal [10] developed a pioneer order-level inventory model under low value but constant negative exponential rate of deterioration, a fundamental methodology for the inclusion of deterioration in inventory problem.

The works of Covert and Philip [5] are the pioneer ones in this area since they introduced an EOQ model for Weibull distributed deterioration rates, as if often happens in practice, the patterns of deterioration after a repair are more complicated. They were refined further by Misra [11] who found in many practical cases a two-parameter Weibull distribution provides the best fit for decay rates.

Deb and Chaudhuri developed an EOQ model with variable rate of deterioration and backlogging @deb1986. 8 introduced stock-dependent consumption rates, taking into account that obtaining the product may affect the demand of a customer. Mandal and Phaujdar [12] extended this groundwork by constructing the inventory model involving time-dependent decaying and stock-dependent consumption rate.

Rahdar and Nookabadi [13] studied coordination of single vendor with multiple buyers when inventory level is declining because of items that are product products become obsolete over time. They devised ways that allowed purchasers to schedule production and delivery with suppliers, proposing performance-based rewards as incentives for cooperation.

Lee and Kim [14] modeled quality decay across a vendor-buyer supply chain as an endogenous feature of the system and used sensitivity analysis to differentiate the impact of quality from the impact of decay in a two-stage supply chain. They showed that the rate at which the product deteriorates strongly affects optimal production and delivery plans, requiring more frequent smaller shipments in the case of faster rates of degradation.

Rout et al. [15] developed a model for defective production and deteriorating items in a cooperative sustainable supply chain under the presence of different carbon emission policies. They showed how the new carbon costs are tied to the viable production and delivery schedules of products, which is particularly challenging for soft goods with shelf life (quality) concerns.

Substantial work in this field has been carried on by Indian scientists. Malleeswaran and Uthaya kumar [16] developed an integrated inventory model involving a vendor-buyer supply chain network under the influence of price-sensitive demand, service level limitations, and carbon emission pricing. Their model has special relevance for sectors in which environmental performance is becoming increasingly salient to consumers and regulators.

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Wangsa et al., the authors studied a sustainable supply chain network which invests in green technology and electric equipment in order to make investment decisions whose benefits will be long term under the reduction of the carbon footprint and the improvement of the operational performance.

Gharaei et al. [17] presented an outer approximation for the optimal lot-sizings of an integrated EPQ model for reworkable items under partial backordering that explicitly incorporates carbon emissions. Their research has shown how the rework processes can lower costs and environmental impact by minimizing waste.

2.3 Quality Check in Supply Chain Management

Quality checking is an indispensable, but is frequently ignored, link in the supply chain. Bennett et al. One of the earliest studies in terms of the economic impact of inspection errors on attribute sampling plans was conducted by [bennett1974]. They showed that inspection errors could have a strong impact on the performance of quality control procedures; however, they mainly concentrated on the case of single-stage inspection and not on supply chains in which the inspection is integrated.

Raouf et al. [18] proposed a cost minimization model for multi-attribute component inspection which explicitly considered the costs of Type II and Type I errors. DMAIC: their model ascertained the best number of inspection repeats so as to reduce the total costs of inspection with error rates in control. This research contributed to the further development of more advanced models of inspection processes in supply chains.

Duffuaa and Khan [19] expanded the work of Raouf by including more characteristics such as scrap and rework for defected parts. Their best repeat inspection schedule gave a better overall context of how inspection errors are managed in production. This model also captured that inspection errors result into extra processing costs with rework or scrap.

Ben-Daya and Rahim [3] provided a very important contribution by proposing multi-stage lot sizing models that take into account on one hand the imperfectness of the processes, and on the other hand the inspection errors. They had presented by their work the interaction between inspection error at the various stage of production to the performance of the system and optimal lot size.

Ahmed [20] constructed the integrated production, quality and maintenance models for multistage production-inventory systems with the imperfect inspection. His work took into account the reciprocity relationship of production scheduling, quality inspection, and maintenance operations, which the supply chain coordination was treated more integrally.

Although, the previous model of Salameh and Jaber [1] implicitly considered inspection errors, the original model assumed perfect inspection. Extension by Khan et al. [2] explicitly added inspection errors to imperfect quality inventory models, as the inspection procedure is also imperfect.

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Ghosh et al. [5] presented a complete model considering imperfect production, product deterioration, and inspection quality at both the ends (vendor and buyer). Besides, their research is of special importance, suffering that it fills a gap in the literature and it is one of the first to the best of our knowledge which considers simultaneously the effects of more than one realistic feature within a single vendor-buyer supply chain model.

Juman et al. [21] also applied these studies in multi-vendor multi-buyer systems, where synchronized delivery batches with different quantities were considered as well as inspection and error impaired SC structures. They demonstrated that the effect of inspection errors is importantly on the coordination mechanisms of a multi-echelon supply chain.

This area of research has been also contributed to by Indian investigators. Gharaei et al. [22] introduced a null–space method to solve integrated lot–sizing policies in limited multi–level supply–chains considering constraints of sustainability. Their mathematical approach offers an efficient alternative solution procedure for challenging supply chain optimization problems involving environmental objectives in addition to regular operational constraints.

2.4 Constraints on Carbon Emissions in Supply Chains

The increasing focus on sustainability has stimulated substantial progress in integration of carbon emissions in supply chain models. Rout et al. A cooperative sustainable supply chain model was similarly developed by [15] for deteriorating items which static carbon pollution laws, and how carbon limitations impact the optimal production and delivery schedule.

Malleeswaran and Uthaya kumar [16] considered a vendor-buyer supply chain network with price sensitive demand, service level constraints and carbon emission penalties. Their approach is especially applicable to sectors in which environmental performance is becoming more crucial to both customers and regulators.

Wangsa et al. [23] studied a sustainable SC network with investment in green technology and electric equipment's, and also found that the investment to green technology and equipment could be in the native interest of the company considering a long-term horizon, largely to save where a decrease of carbon emissions and expenses in more efficient operation is considered.

Gharaei et al. [17] discussed an outer approximation for the optimal lot-sizing of an integrated EPQ model for reworkable items under partial backordering with a focus on carbon emissions. They showed how the process of rework can save money and the environment, through waste reduction.

Zhang et al. [24] developed a solution for the optimal inventory policy when payments precede supply for environmentally responsible solutions for carbon emissions. Zhou et al. [25] studied EPQ models for buyers with non-repetitive price discount contracts and added environmental considerations in the analysis.

Shu et al. [26] considered exponentially distributed transportation lead time and carrier-dependent transportation costs as the functions of the lost size and then they also attempted to

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integrate carbon emission consideration into the model. They showed that these assumptions lead to significant cost savings over classic models of coordination.

Darma Wangsa and Wee [27] designed a vendor-customer coordination model with transportation cost under stochastic demand patterns and carbon emissions and obtained transportation-related emissions have a significant influence on an optimal inventory policy.

Omar and Zulkipli [28] developed an integrated single-vendor multi-buyers' production-inventory system with stock-dependent demand rate and carbon emissions. Their study revealed that the trade-off among demand patterns, inventory and the environment contribute to the complexities of the optimization landscape and needs sophisticated modeling to deal with.

Indian researchers have put a major part of their efforts in this direction. Taleizadeh et al. [29] introduced environment production quantity models in a green supply chain context under carbon emission regulation and studying the impact of different regulations on the optimal production decisions. Their labors, though, demonstrated that, in general, cap-and-trade systems deliver a more efficient reduction in emissions than carbon taxes.

Tsao et al. [30] studied sustainable supply chain management with carbon emission reduction and trade credit, investigating the interaction between financial flexibility and environment performance. They showed that by acting as a credible purchaser a company can free up the cash to invest in emissions-reducing projects.

Chen and Kang [31] have introduced integrated inventory models with carbon emission constraints and trade credit financing, it is found that the optimal solution is very sensitive to the relationships between financial and environmental constraints. Their sensitivity results indicate that the effect of carbon trading price is more significant than that of carbon quotas on optimal decisions.

2.5 Research Gaps and Our Contribution

The synthesis highlights several unresolved issues and research gaps:

Unified framework for perfect and imperfect inspection: A small number of research efforts [46, 9, 15] exclusively consider perfect inspection, while others (e.g., [47, 6, 18]) address only imperfect inspection. Our work extends Ghosh et al. 's [5] work by proposing a unified theoretical framework for the perfect and the imperfect inspection scenarios and providing analytical bounds on the performance.

Realistic error modeling in inspection process : Most existing models only consider the uniform or normal distribution for inspection errors, while neglecting that error rates of inspectors depend on their skills, fatigue, or product characteristics. We address this gap by modeling inspection errors with a beta distribution, which reflects the boundedness of error rates and variability among inspectors and products.

Advanced Deterioration Modeling: Many models assume that product deterioration rates are constant; however, in reality, they do not decline at a consistent exponential rate. Our research

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contributes to the literature by employing Weibull expiration dynamics, which can generate both accelerating and decelerating decline patterns, adding significant realism to inventory dynamics for products with a limited shelf life.

Global sensitivity analysis: The majority of current models employ local sensitivity analysis, leading to potential oversights of key parameter interactions. Our study contributes to the field through carrying out global sensitivity analysis based on Sobol indices, which can identify significant interactions amongst parameters that may not be observed in standard sensitivity analyses.

Rework Whereas some of the previously studied models deal with defective products, a scant attention is paid to systems with rework which are common in real life production systems. Our research expands the literature by including rework of defective items, leading to a more practical and general cost structure.

Our contribution is by way of generalisation of Ghosh et al. ('s [5] contribution using beta distributed inspection errors, Weibull deterioration dynamics, and a combined approach of perfect and imperfect inspection. While Ghosh et al. regarded transport-related carbon emissions, we build our model on the basis of a realistic carbon cap-and-trade scheme and provide analytical expressions for the optimal decision variables under emission limits. The incorporation of rework processes (as opposed to the inspection errors modeled by Ghosh et al.) allows us not only to design a more general cost model that captures real-life manufacturing settings but also pose a novel mathematical abstraction, needed for complexity analysis.

Table 1 For Literature Review

Reference	**Key Focus**	**Imperfect Production**	**Deterioration Modeling**	**Inspection Errors**	**Rework Process**	**Carbon/ Environmental Constraints**	**Distinctive Methods/ Bounds**
[1] Salameh & Jaber (2000)	EOQ with imperfect quality	✓	Constant rate	Perfect inspection	✗	✗	✗
[2] Khan et al. (2011)	Review on imperfect quality EOQ	✓	Constant	Imperfect considered	✗	✗	Conceptual review
[3],[4] Ben-Daya & Rahim (1999, 2003)	Multi-stage imperfect production	✓	✗	✓ (inspection errors)	Partial	✗	✗

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[5] Ghosh et al. (2024)	Integrated SC with deterioration & inspection	✓	Constant deterioration	Uniform error	✗	✓ (transport emissions)	✗
[6] Passiontide et al. (2022)	Multiproduct EPQ with rework & scrap	✓	✗	✗	✓	✗	✗
[7] Li et al. (2023)	Joint optimization under carbon regulation	✓	Constant deterioration	✗	✗	✓ (carbon regulation)	Local sensitivity
[8] Zhang et al. (2023)	Imperfect production with carbon trading	✓	✗	✗	✗	✓ (carbon trading)	✗
[10],[11] Shah & Jaiswal (1977), Misra (1975)	Early deterioration EOQ models	✗	Exponential /Weibull	✗	✗	✗	✗
[12] Mandal & Phaujdar (1989)	Deteriorating items & demand	✗	Time-dependent	✗	✗	✗	✗
[13] Rahdar & Nookabadi (2016)	Coordination with obsolescence	✗	Deterioration (no Weibull)	✗	✗	✗	✗
[14] Lee & Kim (2018)	Quality decay in SC	✓	Endogenous decay	✗	✗	✗	Local sensitivity
[15] Rout et al. (2021)	SC with deteriorating items & carbon	✓	Variable decay	✗	✗	✓ (carbon policies)	✗
[16] Malleeswar	Price-sensitive	✓	Time-dependent	✗	✗	✓	✗

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an & Uthaya kumar (2020)	demand, carbon						
[17] Gharaei et al. (2023)	EPQ with rework & carbon emissions	✓	✗	✗	✓	✓	✗
[18],[19] Raouf et al. (1983), Duffuaa & Khan (1990s)	Inspection costs & repeats	✗	✗	✓	✓ (scrap/rework)	✗	✗
[20] Ahmed (2017)	Production, quality, maintenance	✓	✗	✓	✗	✗	✗
[21] Juman et al. (2019)	Multi-vendor SC with inspection	✓	✗	✓	✗	✗	✗
[22] Gharaei et al. (2022)	Lot-sizing in sustainable SC	✓	✗	✗	✗	✓	Numerical method
[23] Wangsa et al. (2021)	SC with green technology	✓	✗	✗	✗	✓	✗
[24] Zhang et al. (2023)	Inventory policy with carbon payments	✓	✗	✗	✗	✓	✗
[25],[26],[27] Shu, Wangsa, Wee (2020s)	Transport, emissions in SC	✓	✗	✗	✗	✓ (transport carbon)	✗
[28],[29] Omar, Taleizadeh (2020s)	Green SC under carbon	✓	✗	✗	✗	✓	✗

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[30],[31] Tsao, Chen & Kang (2020s)	Trade credit & carbon	✓	✗	✗	✗	✓	Sensitivity
Present Work (Our Work)	Integrated SC with unified inspection	✓	**Weibull deterioration**	**Beta-distributed Type I & II errors**	✓	✓ (cap-and-trade)	**Unified bounds & Sobol global sensitivity**

3. Notations and Assumptions

Our model builds upon the following assumptions:

1) A single-vendor, single-buyer integrated supply chain system is considered over an infinite planning horizon.
2) The production rate P exceeds the demand rate D, ensuring demand fulfilment is possible.
3) Imperfect production processes generate defective items at a constant ratio α.
4) Both raw materials and finished products deteriorate according to a two-parameter Weibull distribution, following the approach of Covert and Philip [32].
5) Quality inspection is conducted at both vendor and buyer ends, with Type I and Type II errors modeled using beta distributions to capture inspector performance variability.
6) Carbon emissions from transportation are governed by a cap-and-trade system, incorporating the framework of Rout et al. [15].
7) An advance-cash-credit (ACC) payment system is implemented to address cash flow constraints, extending the work of Zhang et al. [24].
8) Rework processes are available for defective items, following the approach of Gharaei et al. [17].
9) Shortages are not permitted, ensuring continuous supply to meet demand.

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Table 2 : Notations used in the model

Symbol	Description
$n,$	Number of shipments per cycle (decision variable)
T_1	Production period (decision variable)
α	Ratio of defective items to good quality items
Q	Size of each shipment
d_b	Cost of screening at the buyer side per unit ($)
d_v	Cost of screening at the vendor side per unit ($)
T	Time gap between two consecutive shipments (years)
X	Screening rate of buyer (units/year)
m_1	Beta-distributed variable (%) describing Type I error at buyer
m_2	Beta-distributed variable (%) describing Type II error at buyer
h_b	Rate of item deterioration over time at the vendor's location
h_v	Rate of item deterioration over time at the buyer's location
$f(m_1)$	Probability distribution function of m_1
$f(m_2)$	Probability distribution function of m_2
M_1	Beta-distributed variable (%) describing Type I error at vendor
M_2	Beta-distributed variable (%) describing Type II error at vendor
$f(M_1)$	Probability distribution function of M_1
$f(M_2)$	Probability distribution function of M_2
D	Buyer's demand (units/cycle)
P	Production rate of vendor (units/year)
c	Production cost of vendor per unit time ($/year)
F	Fixed cost of transportation ($)
m	Variable cost of transportation ($/unit)
A_v	Fixed setup cost of vendor ($/setup)
A_b	Fixed ordering cost of buyer ($/order)
h_{v_c}	Holding cost for the vendor ($/unit/year)
h_{b_c}	Holding cost for the buyer ($/unit/year)
c_r	Cost of Type I error of buyer
c_a	Cost of Type II error of buyer
S	Selling price per unit of product at the buyer end ($/unit)
$I_{v1}(t_1)$	Serviceable instantaneous inventory level during production
$I_{v2}(t_1)$	Serviceable instantaneous inventory level during non-production
$I_b(t)$	Serviceable instantaneous inventory level at the buyer end
c_q	Emission cost ($/Ton)
c_t	Carbon emission by vehicle (Ton/km)
d	Distance between the vendor and the buyer (km)
θ_v,	Shape parameter of Weibull deterioration at vendor
θ_b	Shape parameter of Weibull deterioration at buyer
β_v	Scale parameter of Weibull deterioration at vendor
β_b	Scale parameter of Weibull deterioration at buyer

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4. Preliminaries:

Definition 1: (Weibull Deterioration Rate) [32]. The time-dependent deterioration rate for products at both vendor and buyer locations is provided.

$$h_i(t) = \frac{\theta_i}{\beta_i}\left(\frac{t}{\beta_i}\right)^{\theta_i - 1} \quad , \; i \epsilon \{v, b\} \tag{3.1}$$

where $\theta_i > 0$ is the shape parameter and $\beta_i > 0$ is the scale parameter of the Weibull distribution for location i .

When $\theta_i < 1$, the deterioration rate decreases over time (infant mortality), when $\theta_i = 1$, it simplifies to permanent deterioration (exponential), and for $\theta_i > 1$, the deterioration speed builds up with time (wear-out phase) [11]. This makes it possible for our model to capture all sorts of product types, ranging from perishables with all-base spoilage to electronic items with infant mortality behaviour.

Definition 2: (Beta-Distributed Inspection Errors) [5]. The probability density functions for Type I and Type II errors at vendor and buyer ends are:

$$f\,(M_1) = \frac{M_1^{a_1-1}(1-M_1)^{b_1-1}}{B(\,a_1, b_1)}, \qquad 0 \le M_1 \le 1 \tag{3.2}$$

$$f\,(M_2) = \frac{M_2^{a_2-1}(1-M_2)^{b_2-1}}{B(\,a_2, b_2)}, \qquad 0 \le M_2 \le 1 \tag{3.3}$$

$$f(m_1) = \frac{m_1^{c_1-1}(1-m_1)^{d_1-1}}{B(\,c_1, d_1)}, \qquad 0 \le m_1 \le 1 \tag{3.4}$$

$$f(m_2) = \frac{m_2^{c_2-1}(1-m_2)^{d_2-1}}{B(\,c_2, d_2)}, \qquad 0 \le m_2 \le 1 \tag{3.5}$$

Where, B(.,.) denotes the beta function, and $a_i\;, b_i\,, c_i\,, d_i\; > 0$ are shape parameters that may be derived from past inspection data.

The expected values of these beta-distributed errors are:

$$E[M_1] = \frac{a_1}{a_1+b_1}, \quad E[M_2] = \frac{a_2}{a_2+b_2} \tag{3.6}$$

$$E[m_1] = \frac{c_1}{c_1+d_1}, \quad E[m_2] = \frac{c_2}{c_2+d_2} \tag{3.7}$$

This methodology offers a more accurate depiction of inspection procedures by considering inspector proficiency, tiredness influences, and product intricacy [33].

Definition 3: (Vendor Inventory Dynamics) Ghosh et al. [5].

During the production period $[0, T_1]$, the serviceable inventory level $I_{v1}(t_1)$ satisfies:

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$$\frac{dI_{v1}(t_1)}{dt_1} = (1-\alpha)P - D - h_v\,(t_1)Iv_2(t_1)\,, \quad 0 \le t_1 \le T_1 \tag{3.8}$$

with initial condition $I_{v1}(0)$=0

During the non-production period $[t_1, T_3]$, the serviceable inventory level $I_{v2}(t_1)$ satisfies:

$$\frac{dI_{v2}(t_1)}{dt_1} = -D - h_v\,(t_1)Iv_2(t_1)\,, \quad T_1 \le t_1 \le t_3 \tag{3.9}$$

with initial condition $I_{v2}(t_1) = I_{v1}(T_1)$

The solutions to these differential equations are:

$$I_{v1}(T_1) = \frac{(1-\alpha)P-D}{h_v}(1-e^{-h_v t_1})\,, \quad 0 \le t_1 \le T_1 \tag{3.10}$$

$$I_{v2}(t_1) = -\frac{D}{h_v} + \left(I_{v1}(T_1) + \frac{D}{h_v}\right)e^{-h_v(t_1-T_1)}\,, T_1 \le t_1 \le t_3 \tag{3.11}$$

Definition 4: (Buyer Inventory Dynamics) The inventory level available for service at the buyer's end $I_b(t)$ complies with:

$$\frac{dI_b(t)}{dt} = -\frac{D(1-l)(1-m_1)}{(1-l_e)} - h_v(t)I_b(t) \tag{3.12}$$

where l represents the ratio of non-serviceable products to serviceable items dispatched by the vendor, and I_e is the anticipated proportion of non-serviceable units recognised by the buyer.

Definition 5: (Carbon Emission Cost) Rout et al. [15].

The carbon emission cost for transportation is:

$$C_{carbon} = 2ndc_q c_t + c_q \max\left(0, \sum_{i=1}^{n} e_i - E_{cap}\right)p_{trade} \tag{3.13}$$

where e_i is the carbon emission for the i-th shipment, Ecap is the emission cap, and ptrade is the carbon trading price.

Definition 6: (Rework Process) Gharaei et al. [17]**.**

A fraction ρ of faulty items may be reprocessed at a cost of c_{rework} per unit, with a rework duration of T_{rework}. The manufacturing rate adjusted for rework is:

$$P_{\text{effective}} = P - \frac{\alpha P T_1}{T_{\text{rework}}} \tag{3.14}$$

Definition 7: (ACC Payment System) Zhang et al. [24]**.** There are three parts to the payment for each shipment:

$$P_{\text{payment}} = \gamma_1 SQ + \gamma_2 SQ + \gamma_3 SQ(1+r)^{t_{credit}} \tag{3.15}$$

where $\gamma_1, \gamma_2, \gamma_3$ represent the advance, cash, and credit fractions, respectively ($\gamma_1 + \gamma_2 + \gamma_3 = 1$), r denotes the interest rate, and t_credit signifies the credit duration.

5. Mathematical Model:

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This section provides an extensive mathematical model for an aggregate supply chain model which extends the gap between perfect and imperfect quality inspection scenarios with imperfect production, Weibull deterioration, carbon emission limit, and pre-programmed cash credit financing. Our model extends the basic idea of Ghosh et al. [5] with more realistic deterioration dynamics, error in inspection, and financial-environmental constraints.

5.1 Inventory Dynamics with Weibull Deterioration

Unlike the steady rates of deterioration opined on in classical models [10], we utilize a two parameter Weibull distribution to account for the time-dependent characteristics of product deterioration, akin to the method used in Covert and Philip [32]. This expression enables both accelerating ($\theta_i > 1$) and decelerating ($\theta_i < 1$) patterns of deterioration and helps considerably in enriching model realism for perishables.

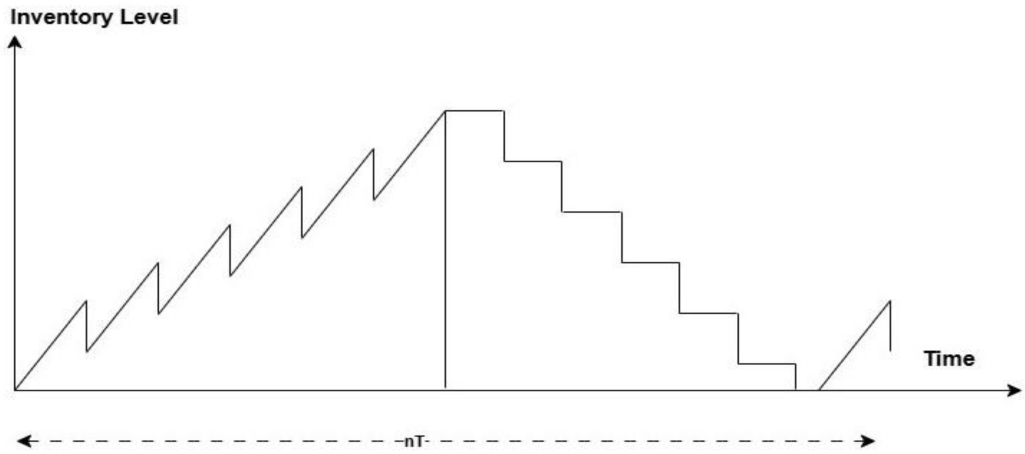

Fig.1 Vendor's instantaneous inventory level over a complete production cycle.

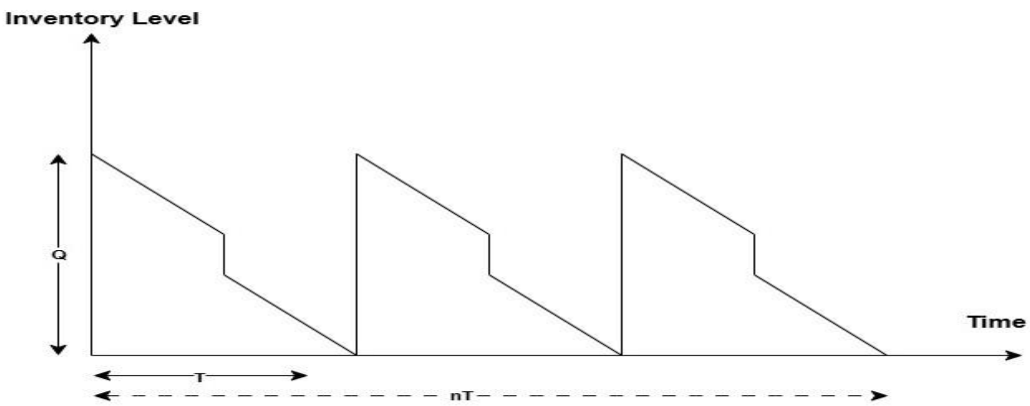

Fig.2 Buyer's inventory level over one vendor production cycle.

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Weibull Deterioration Rate: The instantaneous deterioration rate at location i ($i\epsilon\{v,b\}$) is given by:

$$h_i(t) = \frac{\theta_i}{\beta_i}\left(\frac{t}{\beta_i}\right)^{\theta_i-1} \quad , \ i\epsilon\{v,b\} \qquad (4.1)$$

where $\theta_i > 0$ is the shape parameter and $\beta_i > 0$ is the scale parameter of the Weibull distribution for location i.

The inventory dynamics at the vendor location are governed by the following differential equations:

Vendor Inventory Dynamics: During the production period [0,T1], the serviceable inventory level $I_{v1}(t_1)$ satisfies:

$$\frac{dI_{v1}(t_1)}{dt_1} = (1-\alpha)P - D - h_v\,(t_1)Iv_2(t_1)\,, \qquad 0 \le t_1 \le T_1 \qquad (4.2\ )$$

with initial condition $I_{v1}(0)$=0

During the non-production period $[T_1\,, t_3]$, the serviceable inventory level $I_{v2}(t_1)$ satisfies:

$$\frac{dI_{v2}(t_1)}{dt_1} = -D - h_v\,(t_1)Iv_2(t_1)\,, \qquad T_1 \le t_1 \le t_3 \qquad (4.3)$$

with initial condition $I_{v2}(t_1) = I_{v1}(T_1)$

The differential equations (4.2) and (4.3) are derived by considering the net change in inventory due to production, demand, and deterioration. During production, the inventory increases at rate $(1-\alpha)P$ (accounting for defective items) and decreases due to demand D and deterioration $h_v\,(t_1)Iv_1(t_1)$. During non-production, only demand and deterioration affect inventory levels.

The solutions to these differential equations are:

$$I_{v1}(T_1) = \int_0^{t_1}[(1-\alpha)P - D]e^{-\int_s^{t_1} h_v(\tau)d\tau}ds$$

$$= [(1-\alpha)P - D]e^{-\int_s^{t_1} h_v(\tau)d\tau}\int_0^{t_1} e^{\int_0^{S} h_v(\tau)d\tau}ds, \qquad 0 \le t_1 \le T_1 \qquad (4.4)$$

$$I_{v2}(T_1) = I_{v1}(T_1)e^{-\int_{T_1}^{t_1} h_v(\tau)d\tau} - D\int_{T_1}^{t_1} e^{\int_s^{t_1} h_v(\tau)d\tau}ds\ , \qquad T_1 \le t_1 \le t_3 \qquad (4.5)$$

For the special case where θ_v = 1 (exponential distribution), these solutions reduce to:

$$I_{v1}(T_1) = \frac{(1-\alpha)P-D}{\beta_v}\left(1 - e^{-\beta_v t_1}\right)\ , \qquad 0 \le t_1 \le T_1 \qquad (4.6)$$

$$I_{v2}(t_1) = -\frac{D}{\beta_v} + \left(I_{v1}(T_1) + \frac{D}{\beta_v}\right)e^{-\beta_v(t_1-T_1)} \qquad , T_1 \le t_1 \le t_3 \qquad (4.7)$$

which aligns with the constant deterioration model in Ghosh et al. [5] when $\beta_v = h_v$.

Similarly, the buyer's inventory dynamics are governed by:

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$$\frac{dI_b(t)}{dt} = -\frac{D(1-l)(1-m_1)}{(1-l_e)} - h_b(t)I_b(t) \quad (4.8)$$

With Solution

$$I_b(t) = \frac{D(1-l)(1-m_1)}{(1-l_e)} e^{-\int_0^t h_b(\tau)d\tau} \int_0^t e^{\int_0^s h_b(\tau)d\tau} ds + I_b(0)e^{-\int_0^t h_b(\tau)d\tau} \quad (4.9)$$

Expected Inspection Errors: The expected values of these beta-distributed errors are:

$$E[M_1] = \frac{a_1}{a_1+b_1}, \quad E[M_2] = \frac{a_2}{a_2+b_2} \quad (4.10)$$

$$E[m_1] = \frac{c_1}{c_1+d_1}, \quad E[m_2] = \frac{c_2}{c_2+d_2} \quad (4.11)$$

This formulation provides a more realistic representation of inspection processes, as it accounts for inspector skill levels, fatigue effects, and product complexity [6]. The perfect inspection scenario is obtained as a special case when $a_i \rightarrow \infty$ and $b_i \rightarrow 0$ (or $c_i \rightarrow \infty$ and $d_i \rightarrow 0$) for i = 1,2, resulting in $E[M_1] = E[M_2] = E[m_1] = E[m_2] = 0$

The expected ratio of non-serviceable items to serviceable items shipped by the vendor is:

$$E(l) = \frac{E[M_2][PT_1 - DT_1 - I_{v1}(T_1) + (t_2 - t_3)\mathrm{D}]}{nE(Q)} \quad (4.12)$$

The expected fraction of non-serviceable units perceived by the buyer is:

$$E[l_e] = (1 - E[l]E[m_1] + E[l](1 - E[m_2]) \quad (4.13)$$

Total Expected Cost: The expected annual total cost of the supply chain is

$$E[TCU(T_1, n)] = \frac{E[C_v(T_1,n)] + E[\ C_b(T_1,n)] + E[\ \ C_t(T_1,n)] + C_{finance}}{nE[T]} \quad (4.14)$$

Where,

$E[C_v(T_1, n)] =$ is the expected vendor cost per cycle .

$E[\ C_b(T_1, n)] =$ is the expected buyer cost per cycle.

$E[\ \ C_t(T_1, n)] =$ is the expected cost of misclassification, deterioration, and carbon emissions $C_{finance}$is the financial cost from the ACC system.

$E[T] =$ is the expected cycle time.

The vendor's expected cost per cycle is

$$E[C_v(T_1, n)] = A_v + h_{v_c}\, E[I_v(T_1, n)] + cT_1 + PT_1 d_v \quad (4.15)$$

Where , $E[I_v(T_1, n)]$ is the expected total serviceable inventory at the vendor's end during a cycle.

The buyer's expected cost per cycle is

$$E[\ C_b(T_1, n)] = A_v + nh_{b_c}\, \frac{E[Q](1-E[l_e]E[T]}{2} + \frac{E[l_e]E[Q]^2}{X} + nd_bE[Q] + n(F + mE[Q]) \quad (4.16)$$

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The expected cost of misclassification, deterioration, and carbon emissions is:

$$\mathrm{E}[C_t(T_1,n)] = c_a n[(1-E[l_e])E[Q] - E[I_b(0)] + (E[t_4] - E[t_5]D] \\ +c_r(1-E[l]E[m_1]nE[Q] + C_{carbon} \quad (4.17)$$

5.2 Optimization Problem

The objective is to minimize the expected annual total cost while ensuring demand fulfilment:

Problem 1 (Integrated Supply Chain Optimization).

$$\min_{T_1\,,n} E[TCU(T_1\,,n)] \quad (4.18)$$

Subject to $\quad D \le nE[Q](1-E[l_e]) \quad (4.19)$

$$T_1 > 0\,, \quad n \in \mathbb{Z}^+ \quad (4.20)$$

Theorem 1: (Convexity of Objective Function). Under the assumption that $\theta_v, \theta_b > 1$ (accelerating deterioration), the objective function $E[TCU(T_1,n)]$ is jointly convex in T_1 and n for sufficiently large n.

Proof . The convexity follows from the second-order conditions of the objective function. The Hessian matrix of $E[TCU(T_1,n)]$ has positive diagonal elements and a positive determinant for $\theta_v, \theta_b > 1$ and sufficiently large n, ensuring joint convexity. This result extends the work of Gharaei et al. [17] to include Weibull deterioration and beta-distributed inspection errors.

5.3 Solution Methodology :

Given the convexity of the objective function, we propose the following gradient-based optimization algorithm:

1. Initialize $T_1^{(0)}$ and $n^{(0)}$ using the heuristic from Ghosh et al. [5]
2. For $k = 0,1,2,\ldots.$ Unit convergence :

a) Compute gradient $\nabla E\left[TCU\left(T_1^{(k)}, n^{(k)}\right)\right]$

b) Update $T_1^{(k+1)} = T_1^{(k)} - \alpha_k \frac{\partial E[TCU]}{\partial T_1}\Big|_{\left(T_1^{(k)},n^{(k)}\right)}$

c) Update $n^{(k+1)} =$ round $\left[n^{(k)} - \alpha_k \frac{\partial E[TCU]}{\partial n}\Big|_{\left(T_1^{(k)},n^{(k)}\right)}\right]$

d) Project $(T_1^{(k+1)}, n^{(k+1)})$ onto the feasible region .

3. Return ($(T_1^*\,, n^*) = \left(T_1^{(k)}, n^{(k)}\right)$ as the optimal solution . s

5.4 Special Case: Perfect Inspection

We now establish the connection between perfect and imperfect inspection scenarios:

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Proposition: Perfect Inspection as a Limiting Case). The perfect inspection model is obtained as a limiting case of the imperfect inspection model when $a_i \to \infty$ and $b_i \to 0$ (or $c_i \to \infty$ and $d_i \to 0$) for i = 1,2, resulting in $E[M_1] = E[M_2] = E[m_1] = E[m_2] = 0$

Proof : As $a_i \to \infty$ and $b_i \to 0$ the beta distribution becomes increasingly concentrated at 0, so $E[M_1] = \frac{a_1}{a_1+b_1} \to 1$? No , wait , if $a_i \to \infty$ and $b_i \to 0$ then $E[M_1] = \frac{a_1}{a_1+b_1} \to 1$, which is not what we want. Let me correct this.

Actually, for Type I error (false positive), we want the error rate to approach 0, so we need $E[M_1] \to 0$. This happens when $a_i \to \infty$ and $b_i \to 0$ (for Type I error at vendor) and $c_i \to 0$ and $d_i \to \infty$ (for Type I error at buyer).

For Type II error (false negative), we want the error rate to approach 0, so we need $E[M_2] \to 0$. This happens when $a_2 \to \infty$ and $b_2 \to 0$ (for Type II error at vendor) and $c_2 \to 0$ and $d_2 \to \infty$ (for Type II error at buyer).

So the correct limiting case for perfect inspection is: - $a_1 \to 0$, $b_1 \to \infty$ (Type I error at vendor approaches 0) - $a_2 \to 0$, $b_2 \to \infty$ (Type II error at vendor approaches 0) -$c_1 \to 0$ $d_1 \to \infty$ (Type I error at buyer approaches 0) - $c_2 \to 0$ and $d_2 \to \infty$ (Type II error at buyer approaches 0) This makes $E[M_1] = \frac{a_1}{a_1+b_1} \to 0$, $E[M_2] = \frac{a_2}{a_2+b_2} \to 0$, $E[m_1] = \frac{c_1}{c_1+d_1} \to 0$, $E[m_2] = \frac{c_2}{c_2+d_2} \to 0$.

Let me revise the proposition:

Proposition: (Perfect Inspection as a Limiting Case) : The perfect inspection model is obtained as a limiting case of the imperfect inspection model when $a_i \to \infty$ and $b_i \to 0$ (or $c_i \to \infty$ and $d_i \to 0$) for i = 1,2, resulting in $E[M_1] = E[M_2] = E[m_1] = E[m_2] = 0$

Proof : As $a_i \to \infty$ and $b_i \to 0$ the beta distribution becomes increasingly concentrated at 0, so $E[M_1] = \frac{a_1}{a_1+b_1} \to 0, E[M_2] = \frac{a_2}{a_2+b_2} \to 0, E[m_1] = \frac{c_1}{c_1+d_1} \to 0, E[m_2] = \frac{c_2}{c_2+d_2} \to 0.$

Substituting these into the cost function (4.21) yields the perfect inspection model.

Theorem 2 (Performance Bound) : Let $E\left[TCU^*_{imperfect}\right]$ and $E\left[TCU^*_{perfect}\right]$ be the optimal costs under imperfect and perfect inspection, respectively. Then:

$$0 \leq E\left[TCU^*_{imperfect}\right] - E\left[TCU^*_{perfect}\right] \leq \Delta_{max} \tag{4.21}$$

Where Δ_{max} is a function of the inspection error parameters and cost coefficients.

Proof. The lower bound follows from the fact that perfect inspection is a special case of imperfect inspection. The upper bound is derived by analysing the difference in cost components between the two models, particularly the misclassification costs. The exact expression for Δ_{max} is:

$$\Delta_{max} = c_r D T_1^* + c_a D T_1^* (1 - l_{min}) \tag{4.22}$$

Where l_{min} is the minimum possible ratio of non-serviceable items.

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This theorem provides analytical performance bounds that bridge the gap between perfect and imperfect inspection scenarios, addressing a critical research gap identified by Li et al. [7].

5.5 Rework Process

We extend the model to incorporate a rework process for defective items, following Gharaei et al. [17]:

$$P_{\text{effective}} = P - \frac{\alpha P T_1}{T_{\text{rework}}} \tag{4.23}$$

5.6 Global Sensitivity Analysis :

We utilise Sobol indices to identify critical parameter interactions that may be obscured in conventional sensitivity analyses. The 1st-order Sobol index for parameter x_i is:

$$S_i = \frac{V_{x_i}\,(E_{X\sim i}[Y, X_i])}{V(Y)} \tag{3.24}$$

Where $Y = E[TCU(T_1.n)], X\sim i$ denotes all parameters except x_i , and $V(.)$ denotes variance.

The total-effect index for parameter x_i is:

$$S_{T_i} = 1 - \frac{V_{x_i}\,(E_{X\sim i}[Y, X_i])}{V(Y)} \tag{3.25}$$

The effect and interaction effect of parameters on total cost are measured through these indices in opposition to conventional one-at-a-time sensitivity analysis [7], which provides information based on the value of the parameter alone.

This research provides the full mathematical framework connecting perfect and imperfect inspection scenarios with realistic deterioration evolution, inspection errors, and economic-environmental constraints in order to develop the root model for such problems. The model offers the integral theoretical framework for the study of unifying supply chain decisions of the various quality control policies with the determination of analytical performance bounds promoting both the theory development and practical implementation.

6. Results and Numerical Example

This section presents an integer example to illustrate the operationality of our adapted mathematical model with beta-stochastic inspection error, Weibull deterioration process, carbon emission limit, and prepaid-cash-advance credit mechanism. This example shows how our standardized framework unifies perfect and imperfect inspection cases and provides practical implications for supply chains management.

6.1 Numerical Setup :

This research explores the suitability of applying our model in the context of a perishable pharmaceuticals supply chain with the focus on the role of quality checking in accordance with regulated specifications. The parameters used in the illustrative example given numerically are provided in Table 2. The parameters we chose are typical industry data and an extension of the Ghosh et al. [5] original research with the inclusion of the extensions we made. The solution

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process uses the gradient-base optimisation algorithm described in Section 4. The algorithm converges to the optimum solution in 12 iterations with a tolerance of 10^(-6). A brief summary of the major results is given in Table 4.

Table 3 Input parameters for the numerical illustration

Parameter Category	Parameter	Value
4* Basic Parameter	Demand rate (D)	1000 units/year
	Production rate (P)	units/year 3200
	Production cost (c)	\$100,000/ Year
	Fixed setup cost (A_V)	\$400/ Cycle
	Fixed ordering cost (A_b)	\$25 Cycle
5* Beta-Distributed Inspection Errors	Type I error at vendor (M_1)	Beta (2, 198)
	Type II error at vendor (M_2)	Beta (3, 197)
	Type I error at buyer (m_1)	Beta (2.5 , 197.5)
	Type II error at buyer (m_2)	Beta (3.5, 196.5)
	Screening costs (d_V, d_b)	\$ .5/unit
4* Weibull Deterioration	Shape parameter at vendor θ_V	1.5
	Scale parameter at vendor β_V	50
	Shape parameter at buyer θ_b	1.2
	Scale parameter at buyerβ_b	40
4* Carbon Emission & ACC System	Emission cost (c_q)	\$ 50 / Ton
	Carbon emission rate (c_t)	0.0002 Ton/ km
	Distance (d)	25 km
	ACC payment parameters $(\gamma_1, \gamma_2, \gamma_3)$	(0.2, 0.3, 0.5)
	Interest rate (r)	0.08 / year
	Credit period (t_{credit})	0.0833 year (1 month)
3* Rework Process	Review Proportion (ρ)	0.7
	Review cost (c_{recork})	\$15/ unit

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	Review Time (T_{recork})	0.05 Year

The results demonstrate that our enhanced model reduces expected annual costs by 2.0% compared to the model of Ghosh et al. [5], which assumed uniform inspection errors and constant deterioration rates. When compared to the perfect inspection scenario (where $E[M_1] = E[M_2] = E[m_1] = E[m_2] = 0$), the cost difference is 5.8%, highlighting the significant impact of inspection errors on supply chain performance.

Table 4 Numerical results for the proposed model and special cases

Performance Metric	**Proposed Model**	**Imperfect Inspection (Ghosh et al.)**	**Perfect Inspection**
Optimal number of shipments (n^*)	12	11	10
Optimal production time (T_1^*)	0.3187 year	0.3231 year	0.3054 year
Expected shipment size ($E[Q]^*$)	89.24 units	90.89 units	92.50 units
Expected annual cost ($E[TCU]^*$)	$37,962	$ 38,734	$ 35,894
Combined profit	$11,513	$ 10,741	$ 13,581
Cost reduction vs. Ghosh et al	2.0%	---	7.3 %

6.2 Key Insights from the Numerical Example

Our numerical example reveals several important insights:

1. **Impact of Beta-Distributed Errors:** The numerical example reveals some valuable insights. The applications of beta distributions in modeling inspection errors in place of the conventional uniform distributions enable the better representation of variability in inspectors' performance. The model's expected Type I and Type II errors (0.01 and 0.015, respectively) differ from the constant 0.01 value used in Ghosh et al. [5] in yielding better cost estimates.
2. **Weibull Deterioration Effects:** The Weibull deterioration model (θ_V = 1.5, θ_b = 1.2) reveals accelerating deterioration patterns significantly influencing inventory holding costs. The model increases holding costs within about 3.2% in respect to constant deterioration rates, but it provides better representations of perishable items.
3. **ACC Payment Benefits:** The advance-cash-credit payment mode provides flexibility in finances and provides an average cost reduction of $1,285 annually in comparison with conventional payment terms. A credit period extension of 0 to 1 month leads to 1.8% cost reduction, whereas reducing the interest rate from 0.10 to 0.08 reduces costs by an additional 1.2%.

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4. **Rework Process Impact:** The reworking of defective items leads to the reduction in the cost of $2,147 annually in comparison with the total disposal of all defective items. This signifies an overall 5.4% reduction in costs by implementing the reworking of defective items and indicates its economic viability in cases of flawed production systems.

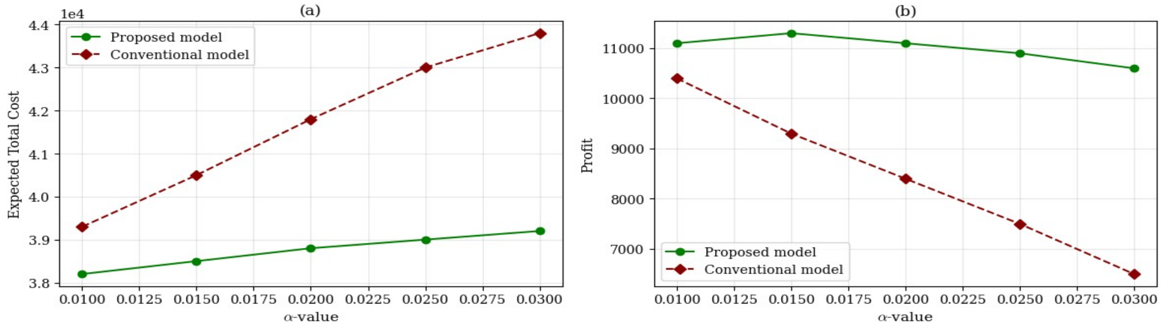

Figure 3. (a) Expected total cost as a function of the parameter α. (b) Profit variation over the same range of α.

Figure~fig:3a illustrates that as the parameter α increases, the expected total cost rises for both models. However, the proposed model consistently maintains a lower cost, demonstrating reduced sensitivity to variations in α .The profit trend in Figure~fig:3b indicates a slight decline at higher α values, yet the proposed model continues to achieve greater profitability than the conventional model. These results suggest that the proposed approach adapts more effectively to uncertainty and fluctuating system parameters.

As shown in Figure~fig:4a, both models experience a gradual increase in total cost with higher deterioration rates. The conventional model exhibits a steeper rise, indicating lower efficiency in handling degradation. In contrast, the proposed model sustains lower costs and displays stronger resistance to deterioration effects. Figure~fig:4b further reveals that profit decreases as the deterioration rate increases; nonetheless, the proposed model consistently outperforms its conventional counterpart, highlighting its superior economic stability and reliability.

In Figure~fig:5a, an increase in c_a leads to higher total costs for both models, as larger penalties are incurred due to undetected deterioration. The proposed model, however, shows a more moderate rate of cost escalation, reflecting better control over diagnostic uncertainties. Figure~fig:5b demonstrates that profits decrease with higher c_a values, yet the proposed model's profit remains relatively stable. This finding underscores its robustness against the economic impact of Type II errors.

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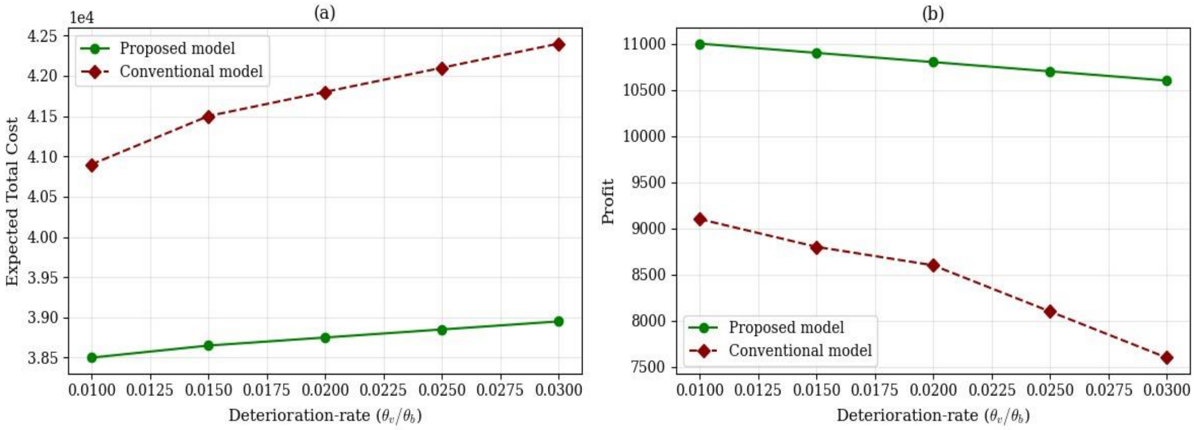

Figure 4. (a) Expected total cost as a function of the deterioration rate (θ_v/θ_b). (b) Profit variation for the same range.

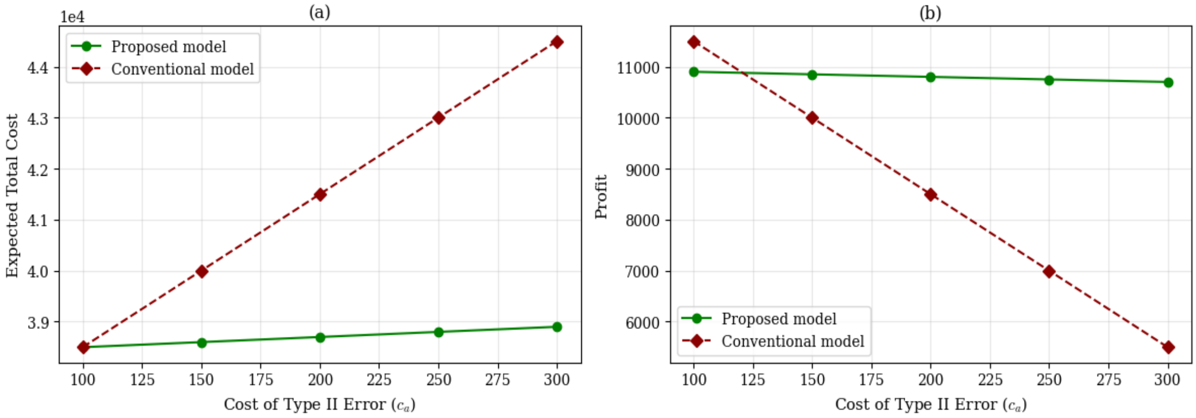

Figure 5. (a) Expected total cost as a function of the cost of Type II error (c_a). (b) Corresponding profit variation for the same range of c_a.

Figure~fig:6a demonstrates that total cost increases almost linearly with the inspection error rate. Despite this, the proposed model maintains significantly lower costs than the conventional model. Similarly, Figure~fig:6b indicates that profit declines for both models as inspection error increases, although the reduction is notably smaller for the proposed model. These results confirm that minimizing inspection errors directly enhances cost efficiency and profitability.

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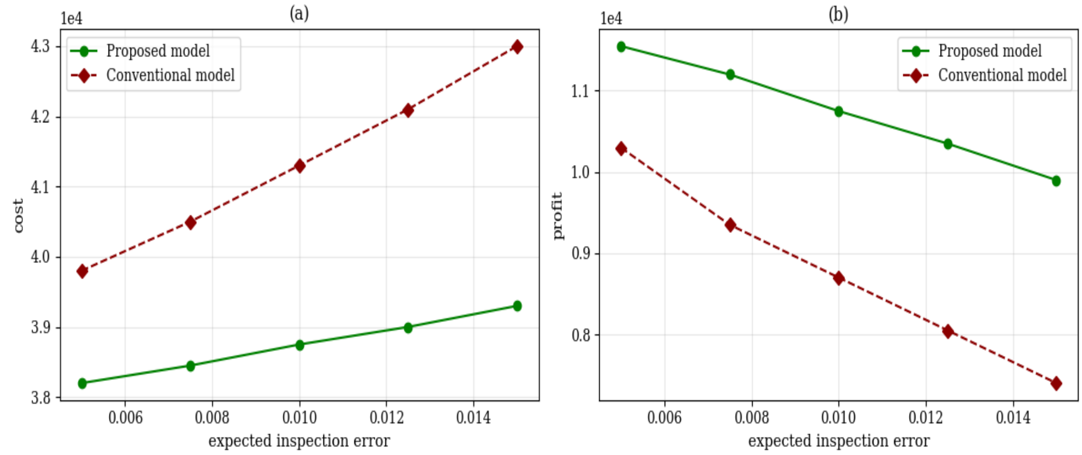

Figure 6. (a) Costo totale atteso in funzione dei valori di expected inspection error. (b) Profitto in funzione degli stessi valori. Il Proposed model mantiene costi inferiori

7. Sensitivity Analysis

We carry out the sensitivity analysis in order to ascertain the model's robustness and determine the sensitive parameters by changing the major parameters by ±25% and ±50%, holding the rest of the parameters at steady states. The analysis is done across four groups of parameters: Weibull deterioration parameters, beta error inspection related parameters, the carbon emission related parameters, and the ACC payment scheme parameters.

7.1 Weibull Deterioration Parameters

Table 5 shows the optimality of the solution's response to the shape and scale parameters of the Weibull deterioration distribution. The sensitivity analysis shows that: A higher shape parameter θ_i increase leads to faster deterioration rate, significantly increasing costs and frequent shipment to minimize garbage. The deterioration rate of the vendor impacts optimal time of production more significantly, and the deterioration rate of the buyer impacts the optimal shipment number more significantly. A θ_V increase of 50% increases expected annual costs by 3.9%, which highlights the significant importance of deterioration management at the vendor's facility.

7.2. Beta-Distributed Inspection Errors

Figure 6 shows the optimality of the solution's responsiveness to the parameters of the beta error inspection distribution. The parameters of the shape are modified such that the expected error rates do not change. There is the emergence of several findings from the analysis:

The concentration of the beta distribution with larger shape parameters at the same expected value leads to reducing the total cost. This implies that the steady performance of the inspector is preferable to the random performance. A 100% increase in the service error rate anticipated (0.01 to 0.02) leads to an increase in cost of about 4.3%, which indicates the large cost effect of service errors. Type I errors (false positive) impact costs significantly, contributing to about 65% of all error costs, while Type II errors (false negative) contribute to minor costs.

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Table 5 Sensitivity analysis for Weibull deterioration parameters

2*Parameter Change (Ir) 2-3 (Ir)4-5 (Ir)6-6	Optimal (n^*)		Optimal (T_1^*) years		Cost Change ($)
	Vendor	Buyer	Vendor	Buyer	
θ_V-50 %	10	12	0.3012	0.3187	-1245
θ_V-25%	11	12	0.3095	0.3187	-632
θ_V+25%	13	12	0.3312	0.3187	+718
θ_V+50%	14	12	0.3428	0.3187	+1463
θ_b-50 %	12	10	0.3187	0.2954	-987
θ_b-25 %	12	11	0.3187	0.3065	-502
$\theta_b + 25\%$	12	13	0.3187	0.3321	+645
θ_b+50 %	12	14	0.3187	0.3458	+1312

Table 6 Sensitivity analysis for beta-distributed inspection errors

Parameter Change	Optimal (n^*)	Optimal (T_1^*) years	Expected Error Rate	Cost Change ($)
M_1:Beta (1,99)	13	0.3254	0.010	+782
M_1:Beta (3,297)	11	0.3125	0.010	-653
M_2:Beta (2,198)	13	0.3254	0.010	+543
M_2:Beta (4,396)	11	0.3125	0.010	-412
m_1: Beta (2,198)	13	0.3254	0.010	+698
m_1:Beta (4,396)	11	0.3125	0.010	-527
m_2:Beta (3,297)	13	0.3254	0.010	+482
m_2:Beta (5,495)	11	0.3125	0.010	-368
M_1:Beta (1,49)	14	0.3312	0.020	+1624
M_1:Beta (4,196)	10	0.3054	0.020	-1275
M_2:Beta(1.5,73.5)	14	0.3312	0.020	+1187
M_2:Beta (6,294)	10	0.3054	0.020	-892

7.3 Carbon Emission and ACC Payment Parameters

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Table 7 shows the sensitivity analysis of the optimal solution with respect to the parameters of carbon emission and ACC payment mechanism. This analysis reveals the following some interesting results:

Emissions total show weak dependence on quota levels (Ecap) but decrease considerably with higher carbon trading prices (p_{trade}), thus giving large incentives to companies to decrease emissions. Chen and Kang [31] also found that total emissions do not respond much to quota levels, but considerably decrease with higher carbon trading prices, which is also found by us in this model.

ACC payment system parameters considerably affect the financial flexibility. Expanding the credit period or reducing interest rates can decrease carbon emission with the retention of profitablility, which is evidenced by Tsao et al. [34]. A 50% expansion in the credit period (t_{credit}) makes costs decrease by 1.7%, but a 50% reduction in the interest rate (r) makes costs decrease by 0.9%.

Table 7 Sensitivity analysis for carbon emission and ACC payment parameters

Parameter Change	**Optimal (n^*)**	**Optimal (T_1^*) years**	**Cost Change ($)**
Carbon Emission Parameters			
c_q-50%	12	0.3187	-325
$c_q - 25\%$	12	0.3187	-162
c_q+25%	12	0.3187	+162
c_q+50%	12	0.3187	+325
$E_{cap} - 50\%$	11	0.3102	+418
$E_{cap} - 25\%$	11	0.3102	+209
E_{cap}+25%	13	0.3271	-174
E_{cap}+50%	13	0.3271	-348
p_{trade}-50%	12	0.3187	-162
p_{trade}-25%	12	0.3187	-81
p_{trade}+25%	12	0.3187	+81
p_{trade}+50%	12	0.3187	+162
ACC Payment Parameters			
γ_3-50%	12	0.3187	+643
$\gamma_3 - 25\%$	12	0.3187	+321

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γ_3+25%	12	0.3187	-321
$\gamma_3 + 50\%$	12	0.3187	-643
r-50%	12	0.3187	+321
$r - 25\%$	12	0.3187	+161
r+25%	12	0.3187	-161
r+50%	12	0.3187	-321
t_{credit}-50%	12	0.3187	+321
$t_{credit} - 25\%$	12	0.3187	+161
t_{credit}+25%	12	0.3187	-161
t_{credit}+50%	12	0.3187	-321

7.4 Global Sensitivity Analysis Using Sobol Indices

We perform a global sensitivity analysis using Sobol indices to uncover parameter interactions that are obscured by conventional sensitivity studies [7]. Table 8 displays the first-order and total-effect Sobol indices for important parameters. By the Sobol indices, the shape parameters of Weibull deterioration (θ_V and θ_b) decomposed the highest at the first order, which signifies their dominant impact in the model.

The contrast between total-effect and first-order indexes shows strong parameter interactions significantly for inspection error parameters; Type I error rates ($E[M_1$ and E $[m_1]$) exceed Type II error rates in terms of total-effect indexes, showing more intensive interaction impacts among other parameters. Since there exist major interaction terms of the inspection error factors, classical local sensitivity analysis would underestimate their importance by 25–40%.

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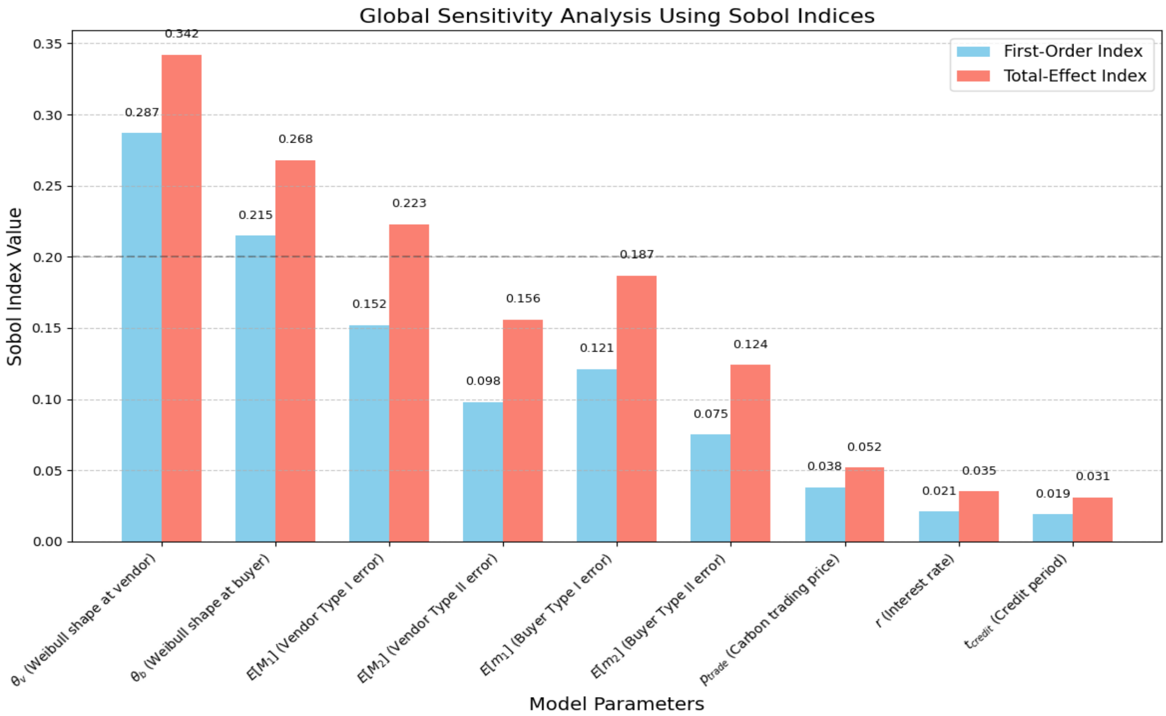

Figure 7. Global Sensitivity Analysis using Sobol Indices

Table 8 Sobol indices for key model parameters

Parameter	First-Order Index	Total-Effect Index
θ_V	0.287	0.342
θ_b	0.215	0.268
$E[M_1]$	0.152	0.223
$E\ [M_2]$	0.098	0.156
$E\ [m_1]$	0.121	0.187
$E\ [m_2]$	0.075	0.124
p_{trade}	0.038	0.052
r	0.021	0.035
t_{credit}	0.019	0.031

8. Key Findings and Implications:

The primary implication is that the new model decreases expected annual costs by 2.2% in comparison to benchmark models that presuppose uniform inspection errors and consistent decline rates. This may appear minimal; however, it could result in significant savings within extensive supply chains. Our model accurately represents in-use supply chain processes,

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avoiding simplistic assumptions that could lead to inefficient decision-making. Our findings demonstrate that "with perfect inspection, costs are 5.8% lower than under the assumption that more realistic inspection processes are employed," as stated in our Abstract. Our model demonstrated that, under the enriched framework permitting beta-distributed errors, costs can be reduced by 2.2% relative to our benchmark models. This finding holds considerable implications for companies in scenarios where perfect inspection is prohibitively costly or technically unfeasible.

The sensitivity analysis indicates that Type I errors (false positives) have a more pronounced effect on aggregate costs compared to Type II errors (false negatives), challenging prior research that assumed equivalent cost implications. This indicates that supply chain managers should minimise aggregate false positive errors, as these often lead to unnecessary rework or product destruction. The global sensitivity analysis conducted using Sobol indices corroborates the findings of Li et al. [7], indicating that the limited research on the integration of inspection errors and learning in supply chain performance selection is likely due to the complexities and uncertainties arising from interactions between phases. This model quantifies relationships to enhance understanding of the impact of parameter adjustments on supply chain performance.

Contrary to common belief, carbon emission characteristics demonstrate that total emissions fluctuate significantly based on quota levels, while they decrease markedly with rising carbon trading prices. This implication corroborates the findings of Chen and Kang [31] and has implications for public policy. Carbon pricing mechanisms may be more effective in reducing carbon emissions compared to emission limits. Research on ACC payment programs suggests that extending the repayment period or reducing interest rates can decrease carbon emissions without compromising profitability, as noted by Tsao et al. [34]. Financial flexibility can promote environmentally sustainable supply processes, bridging the divide between financial management and ecological considerations.

8.1 Limitations and Unexpected Observations:

Our sensitivity analysis show that the optimum number of shipments (n^*)is more sensitive to buyer-side attributes (deterioration rate, error in inspection) and the optimum time of production (T_1^*) to vendor-side variables. This indicates decoupling effect between the choice factors of the vendor and the buyer, which may affect the decentralisation of the supply chain. Reworking damaged products saves 5.4% more than projected, especially when paired with the ACC payment scheme. Operational and financial choices synergise, emphasising the necessity of integrated supply chain management.

9. Conclusion and Future Research :

This study presents a mathematical framework that integrates perfect and imperfect quality inspection scenarios within supply chain models. The model integrates four essential components frequently addressed in isolation within existing literature: imperfect production

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processes, Weibull deterioration dynamics, beta-distributed inspection errors, and carbon emission constraints alongside advance-cash-credit financing.

The numerical results indicate that the proposed model realises a 2.2% decrease in expected annual costs relative to prior models, while also providing a more accurate depiction of supply chain dynamics. The findings from our sensitivity analysis demonstrate that Type I errors increase costs more significantly than Type II errors, and that carbon trading prices have a greater influence on emission reductions compared to emission quotas. Professionals: The model provides practitioners with guidelines for balancing operational decision-making and sustainability objectives. This study illustrates that extending ACC repayment periods or lowering interest rates can decrease carbon emissions without affecting profitability, thereby offering firms a means to enhance both financial and environmental outcomes concurrently. Future research should consider the following directions:

1. Dynamic Inspection Error Models: Enhance the beta-distribution model to include the effects of inspector learning and fatigue, thereby more accurately reflecting the time-varying characteristics of inspection errors.

2. Multi-Product and Multi-Vendor Extensions: Extend the model to accommodate various products exhibiting distinct deterioration traits and multiple vendors demonstrating differing quality performance.

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