

**PARTIAL HUB SETS AND PARTIAL HUB NUMBER IN
GRAPHS**

Shama K.S.¹, Ramakrishnan T.V.², Divya P.M.³

Department of Mathematical Sciences

Kannur University Campus,

Mangattuparamba P.O.

Kannur, Pin-670567, Kerala, INDIA

e-mail: shamaks1551@gmail.com¹

ramakrishnantv@kannuruniv.ac.in²

divyapmadhavan@gmail.com³

Abstract

A partial hub set in a graph $G = (V, E)$ is a set $S \subseteq V$ such that for any two vertices $u, v \notin S$, either u and v are adjacent or there exists a path from u to v such that at least half of the internal vertices of the path belong to S . The minimum cardinality of such a set is called the partial hub number of G , denoted by $h_p(G)$. In this paper, we determine the partial hub number for various standard graphs, examine how vertex contraction affects the partial hub number, and explore bounds relating the hub number of a graph, the partial hub number and other graph parameters.

Math. Subject Classification: 05C40, 05C69, 05C99

Key Words and Phrases: Partial hub set, partial hub number, vertex contraction, connectivity.

1. Introduction

In 2006, Matthew Walsh [8] introduced the concept of the hub number of a graph. Subsequently, Grauman et al. [12] explored the relationship between the hub number, connected hub number, and the connected domination number of a graph. Cuaresma et al. [3] determined the hub numbers for joins, coronas, and Cartesian products of two connected graphs. Shadi Ibrahim Khalaf et al. [10] studied the hub, total hub number and global hub number of graphs. Ahmed M Nour et al. [1] made studies on restrained hub number in graphs.

In the case of a path graph of order n , the hub number is $n - 2$, which is comparatively large. This observation prompted the investigation of a smaller vertex subset that could still maintain a relaxed form of the hub set property. Specifically, we considered sets in which any two non-adjacent vertices outside the set are connected by a path such that more than half of the internal vertices along the path belong to the set. This line of inquiry led to the introduction

of the concept of a partial hub set, which seeks to reduce the size of the set required while preserving essential structural connectivity. A partial hub set is a relaxation of hub set in graphs. The concept of partial hub sets is useful while working with large graphs where a hub set may be too large to use. It is useful in various applications such as network design and optimization, fault-tolerant routing, social network analysis, graph compression and visualization, etc.

2. Preliminaries

By a graph $G = (V, E)$, we mean a finite, simple, and undirected graph with no loops or multiple edges. For standard graph-theoretic terminology, we refer the reader to [6] and [7]. The open neighborhood of a vertex v in G , denoted by $N(v)$, is the set of vertices adjacent to v while the closed neighborhood is defined as $N[v] = N(v) \cup \{v\}$.

The corona $G \odot F$ of two graphs G (with n_1 vertices and m_1 edges) and F (with n_2 vertices and m_2 edges) is defined as the graph obtained by taking one copy of G and n_1 copies of F , and then joining the i -th vertex of G with an edge to every vertex in the i -th copy of F .

A vertex cover of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that every edge $e = (u, v) \in E$ has at least one of its endpoints in C . The vertex cover number $\tau(G)$ is the size of the minimum vertex cover.

A dominating set of a graph G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G . [11]

Let v be a vertex in G . The contraction of v in G , denoted by G/v , is the graph obtained by deleting v and adding edges so that the open neighborhood $N(v)$ forms a clique. If $S \subseteq V(G)$, then G/S is obtained by contracting the vertices in S . An S -path between x and y is a path where all intermediate vertices are from S . A subset $S \subseteq V$ is called a hub set of G , if for any two distinct vertices $u, v \in V \setminus S$, either u and v are adjacent or there exists a $u - v$ path P in G such that all the internal vertices of P belong to S . The hub number of G , denoted by $h(G)$, is the minimum cardinality of a hub set in G . [8]

The following are some of the results we used for the references:

Theorem 1: If $G = P_n$, then $h(P_n) = n - 2$.

Theorem 2: [8] If $G = C_n$ then $h(C_n) = n - 3$.

Theorem 3: [8] Let $S \subseteq V(G)$. Then G/S is complete if and only if S is a hub set of G .

Theorem 4: [8] Let T be a tree with n vertices and l leaves. Then $h(G) = n - l$.

Theorem 5: [3] For any connected graphs G and F such that $|V(G)| \geq 2$, then $h(G \odot F) = |V(G)|$.

3. Partial hub sets and partial hub number of graphs

DEFINITION 3.1. Let $G = (V, E)$ be a graph and $S \subseteq V(G)$. Let $x, y \in V(G)$. An S_p -path from x to y is a path P such that at least half of the internal vertices belong to S . That is, if $P =$

$(x, v_1, v_2, \dots, v_k, y)$, where $k \geq 1$, is an S_p -path, then $|\{v_1, v_2, \dots, v_k\} \cap S| \geq \lceil \frac{k}{2} \rceil$.

An S_p -path is said to be proper if there exists at least one internal vertex, say v_i , that lies outside S .

DEFINITION 3.2. Let $G = (V, E)$ be a graph. $S \subseteq V(G)$ is a partial hub set of G if for any $u, v \in V - S$, either u and v are adjacent in G or there exists an S_p -path from u to v in G . The minimum cardinality of S is called the partial hub number of G , denoted by $h_p(G)$.

Observations:

- Every hub set is a partial hub set.
- If S is a partial hub set, then any super set $S' \supseteq S$ is also a partial hub set.
- If $u \in V(G)$ is an isolated vertex then $u \in S$.
- If S is a hub set of G and $S' \subseteq S$ is such that $|S'| = h_p(G)$, then S' need not be a partial hub set of G .

Throughout this paper we consider simple graphs without isolated vertices only.

THEOREM 3.3. For a graph G , $h_p(G) = h(G) = 0$ if and only if G is complete.

THEOREM 3.4. If $G = P_n$ and $n > 2$, then $h_p(P_n) = \lceil \frac{(n-2)}{2} \rceil$.

Proof. For any $n > 2$, the path graph P_n has $n - 2$ internal vertices, hence $h_p(P_n) = \lceil \frac{(n-2)}{2} \rceil$.

THEOREM 3.5. If $G = C_n$ and $n > 2$, then $h_p(C_n) = \lceil \frac{(n-3)}{2} \rceil$.

Proof. For $n = 3$, C_3 is complete and hence $h_p(C_3) = 0$. For $n > 3$, let C_n be the cycle $(v_1, v_2, \dots, v_n, v_1)$. Note that $h(C_n) = n - 3$ and $S' = V(C_n) - \{v_1, v_2, v_3\}$ forms a minimum hub set for C_n . Define $S = \{v_4, v_6, \dots, v_n\}$, if n is even and $S = \{v_4, v_6, \dots, v_{n-1}\}$, if n is odd. Clearly $|S| = \lceil \frac{(n-3)}{2} \rceil$. Now consider the vertices $v_1, v_2, v_3 \notin V - S$, where $v_1 v_2$ and $v_2 v_3$ are edges. For the pair (v_1, v_3) , there exists an S_p -path with $n - 3$ internal vertices, among which $\lceil \frac{(n-3)}{2} \rceil$ vertices belong to S . Now, for any non-adjacent pair of vertices (v_i, v_j) , $i < j$, in $V - S$, there exist two paths: $P_1 = (v_i v_{i+1} \dots v_j)$ and $P_2 = (v_j v_{j+1} \dots v_n v_1 v_2 \dots v_i)$, such that at least one of these paths is a minimum S_p -path. Hence S is a partial hub set and $h_p(C_n) = \lceil \frac{(n-3)}{2} \rceil$.

THEOREM 3.6. For any graph G of order n , $h_p(G) = n - 1$ if and only if $G \cong \bar{G}(K_n)$.

THEOREM 3.7. Let $G = K_{n_1, n_2, \dots, n_r}$, with $r \geq 3$, then

$$h_p(G) = h(G) = \begin{cases} 0, & \text{if } n_i = 1 \text{ for all } i \\ 1, & \text{if } n_i \leq 2 \text{ for some } i \\ 2, & \text{if } n_i > 2 \text{ for all } i \end{cases}$$

THEOREM 3.8. Let P_n be the path graph and G be any connected graph.

$$\text{Then } h_p(P_n \odot G) = \begin{cases} 0, & \text{if } n = 1 \text{ and } G \text{ is complete} \\ n - \lfloor \frac{n}{3} \rfloor, & \text{if } n \geq 2 \text{ and } G \text{ is complete} \\ n, & \text{if } G \text{ is not complete and } |V(G)| > 1 \end{cases}$$

Proof: If $n = 1$ and G is a complete graph, then $P_n \odot G$ will be complete and $h_p(P_n \odot G) = 0$. Now suppose $n \geq 2$ and G is a complete graph. Let $P_n = (v_1, v_2, \dots, v_n)$ and G_1, G_2, \dots, G_n be the n copies of G incident to v_1, v_2, \dots, v_n respectively. Take $S = V(P_n) - \{v_1, v_4, v_7 \dots v_k\}$, where

$$k = \begin{cases} n - 2, & \text{if } n \equiv 0 \pmod{3} \\ n, & \text{if } n \equiv 1 \pmod{3} \\ n - 1, & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Then $|S| = n - \lfloor \frac{n}{3} \rfloor$. Also note that S is a partial hub set of P_n . Next, we will prove that S is a partial hub set of $P_n \odot G$. Let $u, v \in P_n \odot G$. We now consider the following cases.

Case 1: If $u, v \in G_i$ for some i , then they are adjacent in $P_n \odot G$.

Case 2: If $u, v \in V(P_n) - S$, let $u = v_i$ and $v = v_j$, where $i < j$, then either they are adjacent or there exists an S_p -path $(v_i v_{i+1} \dots v_j)$.

Case 3: If $u \in G_i$ and $v \in G_j$ where $i < j$, then we have the path $u, v_i, v_{i+1}, \dots, v_{j-1}, v_j, v$ from u to v which has $(j - i + 1)$ internal vertices. This leads to the following sub cases.

Sub case 1: If $i = 1$ and $j = n$, then the S_p -path will be the path P_n , where all except $\lfloor \frac{n}{3} \rfloor$ internal vertices are from S .

Sub case 2: If $1 \neq i < j \neq n$ or $i = 1$ and $j \neq n$ or $i \neq 1$ and $j = n$, there are S_p -paths with at least half of internal vertices from S . Hence S is a partial hub set.

To prove S is minimum, since if we take $S' = S - \{v_i\}$, where $1 \leq i \leq n$, then either $v_i v_{i+1}$ or $v_{i-1} v_i$ will be an edge lying outside S . Hence for $u \in G_i, v \in G_{i+1}$ or for $u \in G_{i-1}, v \in G_i$, there does not exist any S_p -path. Thus, we can conclude that S is a minimum partial hub set of $P_n \odot G$.

Finally, suppose G is any connected graph with $|V(G)| > 1$ and let $S = V(P_n)$. Recall that $h(P_n \odot G) = n$ and for any two non-adjacent vertices $u, v \in G_i$, the only S path is $u - v_i - v$ for all $i = 1, 2, \dots, n$. Hence S is a minimum hub set and $h_p = n$.

COROLLARY 3.9. If G is any connected graph and H complete, then

$$h_p(G \odot H) < |V(G)|.$$

COROLLARY 3.10. If G and H are connected graphs and H is not complete with $|V(H)| > 1$, then $h_p(G \odot H) = |V(G)|$.

THEOREM 3.11. Let G be a disconnected graph with components G_1, G_2, \dots, G_n . Then $h_p(G) = \text{Min}\{h_j\}$ where $h_j = \sum_{i=1, i \neq j}^n |G_i| + h_p(G_j), j = 1, 2, \dots, n$.

Proof. Let G_1, G_2, \dots, G_n be the components of G and let S be a partial hubset of G . Let $x, y \in V - S$. If x and y belong to two different components. Then there will be no path between them. Hence G/S become disconnected contradicts that S is a partial hub set of G . This implies $V(G_i) \subseteq S$ for all i except one say j . Also S contains the vertices of any minimum partial hub set of G_j . Hence $h_p(G) = \text{Min}\{h_j\}$.

THEOREM 3.12. If T and U are two components of a graph $G - x$ and S is partial hub set of G , such that $x \notin S$, then either $N(x) \cap T \subseteq S$ or $N(x) \cap U \subseteq S$.

Proof. Suppose x is a cut vertex of G and S be a partial hub set of G such that

$x \notin S$. Let $u \in T$ and $v \in U$ such that $u, v \notin S$. But since x is a cut vertex every path between u and v contains x . Also, if $a \in N(x) \cap T$ and $b \in N(x) \cap U$, then the only path between a and b is $a - x - b$, which implies that either $a \in S$ or $b \in S$. If there exist $a \in N(x) \cap T$ and $b \in N(x) \cap U$ such that $a, b \notin S$, there will not be an S_p -path between them. Hence either $N(x) \cap T \subseteq S$ or $N(x) \cap U \subseteq S$.

COROLLARY 3.13. Let x be a cut vertex of a graph G such that $x \notin S$ and G_1, G_2, \dots, G_k be the components of $G - x$. Then $N(x) \cap V(G_i) \subseteq S$ for all i except possibly one.

Proof. Suppose x is a cut vertex of a graph G , S is a partial hub set of G and let G_1, G_2, \dots, G_k be the components of $G - x$. If $k = 2$ by Theorem 3.12, either $N(x) \cap V(G_1) \subseteq S$ or $N(x) \cap V(G_2) \subseteq S$. If $k > 2$ then if u and v are any two vertices from different components which belong to $N(x)$ then we will have the above situation. Thus, there will be just one component G_j which may not hold the above condition.

COROLLARY 3.14. Let S be a partial hubset of G and C be a cut set in

G which separates the vertex set as T and U , and let $S \cap C = \phi$. Then either $N(C) \cap T \subseteq S$ or $N(C) \cap U \subseteq S$.

Proof. Let $S \cap C = \phi$ and let $u \in T$ and $v \in U$. Since S is a partial hub set there exists an S_p -path from u to v and at least one vertex from C must be in that path. Since C is a cut set, there exists $a \in T$ and $b \in U$ such that $a, b \in N(C)$. But since a and b are non-adjacent and there does not exist an S_p -path from a to b , then either $N(C) \cap T \subseteq S$ or $N(C) \cap U \subseteq S$.

THEOREM 3.15. Let G be a connected graph, then $h_p(G) \leq \tau(G)$.

Proof. Let G be a connected graph and C be a minimum vertex cover of G . If $u, v \notin C$, then by the property of a vertex cover u and v are non-adjacent in G . Since G is connected there exists

a path say $u = v_0, v_1, \dots, v_{k-1}, v_k = v$ from u to v in G . Clearly $v_1, v_k \in C$. Again, using vertex cover property each edge of the path has at least one end point in C . Thus for each i , at least one of v_i or v_{i+1} must be in C . This ensures that at least half of the internal vertices of the path belong to C . Hence C is a partial hub set of G . Therefore $h_p(G) \leq \tau(G)$.

The above theorem says that every vertex cover of a connected graph is a partial hub set of that graph. But the converse is not true.

4. Relation with other graph parameters

Vertex contraction can reduce the partial hub number by creating shortcuts and simplifying connectivity. Generally, $h_p(G/v) \leq h_p(G)$. The following result is straight forward and can be easily verified.

THEOREM 4.1.

- a) If $n > 2$ and v is any vertex in P_n , $h_p(P_n/v) = \left\lfloor \frac{(n-3)}{2} \right\rfloor$.
- b) If $n > 3$ and v is any vertex in C_n , $h_p(P_n/v) = \left\lfloor \frac{(n-4)}{2} \right\rfloor$.

THEOREM 4.2. Let $G = (V, E)$ be a connected graph. Then $h_p(G) = 1$ and $S = \{u\}$ is a partial hubset of G , if and only if G satisfies one of the following conditions.

- a) $h(G/u) = 0$
- b) $h(G/v) = 1$, for some vertex v in G .
- c) $V - N[u]$ induces a clique and for any $v \in N(u), w \in V - N(u)$ either v and w are adjacent or there exists an S_p -path of length at most three from v to w .

Proof.

If $G \cong K_{1,n}$ then $h_p(G) = h(G) = 1$.

Let $h_p(G) = 1$ and $S = \{u\}$ be a partial hub set, then we have the following three cases based on the hub number of G .

Case 1: If $h(G) = 1$, then a spanning subgraph of G will be isomorphic to $K_{1,n}$. Hence G/u becomes complete and therefore

$$h(G/u) = 0.$$

Case 2: Suppose $h(G) = 2$, let $S' = \{u, v\}$ be a hub set of G . Then clearly $N(v)$ is adjacent to u in G/v . Hence for any two vertices $x, y \notin S$ either x and y are adjacent or there exists a path $x - u - y$ in G/v , proving that $\{u\}$ is a hub set of G/v . Hence $h(G/v) = 1$.

Case 3: Suppose $h(G) > 2$ and S' is a hubset of G . Consider $v \in N(u)$ and $w \in V - N[u]$. If v and w are not adjacent, then there must be an S_p -path between them and any S_p -path from v to w will be of at most length three. Otherwise, $h_p(G) > 1$. Also for any vertices $x, y \in V -$

$N[u]$, any S' -path from x to y will be of length at least four which is not possible. Hence x and y are adjacent and $V - N[u]$ induces a clique in G .

For the Converse part we have the following three cases.

Case 1: Suppose that $h(G/u) = 0$ then clearly G/u is complete and hence a spanning subgraph of $G \cong K_{1,n}$. Which implies $h_p(G) = 1$.

Case 2: Suppose there exists $v \in V(G)$ such that $h(G/v) = 1$. Let $S' = \{u\}$ be a hubset of G/v . Take $S = \{u, v\}$. We will prove that S' is a partial hub set of G . Let $x, y \notin S$ be non-adjacent in G then either they are adjacent in G/v or non-adjacent in G/v . If x and y are adjacent in G/v then there exist a path $x - v - y$ in G . Suppose x and y are non-adjacent in G/v then since S is a hub set of G/v there exist a path $x - u - y$ which gives an S'_p -path for x and y . Hence S' is a partial hub set of G which is minimum.

Case 3: Suppose that $V - N[u]$ induces a clique and for any $v \in N(u), w \in V - N(u)$ either v and w are adjacent or there exists an S_p -path of length at most three from v to w . Then $S = \{u\}$ will be a partial hubset of G .

THEOREM 4.3. If S is a partial hub set of $G = (V, E)$, then $h(G/S) \leq h(G) - h_p(G)$.

Proof. Let S be a partial hub set of G and let $S' = S \cup S_1$ be a hubset of G . Now consider G/S , since S' is a hub set of G , for any non-adjacent vertices $x, y \in V - S'$ in G , there exists a path P whose internal vertices are from S and S_1 . Hence there exists a path P' in G/S from u to v which contains all the internal vertices from S_1 . Which implies S_1 is a hubset of G/S . Thus $h(G/S) \leq |S_1| = h(G) - h_p(G)$.

THEOREM 4.4. Let $S \subseteq V(G)$ be a partial hub set of a graph G , and let $x \in S$. Then $S - \{x\}$ forms a partial hub set of the contracted graph G/x if and only if every S_p -path in G that includes x contains strictly more than half of its internal vertices in S .

Proof. Let S be a partial hub set of G and $x \in S$. Suppose $S - \{x\}$ is a partial hub set of G/x . For non-adjacent vertices $a, b \notin S - \{x\}$, if x is a vertex of an S_p -path P of length three from a to b , then the path will be of the form say $a - x - y - b$ where $x \in S$ and $y \notin S$ in G . On contracting x , P becomes the path $a - y - b$ in G/x , where $y \notin S - \{x\}$ and a, b are non-adjacent in G/x . Hence $S - \{x\}$ is not a partial hub set of G/x . We can extend this case for any S_p -path P containing x , only if P contain strictly more than half of its internal vertices in S .

Conversely if $x \in P$, where P is an S_p -path containing exactly half of its internal vertices, then clearly $P - \{x\}$ becomes a path containing less than half of its internal vertices in $S - \{x\}$. Hence $S - \{x\}$ is not a partial hub set of G/x .

THEOREM 4.5. Let S be a partial hub set of a connected graph G , then $d(G/S) \leq \lceil d(G)/2 \rceil$.

Proof. Let S be a partial hub set of G and let $d(G/S) > \lfloor d(G)/2 \rfloor$. Suppose $x, y \notin S$ is such that $d(x, y) \geq \lfloor d(G)/2 \rfloor + 1$ in G/S . Note that each single vertex contraction reduces the diameter at most one. But because S is a partial hub set there does not exist such a path, since then $d(x, y) > d(G)$ in G . Thus $d(G/S) \leq \lfloor d(G)/2 \rfloor$.

COROLLARY 4.6. Let G be a connected graph, then $h_p(G) \geq \lfloor d(G)/2 \rfloor$.

Proof. Let S be a partial hub set of a connected graph G . By above theorem $d(G/S) \leq \lfloor d(G)/2 \rfloor$ and by Walsh every single vertex contraction decreases the diameter by at most one. So, we need at least $\lfloor d(G)/2 \rfloor$ vertex contractions to reach the diameter of G/S . Therefore $h_p(G) \geq \lfloor d(G)/2 \rfloor$.

The following theorem provides an example of a connected graph with an arbitrarily large hub number, while its partial hub number remains equal to one.

THEOREM 4.7. For any positive integer n there exists a connected graph G with $h(G) = n$ and $h_p(G) = 1$.

Proof. For $n = 1$, let $G = K_{1,n}$ then $h(G) = h_p(G) = 1$.

For $n \geq 2$, let G be a graph with vertex set $V = \{a_i, b_i : 1 \leq i \leq n\}$ and edge set $E = \{a_1 b_1, a_1 a_i, a_j b_j, b_i b_j : 2 \leq i, j \leq n\}$. Thus $|V| = 2n$ and $\{b_i : 2 \leq i \leq n\}$ induces a clique in G . Also note that G is connected. Now let $S' = \{a_1, a_2, \dots, a_n\}$. Since G/S' is complete, S' is a hub set of G . S' is minimal, as the removal of any vertex from S' will not make G/S' complete. Further $h(G) = n$, since any minimum hub set will contain either all a_i 's or all b_i 's. Take $S = \{a_1\}$. We will show that S is a partial hub set. For any pair $a_i, b_j \notin S$ there exists a path $a_i - a_1 - a_j$. All the pairs b_i, b_j where $i \neq j \neq 2$ are adjacent and all a_i, b_i are adjacent. For the pair $a_i, b_j, i \neq j$, the S_p -path is $a_i - a_1 - a_j - b_j$. Finally, for b_1, a_i and b_1, b_j the paths are $b_1 - a_1 - a_i$ and $b_1 - a_1 - a_j - b_j$. Hence $h_p(G) = 1$.

The following theorem presents an algorithm for constructing a partial hub set from a given connected hub set.

THEOREM 4.8. Let S be a minimum connected hub set of G and let $v \in S$ such that $N(v) \subseteq S$, then $S - \{v\}$ is a partial hubset of G and $h_p(G) \leq h_c(G) - 1$, where $h_c(G)$ is the connected hub number of G .

Proof. Let S be a minimum hubset of G with the given conditions. Then $N(v) \cap (V - S) = \emptyset$. Let $S' = S - \{v\}$ and $u \in V - S$. Then $u, v \notin S'$ and since $N(v) \subseteq S$, they are non-adjacent in G . We examine the following cases.

Case 1: If v is not an internal vertex of any S -path in G , then if $u \notin S'$, either $u \in N(S)$ or $u \in V - N[S]$. If $u \in N(S)$, there exists a vertex $w \in S'$ such that uw is an edge and since $G[S]$ is connected, there exists a path from w to v in S . Therefore u and v are connected by an S -path. If $u \in V - N[S]$, then u is adjacent to some $w \in N(S)$, which gives an S'_p -path from u to v through w . Thus S' is a partial hub set of G .

Case 2: If v is an internal vertex of any S -path P from u to w , where $w \notin S$, then the length of P must be at least four. Therefore $P - \{v\}$ will be an S'_p path in G from u to w .

Case 3: If v is not an internal vertex of any S -path from u to w , we consider an S -path between x and y say, $xv_1v_2\dots v\dots v_kv_ky$ containing v as an internal vertex, where $x, y \notin S$. Now we have the following sub cases.

Sub case 1: If x and u are adjacent then there exists an S'_p -path of length at least four from u to v through x with vertex x outside of S' .

Sub case 2: If x and u are connected by an S -path then also there is an S'_p path from u to v through x . Thus, in all the cases S' forms a partial hub set of G .

THEOREM 4.9. Let G be a tree with n vertices and l leaves. Then $h_p(G) = h(G) = n - l$ if and only if for any non-pendant vertex v , $N(v)$ contains more than one pendant vertex and $h_p(G) < n - l$, otherwise.

Proof. Let S be the set of all non-leaf vertices in G , then S is a minimum hub set of G . Let S' be a minimum partial hub set of G . Then it follows that $h(G) = n - l$ and $S' \subseteq S$. Suppose that for any non-pendant vertex $v \in S$, the neighbourhood $N(v)$ contains more than one pendant vertex. Let $x, y \in N(v)$ be two pendant vertices in G . In this case it is clear that $v \in S'$, and thus $S' = S$, implying that $h_p(G) = n - l$.

Conversely, assume that $h_p(G) = n - l$ and S be a minimum partial hubset of G . Clearly S does not contain any pendant vertex. Let v be a non- pendant vertex in G . We consider the following two cases

Case 1: If $N(v)$ contains no pendant vertex then by Theorem 4.8, $S - \{v\}$ will be a partial hub set of G .

Case 2: If $N(v)$ contains exactly one pendent vertex, then the diameter of G will be at least three. Therefore $S - \{v\}$ will also be a partial hubset. Both of these cases contradict the assumption that $h_p(G) = n - l$.

Therefore, all non-pendant vertices v of G will satisfy the condition that $N(v)$ contains more than one pendant vertex. Further for the graphs with above two cases $h_p(G) < n - l$.

Recall that for any graph G , the domination number satisfies the inequality $\gamma(G) \leq h(G) + 1$, where $h(G)$ denotes the hub number of G [8]. For the graph G constructed in Theorem 4.7, it can be verified that $\gamma(G) = 2 = h_p(G) + 1$, but $h(G) = n$ (n can be any positive integer). The following theorem gives an upper bound for $\gamma(G)$ in terms of $h_p(G)$.

THEOREM 4.10. Let S be a partial hub set of a graph $G = (V, E)$ such that $V - N[S] = \phi$. Then $\gamma(G) \leq h_p(G)$. If $G[V - N[S]]$ is complete, then $\gamma(G) \leq h_p(G) + 1$. In general, $\gamma(G) \leq h_p(G) + \gamma(G[V - N[S]])$.

Proof. Let S be a partial hub set of G . Consider the following cases.

Case 1: Let $V - N[S] = \emptyset$. Since S is a partial hub set, for any $u \notin S$, there exists $v \in S$ such that uv is an edge. Hence S is a dominating set and $\gamma(G) \leq h_p(G)$.

Case 2: Let $G[V - N[S]]$ be complete, then take $S = S \cup \{x\}$, where $x \in V - N[S]$. Clearly any vertex in $V - N[S]$ is dominated by x and any vertex in $N(S)$ is dominated by some vertex in S . Hence $\gamma(G) \leq h_p(G) + 1$.

Case 3: For any other cases, let S' be a dominating set of $G[V - N[S]]$. Then $S \cup S'$ will be a dominating set for G .

A dominating set $D \subseteq V(G)$ ensures that every vertex not in D is adjacent to at least one vertex in D , thereby achieving local control or visibility. On the other hand, a partial hub set $S \subseteq V(G)$, ensures that any two non-adjacent vertices outside the set are connected by a path whose internal vertices are mostly (at least half) within S , achieving a form of path-based global control.

References

- [1] Ahmed M. Nour, M. Manjunatha, Ahmed M. Naji, On Restrained Hub Number in Graphs, *Italian Journal of Pure and Applied Mathematics*, 45(2021), pp. 388–396.
- [2] Carmelito E. Go, Sergio R. Canoy Jr., Domination in the Corona and Join of Graphs, *International Mathematical Forum*, 6(2011), pp. 763–771.
- [3] E.C. Cuaresma Jr., R.N. Paluga, On the Hub Number of Some Graphs, *Annals of Studies in Science and Humanities*, 1(1)(2015), pp. 17–24.
- [4] F. Harary, Graph Theory, *Addison-Wesley Publishing House*(1969).
- [5] R. Frucht, F. Harary, On the Corona of Two Graphs, *Aequat.Math.*, 4(1970), pp. 322–325.
- [6] Gary Chartrand, Ping Zhang, Introduction to Graph Theory, *Tata McGraw-Hill*(2006).
- [7] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, *The Macmillan Press Ltd*(1976).
- [8] Matthew Walsh, The Hub Number of a Graph, *International Journal of Mathematics and Computer Science*, 1(2006), pp.117–124.
- [9] R.B. Grageda, B.H. Arriola, 1/2-Domination in Graphs, *Journal of Algebra and Applied Mathematics*, 16(1),(2018) pp. 63–74.
- [10] Shadi Ibrahim Khalaf, Veena Mathad, Sultan Senan Mahde, Hub and Global Hub Number of a Graph, *Proc. Jangjeon Math. Soc.*, 23(2),(2020) pp. 231–239.
- [11] Teresa W. Haynes, Stephan T. Hedetniemi, Peter J. Slater, Fundamentals of Domination in Graphs, *Marcel Dekker, New York*(2008).
- [12] Tracy Grauman, Stephan A. Hartke, Adam Jacobson, et al., The Hub Number of Graph, *Information Processing Letters*, 108(2008), pp. 226–228.