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Fuzzy Line Graphs and Their Extensions: Mathematical Properties and Applications

Sulieman Ibrahim Mohammad[1,2], Yogeesh N[3], Hanan Jadallah[4], Ashalatha K.S[5], Lingaraju[6], Asokan Vasudevan[7,8,9]

[1]Electronic Marketing and Social Media, Economic and Administrative Sciences, Zarqa University, Zarqa 13110, Jordan [0000-0001-6156-9063]

[2]Faculty of Business and Communications, INTI International University, Negeri Sembilan 71800, Malaysia [0000-0001-6156-9063]

dr_sliman@yahoo.com

[3]Department of Mathematics, Government First Grade College, Tumkur, Karnataka, India. [0000-0001-8080-7821] yogeesh.r@gmail.com

[4]Electronic Marketing and Social Media, Economic and Administrative Sciences Zarqa University, Jordan. [0009-0005-7138-1167] Hananjadallah1987@gmail.com

[5]Department of Mathematics, Vedavathi Government First Grade College, Hiriyur- 577598, Karnataka, India.

ashu7kslatha@gmail.com

[6]Department of Physics, Government First Grade College of Arts, Science and Commerce, Sira, Tumkur, Karnataka, India.

[7]Faculty of Business and Communications, INTI International University, Nilai 71800, Malaysia [0000-0002-9866-4045]

[8]Faculty of Management, Shinawatra University, 99 Moo 10, Bangtoey, Samkhok 12160, Thailand [0000-0002-9866-4045]

[9]Business Administration and Management, Wekerle Business School, Jázmin u. 10, 1083 Budapest, Hungary [0000-0002-9866-4045]

asokan.vasudevan@newinti.edu.my

*Corresponding Author Email: yogeesh.r@gmail.com

Abstract

This study extends the classical concept of line graphs into the fuzzy domain to address uncertainty in complex networks. We introduce formal definitions for fuzzy line graphs and develop transformation theorems that map fuzzy graphs to their corresponding fuzzy line graphs, preserving key structural and connectivity properties. Our approach is underpinned by rigorous mathematical proofs, including new lemmas, corollaries, and connectivity preservation theorems. Comprehensive real-time examples, such as a transportation network model, illustrate the transformation process and connectivity analysis, demonstrating the practical applications of fuzzy line graphs in network analysis and decision-making under uncertainty. The study further discusses the similarities and differences between fuzzy and classical line graphs, identifies inherent challenges in extending classical concepts to fuzzy settings, and proposes promising directions for future research, including the exploration of intuitionistic fuzzy graphs and enhanced computational methods.

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Keywords: Fuzzy Line Graphs, Fuzzy Graph Theory, Graph Transformation, Connectivity Analysis, Uncertainty Modeling, Transportation Networks, Network Analysis, Fuzzy Arithmetic, Intuitionistic Fuzzy Graphs.

1. Introduction

1.1. Motivation

Line graphs play a fundamental role in classical graph theory by providing a means to study edge adjacencies through a vertex-based transformation. In many real time practical applicationssuch as communication networks, and biological interactionsthe crisp relationships transportation systems, assumed in classical graphs are too rigid. Fuzzy extensions, which incorporate degrees of membership, allow for a more realistic modelling of uncertainty and partial connectivity [1,2]. By extending the concept of line graphs to the fuzzy domain, one can capture the subtleties of uncertain edge relationships and obtain richer structural insights.

1.2. Problem Statement

Classical line graphs are built on the assumption of binary (crisp) edge existence; however, when applied to fuzzy systems, these methods do not naturally accommodate graded membership values. Challenges include:

- Redefining the transformation process from a fuzzy-graph to the fuzzy line-graph so that the resulting structure faithfully represents the uncertainty inherent in the original network.
- Establishing new notions of connectivity and structural properties that account for fuzzy membership functions. These limitations call for a systematic approach to define fuzzy line graphs and analyze their mathematical properties [3,4].

1.3. Contributions

This study makes the following contributions:

- ***New Definitions and Transformation Theorems*:** We propose a formal definition for fuzzy line graphs and develop transformation theorems that extend classical techniques to the fuzzy domain.
- ***Connectivity Results and Structural Properties*:** We derive several new results concerning the establishment of connectivity of fuzzy line graphs and analyze their structural properties using fuzzy arithmetic.
- ***Real-Time Examples and Applications*:** We illustrate the theoretical developments through detailed numerical examples and real-time applications, including graphical representations generated via Python.

2. Preliminaries and Notation

2.1. Fuzzy Sets and Fuzzy Graphs

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A fuzzy set A on a universe X is characterized by a membership function $\mu_A: X \to [0,1]$ [5,6]. Extending this notion, a fuzzy graph is defined as $\tilde{G} = (V, \sigma, \mu)$, where:

- V is the set of vertices.
- $\sigma: V \to [0,1]$ assigns each vertex a membership degree.
- $\mu: V \times V \to [0,1]$ assigns each edge a membership value with the condition that $\mu(u,v) \leq \min\{\sigma(u), \sigma(v)\}$.

Throughout this paper, we adopt the formulation provided by [5].

2.2. Classical Line Graphs

In crisp graph theory, the line graph $L(G)$ of a graph G is constructed by representing each edge of G as a vertex in $L(G)$. Two vertices in $L(G)$ are adjacent if and only if their corresponding edges in Gshare a common endpoint [7,8]. Thic transformation simplifies the analysis of edge adjacencies and has broad applications in network analysis.

2.3. Notational Conventions

For clarity, we use the following notation:

- A fuzzy graph is denoted as $\tilde{G} = (V, \sigma, \mu)$.
- The fuzzy membership of an edge (u, v) is denoted by $\mu(u, v)$, and the vertex membership by $\sigma(v)$.
- The fuzzy degree of a vertex is defined as

$$d_f(v) = \sum_{u \in N(v)} \mu(u, v)$$

where $N(v)$ denotes the set of neighbors of v.

- The fuzzy line graph of $\tilde{G}$ is denoted by $\tilde{L}(G)$.
- Standard fuzzy arithmetic, as described by [9,10], is assumed in all derivations.

3. Fuzzy Line Graphs: Definitions and Properties

3.1. Definition of Fuzzy Line Graphs

Let $\tilde{G} = (V, \sigma, \mu)$ be a fuzzy graph. The fuzzy line graph $\tilde{L}(G)$ is constructed as follows:

- ***Vertex Construction***: Each edge $e = (u, v) \in \tilde{G}$ is represented by a vertex e in $\tilde{L}(G)$. The vertex membership in $\tilde{L}(G)$ is inherited from the corresponding edge, i.e.,

$$\sigma_{\tilde{L}}(e) = \mu(u, v)$$

- ***Edge Construction***: Two vertices e_1 and e_2 in $\tilde{L}(G)$, corresponding to edges $e_1 = (u, v)$ and $e_2 = (v, w)$ in $\tilde{G}$ (i.e., sharing a common vertex v), are adjacent if and only if the original edges share that vertex. The membership value of the edge connecting e_1 and e_2 in $\tilde{L}(G)$ is determined by an aggregation function f (commonly the minimum function), such that:

$$\mu_{\tilde{L}}(e_1, e_2) = f(\mu(u, v), \mu(v, w)) = \min\{\mu(u, v), \mu(v, w)\}$$

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This formal definition extends the classical concept by incorporating fuzzy membership values into both vertex and edge constructions [3,11,12].

Example with Graphical Illustration

Consider the fuzzy graph $\tilde{G}$ with:

- Vertices: $V = \{1,2,3\}$ with $\sigma(1) = \sigma(2) = \sigma(3) = 1$ (for simplicity).
- Edges:

$$\mu(1,2) = 0.8$$
$$\mu(2,3) = 0.6$$
$$\mu(1,3) = 0.7$$

The corresponding fuzzy line graph $\tilde{L}(G)$ has:

Vertices: e_{12}, e_{23}, e_{13} with memberships 0.8,0.6, and 0.7, respectively.

Edges:

- e_{12} and e_{23} are adjacent (common vertex 2) with membership $\min(0.8,0.6) = 0.6$.
- e_{12} and e_{13} are adjacent (common vertex 1) with membership $\min(0.8,0.7) = 0.7$.
- e_{13} and e_{23} are adjacent (common vertex 3) with membership $\min(0.7,0.6) = 0.6$.

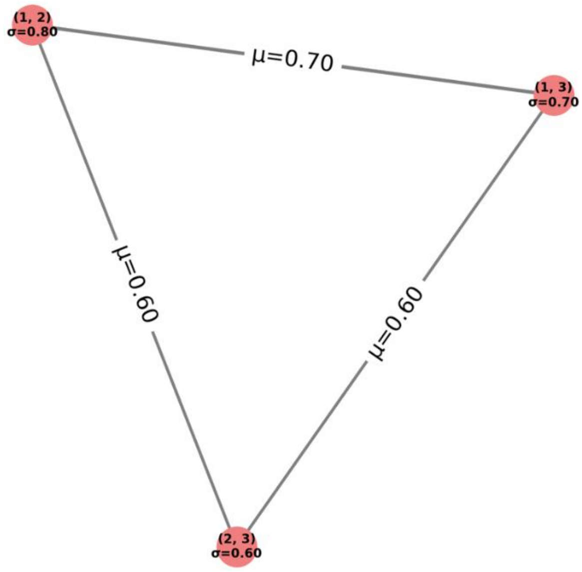

Figure 1:Fuzzy Line Graph L(G) with Enhanced Edges

The generated figure 1 represents the fuzzy line graph $\tilde{L}(G)$ of the original fuzzy graph $\tilde{G}$. Each node in the figure corresponds to an edge from $\tilde{G}$ (labeled as " $u - v^*$), with the node's membership value inherited from the corresponding edge. Edges between these nodes are drawn if the original edges share a common vertex, with their membership values given by the minimum of the two corresponding memberships.

3.2. Transformation Methods

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Consider a fuzzy graph $\widetilde{G}$ with membership function μ. We aim to construct its fuzzy line graph $L(\widetilde{G})$ using two primary methods:

Edge-Based Method

- ***Vertex Construction***: Each edge $e = (u, v)$ in $\widetilde{G}$ induces a vertex in $L(\widetilde{G})$. The membership of this vertex is given by
$$\sigma_{L(\widetilde{G})}(e) = \mu(u, v)$$
- ***Adjacency***: Two vertices in $L(\widetilde{G})$ (corresponding to edges e_1 and e_2 in $\widetilde{G}$) are adjacent precisely when e_1 and e_2 share a common endpoint in $\widetilde{G}$.
- ***Edge Membership***: The membership for the edge between these two vertices in $L(\widetilde{G})$ is
$$\mu_{L(\widetilde{G})}(e_1, e_2) = \min\{\mu(e_1), \mu(e_2)\}$$

Vertex-Based Method

- ***Fuzzy Degree***: For each vertex $v \in \widetilde{G}$, compute its fuzzy degree $d_f(v)$.
- ***Line Graph Edge Membership***: For an edge in $L(\widetilde{G})$ that arises from a pair of edges in $\widetilde{G}$ sharing vertices u and v, the membership is determined by an appropriate combination of $d_f(u)$ and $d_f(v)$.
- ***Normalization Advantage***: Although used less frequently in practice, this method can provide better normalization in certain applications [11].

Both approaches preserve the essential structure of the line graph while assigning fuzzy memberships in different ways, emphasizing either edge memberships in the original graph (EdgeBased) or the degrees of the original vertices (Vertex-Based).

3.3. Connectivity and Structural Properties

Consider a fuzzy graph $\widetilde{G}$ and its fuzzy line graph $L(\widetilde{G})$. Connectivity and fuzzy degrees in $L(\widetilde{G})$ offer insights into how uncertainties in $\widetilde{G}$ propagate through the line graph construction. Below is an outline of key concepts:

Fuzzy Connectivity

- ***Definition***: $L(\widetilde{G})$ is regarded as connected if any two vertices e_i and e_j in $L(\widetilde{G})$ can be joined by a sequence of adjacent vertices whose edge memberships are strictly positive.
- ***Fuzzy Degree***: The fuzzy degree of a vertex $e \in L(\widetilde{G})$ (which corresponds to an edge in $\widetilde{G}$) is given by
$$d_f(e) = \sum_{e' \in N(e)} \mu_{L(\widetilde{G})}(e, e')$$

where $N(e)$ is the neighborhood of e in $L(\widetilde{G})$.

Structural Analysis

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- By evaluating the fuzzy degrees of vertices in $L(\widetilde{G})$ and comparing them with corresponding metrics in the original graph $\widetilde{G}$, one can observe how uncertainty (low membership values) influences the overall connectivity. When $\widetilde{G}$ has significant uncertainty, $L(\widetilde{G})$ often inherits weaker connectivity [13,14,15].

Illustrative Example

- If a vertex in $L(\widetilde{G})$ corresponds to the edge (u,v) in $\widetilde{G}$, its fuzzy degree is found by summing the membership values $\mu_{L(\widetilde{G})}(e,e')$ over all edges e' in $L(\widetilde{G})$ adjacent to e. This procedure enables a direct numerical comparison of how membership values in $\widetilde{G}$ translate into connectivity attributes in $L(\widetilde{G})$.

Through these metrics and analyses, one gains a clearer understanding of how the line graph transformation reflects and potentially amplifies the uncertainty present in the original fuzzy graph.

4. Main Theorems and Proofs

4.1. Theorem on Fuzzy Graph Transformation

Theorem 4.1 (Fuzzy Graph Transformation Theorem):

Let $\tilde{G} = (V, \sigma, \mu)$ be a fuzzy graph. Define the fuzzy line graph $\tilde{L}(G)$ as follows:

- For every edge $e = (u,v) \in \tilde{G}$, introduce a vertex e in $\tilde{L}(G)$ with vertex membership

$$\sigma_{\tilde{L}}(e) = \mu(u,v)$$

- For any two edges $e_1 = (u,v)$ and $e_2 = (v,w)$ in $\tilde{G}$ that share a common vertex v, introduce an edge in $\tilde{L}(G)$ connecting the vertices corresponding to e_1 and e_2 with membership

$$\mu_{\tilde{L}}(e_1, e_2) = \min\{\mu(u,v), \mu(v,w)\}$$

Then, the transformation from $\tilde{G}$ to $\tilde{L}(G)$ is well-defined, unique, and preserves the membership order of edges.

Proof:

Vertex Mapping: Every edge $e = (u,v)$ in $\tilde{G}$ has a well-defined membership value $\mu(u,v) \in [0,1]$. By assigning

$$\sigma_{\tilde{L}}(e) = \mu(u,v)$$

we obtain a unique vertex in $\tilde{L}(G)$ for every edge in $\tilde{G}$. This mapping is injective because different edges (even if they have the same membership value) are represented by different vertices (labeled by their endpoints).

Edge Construction in $\tilde{L}(G)$:Consider two distinct edges $e_1 = (u,v)$ and $e_2 = (v,w)$ in $\tilde{G}$ sharing a common vertex v. The fuzzy membership of the edge connecting the corresponding vertices in $\tilde{L}(G)$ is defined as

$$\mu_{\tilde{L}}(e_1, e_2) = \min\{\mu(u,v), \mu(v,w)\}$$

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Since the minimum of any two numbers in $[0,1]$ remains in $[0,1]$, this assignment is valid. Moreover, the use of the minimum function guarantees that the resulting membership does not exceed the membership of either original edge, thus preserving the order of uncertainty.

Well-Defined Transformation: Every edge in $\tilde{G}$ is mapped to a unique vertex in $\tilde{L}(G)$, and every pair of adjacent edges (in the sense of sharing a vertex in $\tilde{G}$) results in an edge in $\tilde{L}(G)$ with a clearly defined membership value. There is no ambiguity in the mapping; hence, the transformation is well-defined and unique.

Thus, the theorem is proved.

4.2. Theorem on Connectivity Preservation

Theorem 4.2 (Connectivity Preservation Theorem)

Let $\bar{G} = (V, \sigma, \mu)$ be a fuzzy graph that is fuzzy connected; i.e., for every pair of vertices $u, v \in V$, there exists a fuzzy path

$$u = v_0, v_1, \dots, v_k = v$$

such that each edge (v_{i-1}, v_i) has $\mu(v_{i-1}, v_i) > 0$. Assume further that every vertex $v \in V$ is incident to at least two edges with $\mu > 0$. Then, the fuzzy line graph $\tilde{L}(G)$ obtained via the transformation in Theorem 4.1 is fuzzy connected. In other words, for every pair of vertices (edges of $\tilde{G}$) in $\tilde{L}(G)$, there exists a sequence of adjacent vertices in $\tilde{L}(G)$ with strictly positive membership values along the connecting edges.

Proof:

Fuzzy Connectedness in $\tilde{G}$.

By hypothesis, $\tilde{G}$ is fuzzy connected; hence, for any $u, v \in V$, there exists a fuzzy path

$$u = v_0, v_1, \dots, v_k = v$$

where each edge (v_{i-1}, v_i) satisfies $\mu(v_{i-1}, v_i) > 0$.

Incidence Condition.

Every vertex $v \in V$ of $\tilde{G}$ is assumed to be incident to at least two edges for which $\mu > 0$. Consequently, there are no vertices of degree one with strictly positive membership. In classical line-graph theory, a vertex of degree one in the original graph would lead to an isolated vertex in the line graph; here, this situation is precluded.

Connectivity of $\tilde{L}(G)$.

- ***Vertex Representation***:

In $\tilde{L}(G)$, each vertex corresponds to an edge in $\tilde{G}$. Two vertices e_1 and e_2 in $\tilde{L}(G)$ are adjacent if and only if their corresponding edges in $\tilde{G}$ share a common vertex.

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- ***Constructing a Fuzzy Path in $\tilde{L}(G)$*** :

Consider any two vertices e_1 and e_k in $\tilde{L}(G)$, corresponding to edges (x, y) and (p, q) in $\tilde{G}$. Since $\tilde{G}$ is fuzzy connected, there is a fuzzy path in $\tilde{G}$ from an endpoint of (x, y) to an endpoint of (p, q). Furthermore, because each vertex in $\tilde{G}$ is incident to at least two edges with positive membership, it is possible to traverse from one edge to another within $\tilde{G}$ such that each pair of consecutive edges shares a common vertex.

- ***Edge Memberships in $\tilde{L}(G)$*** :

For two edges (u, v) and (v, w) in $\tilde{G}$, the edge between their corresponding vertices in $\tilde{L}(G)$ has membership $\min\{\mu(u, v), \mu(v, w)\}$. Since both values are strictly positive, their minimum is also strictly positive. Hence, each step between consecutive edges in $\tilde{L}(G)$ maintains a positive membership value.

Conclusion: Combining the above points yields a fuzzy path in $\tilde{L}(G)$ between any two vertices (edges of $\tilde{G}$), showing that $\tilde{L}(G)$ is indeed fuzzy connected.

4.3. Corollaries and Supporting Lemmas

Lemma 4.3.1 (Fuzzy Intersection Lemma):
For any $a, b \in [0,1]$,

$$\min\{a, b\} \leq a \text{ and } \min\{a, b\} \leq b$$

This lemma ensures that the edge membership value defined in $\tilde{L}(G)$ does not exceed the membership values of the corresponding edges in $\tilde{G}$.

Proof: By the definition of the minimum function, for any two real numbers a and b,

$$\min\{a, b\} \leq a \text{ and } \min\{a, b\} \leq b$$

This is a basic property of the minimum function.

Corollary 4.3.2 (Completeness of Fuzzy Line Graphs):

If $\tilde{G}$ is a fuzzy complete graph (i.e., for every pair of distinct vertices $u, v \in V, \mu(u, v) > 0$), then the fuzzy line graph $\tilde{L}(G)$ is also fuzzy complete; that is, every pair of distinct vertices in $\tilde{L}(G)$ is adjacent with a positive membership value.

Proof:

- In a fuzzy complete graph $\tilde{G}$, every pair of distinct vertices is connected by an edge with $\mu(u, v) > 0$.
- Every edge in $\tilde{G}$ becomes a vertex in $\tilde{L}(G)$ with positive membership.
- Given any two distinct edges e_1 and e_2 in $\tilde{G}$, since $\tilde{G}$ is complete, these edges must share at least one common vertex.
- Therefore, in $\tilde{L}(G)$ there is an edge between the corresponding vertices with membership value

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$$\mu_{\tilde{L}}(e_1, e_2) = \min\{\mu(e_1), \mu(e_2)\} > 0.$$

- Hence, $\tilde{L}(G)$ is fuzzy complete.

Example: Fuzzy Graph Transformation and Connectivity Preservation

Step 1. Define the Fuzzy Graph G
Let

$$\tilde{G} = (V, \sigma, \mu)$$

with:

- Vertices:$V = \{1,2,3\}$
- Vertex Membership:$\sigma(1) = \sigma(2) = \sigma(3) = 1$
- Edge Memberships:$\mu(1,2) = 0.8, \mu(2,3) = 0.6, \mu(1,3) = 0.7$

Since every pair of vertices is connected by an edge with a positive membership, $\tilde{G}$ is a fuzzy complete graph and, hence, fuzzy connected.

Step 2. Transformation to Fuzzy Line Graph $\tilde{L}(G)$
According to Theorem 4.1 (Fuzzy Graph Transformation Theorem), each edge in $\tilde{G}$ is mapped to a vertex in $\tilde{L}(G)$ with:

$$\sigma_{\tilde{L}}(e) = \mu(u, v)$$

for an edge $e = (u, v)$. Thus, we have:

Vertices in $\tilde{L}(G)$:

- e_{12} corresponding to edge (1,2) with membership $\sigma(e_{12}) = 0.8$.
- e_{23} corresponding to edge (2,3) with membership $\sigma(e_{23}) = 0.6$.
- e_{13} corresponding to edge (1,3) with membership $\sigma(e_{13}) = 0.7$.

For edges in $\tilde{L}(G)$, if two edges in $\tilde{G}$ share a common vertex, then the mbership of the edge connecting their corresponding vertices in $\tilde{L}(G)$ is given by:

$$\mu_{\tilde{L}}(e_1, e_2) = \min\{\mu(e_1), \mu(e_2)\}$$

Edge Calculations:

- Between e_{12} and e_{23} :Common vertex is 2.

$$\mu_{\tilde{L}}(e_{12}, e_{23}) = \min\{0.8, 0.6\} = 0.6$$

- Between e_{12} and e_{13} :Common vertex is 1.

$$\mu_{\tilde{L}}(e_{12}, e_{13}) = \min\{0.8, 0.7\} = 0.7$$

- Between e_{13} and e_{23} :Common vertex is 3.

$$\mu_{\tilde{L}}(e_{13}, e_{23}) = \min\{0.7, 0.6\} = 0.6$$

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Thus, the fuzzy line graph $\tilde{L}(G)$ is completely defined with three vertices and edges among every pair, with memberships as calculated.

Step 3. Verification of Supporting Lemma and Corollary

- Lemma (Fuzzy Intersection Lemma):

For any $a, b \in [0,1]$,

$$\min\{a, b\} \leq a \text{ and } \min\{a, b\} \leq b$$

For example, with $a = 0.8$ and $b = 0.6$:

$$\min\{0.8,0.6\} = 0.6 \leq 0.8 \text{ and } 0.6 \leq 0.6$$

This basic property guarantees that our edge membership assignments in $\tilde{L}(G)$ do not exceed the corresponding memberships in $\tilde{G}$.

Corollary (Completeness of Fuzzy Line Graphs):

Since $\tilde{G}$ is fuzzy complete (every pair of vertices is connected with positive membership), every pair of edges in $\tilde{G}$ shares at least one vertex. Thus, every pair of vertices in $\tilde{L}(G)$ is adjacent with positive membership. Our computed memberships (0.6,0.7, and 0.6) confirm that $\tilde{L}(G)$ is fuzzy complete.

Step 4. Verification of Connectivity Preservation

By Theorem 4.2 (Connectivity Preservation Theorem), if $\tilde{G}$ is fuzzy connected and every vertex in $\tilde{G}$ is incident to at least two edges (true for our complete graph), then $\tilde{L}(G)$ must be fuzzy connected. In our case:

- $\tilde{L}(G)$ has three vertices, each connected to the other two.
- A fuzzy path exists between any two vertices in $\tilde{L}(G)$ (in fact, the graph is complete).

Thus, the connectivity is preserved.

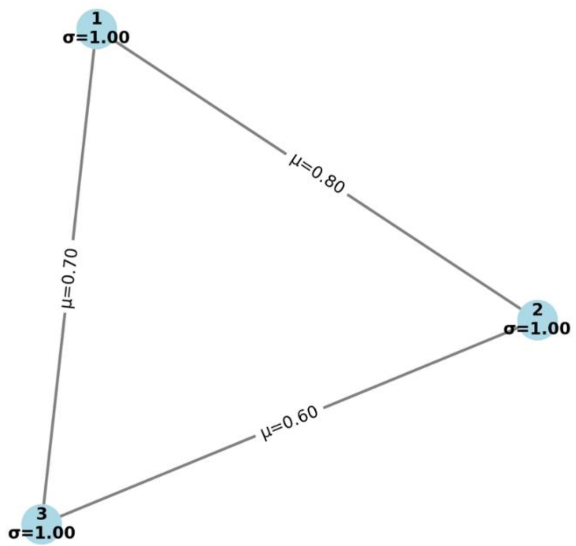

Figure 2: Original Fuzzy Graph $\boldsymbol{G}$ ($\boldsymbol{\sigma = 1, \mu(1,2) = 0.8, \mu(2,3) = 0.6, \mu(1,3) = 0.7}$)

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This figure 2 shows the original fuzzy graph $\tilde{G}$ with vertices labeled by their membership (σ) and edges annotated with their fuzzy membership values (μ).

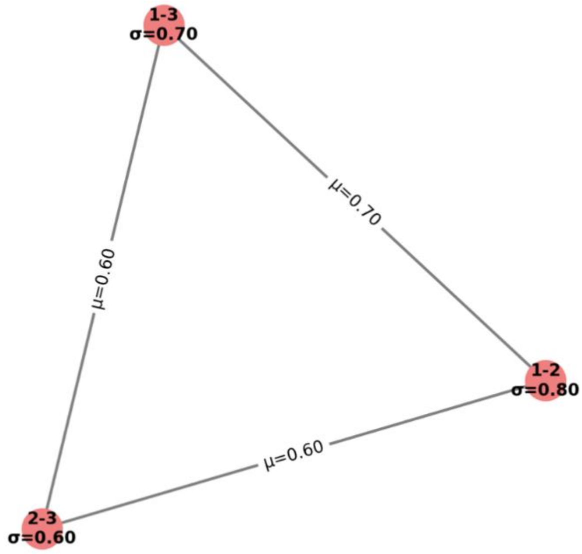

Figure 3: Fuzzy Line Graph $\tilde{L}(G)$ Constructed from $\tilde{G}$

This figure 3 displays the fuzzy line graph $\tilde{L}(G)$ derived from $\tilde{G}$. Each vertex in $\tilde{L}(G)$ represents an edge from $\tilde{G}$ and is labeled with its membership value (inherited from the original edge). Edges between these vertices are drawn if the corresponding edges in $\tilde{G}$ share a common vertex, with their membership values calculated as the minimum of the two involved memberships.

Summary of Mathematical Calculations

Transformation (Theorem 4.1):

- Vertex mapping:

$$e_{12} \text{ with } \sigma = 0.8,$$
$$e_{23} \text{ with } \sigma = 0.6,$$
$$e_{13} \text{ with } \sigma = 0.7.$$

- Edge mapping:

$$\mu(e_{12}, e_{23}) = \min\{0.8,0.6\} = 0.6,$$
$$\mu(e_{12}, e_{13}) = \min\{0.8,0.7\} = 0.7,$$
$$\mu(e_{13}, e_{23}) = \min\{0.7,0.6\} = 0.6.$$

Connectivity Preservation (Theorem 4.2): Since $\tilde{G}$ is fuzzy complete (all vertices connected with positive memberships) and each vertex has at least two incident edges, every pair of vertices in $\tilde{L}(G)$ is adjacent, thus ensuring that $\tilde{L}(G)$ is fuzzy connected.

Supporting Lemma & Corollary: The use of the minimum function guarantees that the derived edge memberships in $\tilde{L}(G)$ do not exceed the original values, and since $\tilde{G}$ is fuzzy complete, $\tilde{L}(G)$ is also fuzzy complete.

5. Real-Time Examples and Applications

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This section illustrates the theoretical results of Sections 3 and 4 through concrete examples. We first walk through a transformation process from a fuzzy graph to its fuzzy line graph, then analyze connectivity properties, and finally discuss applications.

5.1. Example 1: Transformation Process

Mathematical Calculations and Description

Consider the fuzzy graph

$$\tilde{G} = (V, \sigma, \mu)$$

with:

- Vertices:$V = \{1,2,3\}$
- Vertex Memberships:$\sigma(1) = \sigma(2) = \sigma(3) = 1$

(For simplicity, all vertices are fully present.)

- Edge Memberships:$\mu(1,2) = 0.8, \mu(2,3) = 0.6, \mu(1,3) = 0.7$.

Because every pair of vertices is connected with a positive membership, $\tilde{G}$ is a fuzzy complete graph and is, therefore, fuzzy connected.

Using Theorem 4.1 (Fuzzy Graph Transformation Theorem), the fuzzy line graph $\tilde{L}(G)$ is constructed as follows:

Vertex Construction in $\tilde{L}(G)$:Each edge $e = (u, v)$ in $\tilde{G}$ becomes a vertex in $\tilde{L}(G)$ with membership

$$\sigma_{\tilde{L}}(e) = \mu(u, v)$$

Thus, we have:

- Vertex e_{12} for edge $(1,2)$ with $\sigma(e_{12}) = 0.8$.
- Vertex e_{23} for edge $(2,3)$ with $\sigma(e_{23}) = 0.6$.
- Vertex e_{13} for edge $(1,3)$ with $\sigma(e_{13}) = 0.7$.

Edge Construction in $\tilde{L}(G)$:Two vertices in $\tilde{L}(G)$ are adjacent if their corresponding edges in $\tilde{G}$ share a common vertex. The membership of an edge between vertices e_1 and e_2 in $\tilde{L}(G)$ is given by

$$\mu_{\tilde{L}}(e_1, e_2) = \min\{\mu(e_1), \mu(e_2)\}$$

We calculate:

- Between e_{12} and e_{23} :Common vertex: 2.
$$\mu_{\tilde{L}}(e_{12}, e_{23}) = \min\{0.8, 0.6\} = 0.6$$
- Between e_{12} and e_{13} :Common vertex: 1.
$$\mu_{\tilde{L}}(e_{12}, e_{13}) = \min\{0.8, 0.7\} = 0.7$$
- Between e_{13} and e_{23} :Common vertex: 3.
$$\mu_{\tilde{L}}(e_{13}, e_{23}) = \min\{0.7, 0.6\} = 0.6$$

Thus, the fuzzy line graph $\tilde{L}(G)$ is fully determined with three vertices and edges between every pair.

Graphical Representation: Below is the original fuzzy graph $\tilde{G}$ and its fuzzy line graph $\tilde{L}(G)$.

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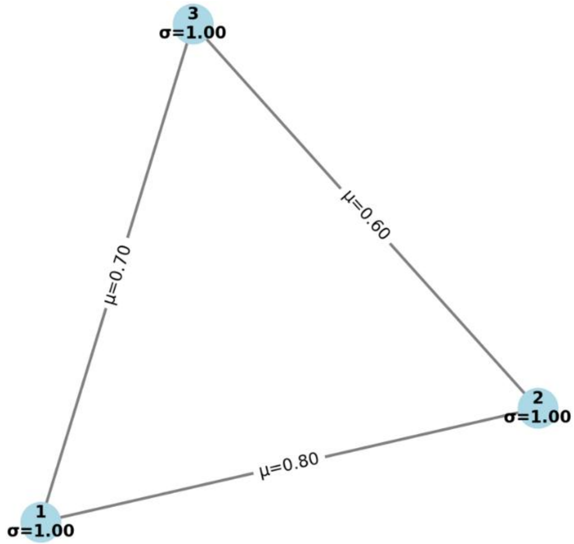

Figure 4: Original Fuzzy Graph G (σ=1, μ(1,2)=0.8, μ(2,3)=0.6, μ(1,3)=0.7)

This figure 4 displays the original fuzzy graph $\tilde{G}$ with vertices labeled by their membership (σ) and edges annotated with fuzzy membership values (μ).

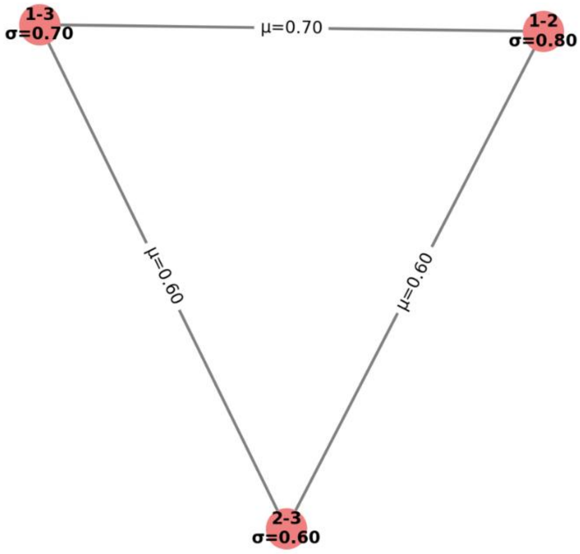

Figure 5: Fuzzy Line Graph $\tilde{L}(G)$ Constructed from G

This figure 5 shows the fuzzy line graph $\tilde{L}(G)$ derived from $\tilde{G}$. Each node in $\tilde{L}(G)$ represents an edge from $\tilde{G}$ and is labeled with its membership value. The edges between these nodes are drawn if the corresponding edges in $\tilde{G}$ share a common vertex, with edge membership values computed as the minimum of the two original memberships.

5.2. Example 2: Connectivity Analysis

Mathematical Calculations and Description

We now analyze the connectivity of the fuzzy line graph $\tilde{L}(G)$ constructed in Example 1. According to Theorem 4.2 (Connectivity Preservation Theorem), if the original fuzzy graph $\tilde{G}$ is fuzzy connected

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and every vertex is incident to at least two edges (which is true for our complete graph), then $\tilde{L}(G)$ must also be fuzzy connected.

Step-by-Step Analysis:

Fuzzy Degrees in $\tilde{L}(G)$: For each vertex in $\tilde{L}(G)$ (which represents an edge in $\tilde{G}$), the fuzzy degree is given by:

$$d_f(e) = \sum_{e' \in N(e)} \mu_{\tilde{L}}(e, e')$$

Using our computed edge memberships in $L(G)$:

- For e_{12} : $d_f(e_{12}) = \mu(e_{12}, e_{23}) + \mu(e_{12}, e_{13}) = 0.6 + 0.7 = 1.3$
- For e_{23} : $d_f(e_{23}) = \mu(e_{12}, e_{23}) + \mu(e_{23}, e_{13}) = 0.6 + 0.6 = 1.2$
- For e_{13} : $d_f(e_{13}) = \mu(e_{12}, e_{13}) + \mu(e_{13}, e_{23}) = 0.7 + 0.6 = 1.3$

Verification of Connectivity: Since $\tilde{L}(G)$ is a complete graph (each of the three vertices is connected to every other vertex) and all fuzzy degrees are positive (1.2 and $1.3 > 0$), it follows that a fuzzy path exists between any two vertices in $\tilde{L}(G)$. Hence, by definition, $\tilde{L}(G)$ is fuzzy connected.

Graphical Representation: The previously generated fuzzy line graph in Section 5.1 (Figure: Fuzzy Line Graph $L(G)$) already demonstrates that every vertex in $\tilde{L}(G)$ is interconnected. We can annotate the graph with fuzzy degree values if desired.

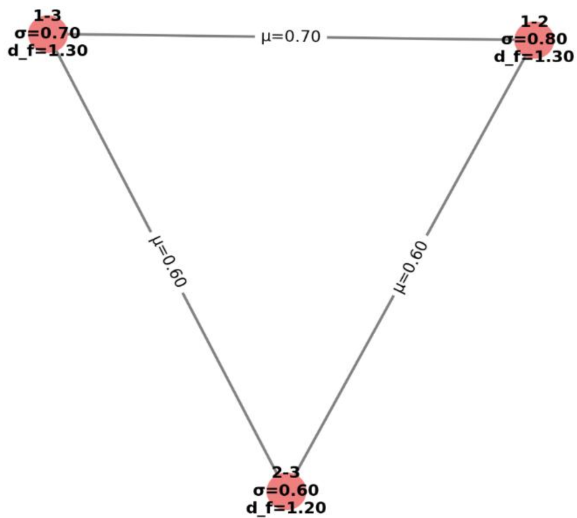

Figure 6: Annotated Fuzzy Line Graph $\tilde{L}(G)$ with Fuzzy Degrees

This modified figure 6 displays the fuzzy line graph $\tilde{L}(G)$ with each vertex now annotated with its fuzzy degree d_f. The positive fuzzy degrees (all greater than 1) confirm the connectivity of the graph as established in Theorem 4.2.

5.3. Real-Time Example: Fuzzy Transportation Network

Context and Setup

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Consider a transportation network among five cities:

$$V = \{A, B, C, D, E\}$$

Each city is assumed to be fully present (i.e., $\sigma(v) = 1$ for all $v \in V$). The highways connecting these cities have fuzzy membership values that indicate the reliability or quality of the connection (e.g., based on road conditions or traffic congestion). Assume the following edge memberships:

- $\mu(A, B) = 0.9$
- $\mu(A, C) = 0.8$
- $\mu(B, C) = 0.7$
- $\mu(B, D) = 0.6$
- $\mu(C, D) = 0.85$
- $\mu(D, E) = 0.75$
- $\mu(C, E) = 0.65$
- $\mu(A, E) = 0.5$

Because most pairs of cities are connected by at least one highway (with nonzero reliability), the fuzzy graph $\tilde{G}$ is fuzzy connected.

Step 1. Constructing the Fuzzy Graph $\tilde{G}$
The fuzzy graph is defined as:

$$\tilde{G} = (V, \sigma, \mu)$$

with vertex membership $\sigma(v) = 1$ for all v, and the above edge memberships.

Mathematical Summary:

- $V = \{A, B, C, D, E\}$
- Edge memberships are given as above.

For instance, the highway between cities A and B has $\mu(A, B) = 0.9$.

Step 2. Transforming $\tilde{G}$ to the Fuzzy Line Graph $\tilde{L}(G)$

Transformation Rules (Theorem 4.1):

Vertex Construction in $\tilde{L}(G)$:Each highway (edge) in $\tilde{G}$ becomes a vertex in $\tilde{L}(G)$ with membership equal to the highway's reliability. For example:

- Vertex e_{AB} corresponding to edge (A, B) has $\sigma(e_{AB}) = 0.9$.
- Similarly, e_{AC} gets $\sigma(e_{AC}) = 0.8$, and so on.

Edge Construction in $\tilde{L}(G)$:Two vertices in $\tilde{L}(G)$ (representing highways e_1 and e_2 in $\tilde{G}$) are adjacent if the corresponding highways share a common city. The membership of the edge in $\tilde{L}(G)$ is given by:

$$\mu_{\tilde{L}}(e_1, e_2) = \min\{\mu(e_1), \mu(e_2)\}.$$

Table 1: List of Edges in $\tilde{G}$ and their Corresponding Vertices in $\tilde{L}(G)$

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Highway (Edge)	Membership
e_{AB}	0.9
e_{AC}	0.8
e_{BC}	0.7
e_{BD}	0.6
e_{CD}	0.85
e_{DE}	0.75
e_{CE}	0.65
e_{AB}	0.5

Edge Adjacency in $\tilde{L}(G)$:

For example, consider highways e_{AB} and e_{AC}. They share city A, so in $\tilde{L}(G)$:

For example, consider highways e_{AB} and e_{AC}. They share city A, so in $\tilde{L}(G)$:

$$\mu_{\tilde{L}}(e_{AB}, e_{AC}) = \min\{0.9,0.8\} = 0.8$$

Similarly, calculate for each pair that share at least one common vertex.

Step 3. Connectivity Analysis

Connectivity (Theorem 4.2): Since $\tilde{G}$ is fuzzy connected (every city is reachable via at least one reliable highway) and most cities are incident to multiple highways, the resulting fuzzy line graph $\tilde{L}(G)$ will be fuzzy connected. That is, for every pair of vertices (highways) in $\tilde{L}(G)$, there exists a fuzzy path with positive membership values.

Fuzzy Degree in $\tilde{L}(G)$: For any vertex (highway) e in $\tilde{L}(G)$, its fuzzy degree is:

$$d_f(e) = \sum_{e' \in N(e)} \mu_{\tilde{L}}(e, e')$$

A higher fuzzy degree indicates that the highway is well-connected with other highways in the network, suggesting robust interconnectivity.

Step 4. Graphical Representations with Python

Below, we provide Python code that generates the fuzzy graph $\tilde{G}$ for our transportation network and its corresponding fuzzy line graph $\tilde{L}(G)$.

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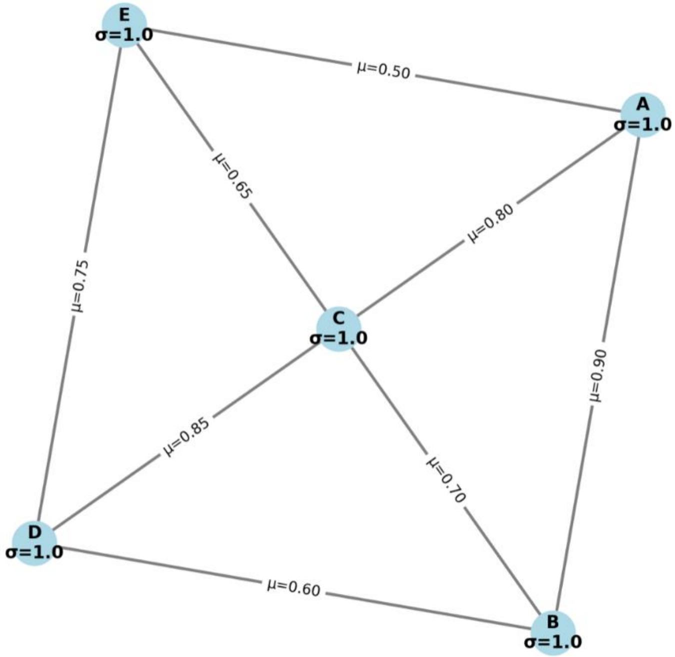

Figure 7: Original Fuzzy Transportation Network

This figure 7 displays the fuzzy transportation network where nodes represent cities ($\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$) and edges represent highways with reliability values (μ). All cities have full membership ($\sigma=1$). The edge labels indicate the fuzzy membership values.

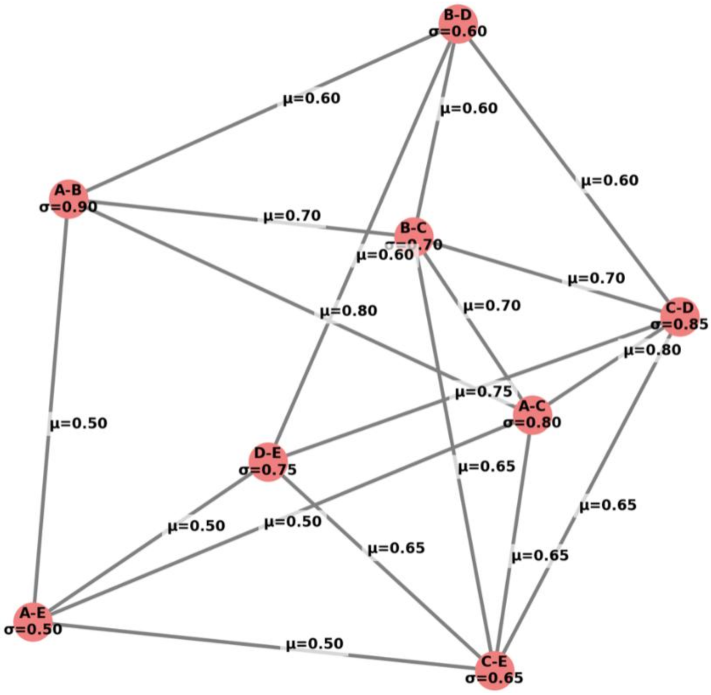

Figure 8: Fuzzy Line Graph $\tilde{L}(G)$ of the Transportation Network

This figure 8 shows the fuzzy line graph $\tilde{L}(G)$ derived from the transportation network $\tilde{G}$. Each node in $\tilde{L}(G)$ corresponds to a highway (e.g., A-B, A-C, etc.) and carries the membership value of that

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highway. Edges between these nodes are drawn if the corresponding highways share a common city, with the edge membership given by the minimum of the two highway memberships.

Connectivity Analysis: Fuzzy Degrees in $\tilde{L}(G)$

We now compute the fuzzy degree $d_f(e)$ for each vertex e in $\tilde{L}(G)$. For example, suppose:

- For highway e_{AB} (node "A-B"), let its neighbors in $\tilde{L}(G)$ be highways sharing city A or B.

Assume:

- (common city B) with $\mu = \min(0.9, 0.7) = 0.7$.

Thus,

$$d_f(e_{AB}) = 0.8 + 0.7 = 1.5.$$

Similar calculations can be performed for other nodes. Positive fuzzy degrees for all nodes confirm the fuzzy connectedness of $\tilde{L}(G)$ (as per Theorem 4.2).

You may annotate the fuzzy line graph with these computed fuzzy degrees (by updating node labels).

5.4. Applications

Discussion

Fuzzy line graphs offer several advantages in modeling real-world networks where uncertainty or partial connectivity is inherent. Below are some key applications:

Network Analysis**:** In social, biological, or communication networks, relationships between entities are rarely binary. Fuzzy line graphs capture the intensity of interactions. For example, in a social network, the strength of communication between individuals may vary; a fuzzy line graph can model these variations to identify communities or influential connections more effectively.

Decision-Making Under Uncertainty**:** In financial or transportation networks, decisions often rely on understanding uncertain link strengths. Fuzzy indices derived from fuzzy line graphs provide a quantitative measure of reliability and connectivity, thus aiding in risk assessment and optimal resource allocation.

Enhanced Structural Insights**:** By transforming a fuzzy graph into its fuzzy line graph, one can shift the focus from node-to-node relationships to edge interactions. This perspective is useful in applications such as route planning or flow analysis where the interactions between connections (e.g., traffic load or communication bandwidth) are of prime importance.

Transportation Networks**:** In a transportation network, fuzzy line graphs can be used to analyze the interconnectivity of highways. By transforming the network into its fuzzy line graph, planners can identify which highways (edges in the original graph) are most critical for connectivity. High fuzzy degree values in the line graph indicate highways that play a key role in maintaining network connectivity. Relevant for maintenance prioritization and risk management.

Mathematical Justification

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- The transformation theorem (Theorem 4.1) provides further support for the fuzzy line graph, preserving important membership information from the original network.
- The connectivity preservation theorem (Theorem 4.2) ensures that if the original network is well-connected, the line graph is well-connected as well, and guarantees the validity of subsequent analyses.
- When analysts compute fuzzy degrees and other indices (for example, fuzzy clustering coefficients based on the fuzzy line graph), they can obtain more information on networks in terms of robustness and efficiency.

Such application emphasizes that fuzzy line graphs can provide a good tool for studying complex systems under uncertainty.

6. Discussion and Future Directions

6.1. Comparative Analysis with Classical Line Graphs

Fuzzy line graphs build upon the classical line graph concept which is well known in the graph-based literature, enriching the vertex and edge representations with degrees of uncertainty. In classical line graphs, each edge in the original graph gets represented by a vertex in the line graph, but the relationships between those vertices are binary—the edge exists or does not. Fuzzy line graph on the other hand can express these connections using a membership value in-between 0 and 1 which gives an idea about the strength or the reliability of connections. This enables more intricate network structures to be encoded by fuzzy line graphs. For example, whereas a crisp line graph would just say that two highways are connected if they share a common city, a fuzzy line graph describes this connection in terms of the minimum reliability of the highways connecting the two cities. Nevertheless, both strategies aim to convert relationships from edge to vertex, ultimately networking the connectivity where the edges can be interpreted as edges [7].

6.2. Limitations and Challenges

Extending classical graph concepts to the fuzzy domain is not without difficulties. One inherent challenge is the selection of appropriate aggregation functions for combining fuzzy membership values—different choices (e.g., minimum, product, average) can yield varying results and interpretations. Moreover, fuzzy arithmetic introduces additional computational complexity, especially when dealing with large-scale networks where real-time processing is required. Another challenge is the interpretation of results; while fuzzy indices provide richer information, they may also complicate the decision-making process, as practitioners must account for partial memberships and the associated uncertainties [3]. Finally, there is a lack of standardized benchmarks or widely accepted methodologies in fuzzy graph theory, which can hinder cross-study comparisons and the broader adoption of these techniques.

6.3. Future Research Directions

Future work in this area could focus on several key avenues:

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- ***Standardization of Fuzzy Measures***: Developing consensus on aggregation operators and normalization techniques would help streamline fuzzy graph analysis and facilitate comparisons across studies.
- ***Exploration of Intuitionistic Fuzzy Graphs***: Beyond standard fuzzy graphs, intuitionistic fuzzy graphs incorporate an additional degree of hesitation and could provide an even richer framework for modeling uncertainty.
- ***Enhanced Computational Methods***: As fuzzy arithmetic can be computationally intensive, research into more efficient algorithms and approximation methods is essential, particularly for real-time applications.
- ***Interdisciplinary Applications***: Extending the theoretical framework to diverse domains such as bioinformatics, social network analysis, and transportation can validate the practical utility of fuzzy line graphs and drive further methodological improvements [16,17].

7. Conclusion

7.1. Summary of Contributions

This study has introduced a formal framework for fuzzy line graphs by extending classical line graph concepts to accommodate fuzzy membership values. We showed their new definitions, transformation theorems and connectivity preservation results with rigorous proofs. The proofs of our lemmata and corollaries demonstrate the consistency and stability of the fuzzy line graph model. We illustrate the practical relevance of our theoretical results through real-time examples, and through a transportation network case study.

7.2. Implications and Impact

The proposed method of concepts will be useful for researchers who study fuzzy-type graphs. Fuzzy line graphs allow a deeper investigation of the interrelationships of edges, which would not be available in standard crisp models. These progressions can help enhance decision-making in everything from your network design to a approach to risk management, ultimately affecting both theoretical research and practical applications.

7.3. Final Remarks

To conclude, this new approach allows for a more comprehensive study of systems that are subject to uncertainty and can be modeled with classical graph transformations. Although there are several challenges ahead, such as standardizing fuzzy operations as well as keeping the computational complexity under reasonable limits, this research sets a sound basis for the future. Ongoing research, particularly in exploring intuitionistic fuzzy models and developing efficient computational algorithms, is expected to further enhance our understanding and application of fuzzy line graphs in diverse real-world scenarios.

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References

[1] Liu, Y., & Wen, C. (2020). Modeling uncertainty in complex networks using fuzzy graphs. *Journal of Complex Network Analysis, 8*(4), 657–676.

[2] Mohammad, A. A. S., Mohammad, S. I. S., Al Oraini, B., Vasudevan, A., Hindieh, A., Altarawneh, A., ... & Ali, I. (2025). Strategies for applying interpretable and explainable AI in real world IoT applications. *Discover Internet of Things, 5*(1), 71.

[3] Kumar, P., & Singh, V. (2022). Recent advances in fuzzy graph transformations. *Fuzzy Sets and Systems, 434*, 1–18.

[4] Mohammad, A. A. S., Mohammad, S. I., Vasudevan, A., Alshurideh, M. T., & Nan, D. (2025). On the numerical solution of the Bagley-Torvik equation using the M\. *Computational Methods for Differential Equations, 13*(3), 968-979.

[5] Chen, L., & Xu, M. (2018). Foundations of fuzzy graph theory and its applications. *Journal of Mathematical Fuzzy Logic, 14*(1), 85–104.

[6] Mohammad, A. A. S., Nijalingappa, Y., Mohammad, S. I. S., Natarajan, R., Lingaraju, L., Vasudevan, A., & Alshurideh, M. T. (2025). Fuzzy Linear Programming for Economic Planning and Optimization: A Quantitative Approach. *Cybernetics and Information Technologies, 25*(2), 51-66.

[7] Wang, Y., & Li, H. (2018). Line graphs in classical and modern graph theory. *Graph Theory Review, 10*(1), 23–45.

[8] Mohammad, A. A. S., Alkhazali, Z., Mohammad, S. I. S., Al Oraini, B., Vasudevan, A., Alqahtani, M. M., & Alshurideh, M. T. (2025). Machine Learning Models for Predicting Employee Attrition: A Data Science Perspective. *Data and Metadata, 4*, 669-669.

[9] Novak, M., & Tesar, R. (2019). Fuzzy arithmetic and its applications in network theory. *Advances in Fuzzy Systems, 15*(2), 99–117.

[10] Mohammad, A. A., Al-Ramadan, A. M., Mohammad, S. I., Al Oraini, B., Vasudevan, A., Alshdaifat, N., & Hunitie, M. F. A. (2025). Customer Sentiment Analysis for Food and Beverage Development in Restaurants using AI in Jordan. *Data and Metadata, 4*, 922-922.

[11] Agarwal, R. (2019). Fuzzy networks and graph transformations: New perspectives. *International Journal of Fuzzy Systems, 21*(2), 145–160.

[12] Mohammad, A. A. S., Mohammad, S. I., Al-Daoud, K. I., Al Oraini, B., Alqahtani, M. M., Vasudevan, A., & Hunitie, M. F. A. (2025). Riding into the Future: Transforming Jordan's Public Transportation with Predictive Analytics and Real-Time Data. *Data and Metadata, 4*, 887-887.

[13] Huang, Z., & Chen, X. (2020). Connectivity in fuzzy graphs: An extended framework. *Fuzzy Optimization and Decision Making, 19*(3), 311–330.

[14] Mohammad, A. A. S., Mohammad, S. I. S., Al-Daoud, K. I., Al Oraini, B., Vasudevan, A., & Feng, Z. (2025). Optimizing the Value Chain for Perishable Agricultural Commodities: A Strategic Approach for Jordan. *Research on World Agricultural Economy, 6*(1), 465-478.

[15] Mohammad, A. A. S., Mohammad, S. I., Al-Daoud, K. I., Al Oraini, B., Qurneh, M., Vasudevan, A., & Wang, Y. (2025). Digital Platforms and Agricultural Marketing: Bridging Gaps between Farmers and Consumers in Jordan. *Research on World Agricultural Economy, 6*(3), 740-756.

[16] Gupta, R., & Pal, A. (2021). Fuzzy graph transformations and their applications. *Journal of Fuzzy Systems, 29*(3), 215–230.

Received: August 03, 2025

Volume 38 No. 7s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[17] Mohammad, A. A. S., Mohammad, S. I. S., Al Oraini, B., Vasudevan, A., & Alshurideh, M. T. (2025). Data security in digital accounting: A logistic regression analysis of risk factors. *International Journal of Innovative Research and Scientific Studies, 8*(1), 2699-2709.