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ESTIMATION OF CONVERGENCE RATE IN LIMIT THEOREMS ON TRANSIENT PHENOMENA FOR BRANCHING RANDOM PROCESSES STARTING WITH A RANDOM NUMBER OF PARTICLES

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Abstract

The present work investigates estimates of the rate of convergence in transitional phenomena, i.e., in limit theorems for branching stochastic processes as $A \to 1$, $n \to \infty$. The rate of convergence of the remainder term in the limit theorems concerning transitional phenomena for branching processes starting with a random number of particles is obtained.

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1. Introduction. Preliminary results

The theory of branching processes is a model of numerous real phenomena of multiplication and transformation of particles in natural and social sciences. Branching random processes play an important role in the study of problems in demography, medicine, chemistry, biology, as well as in genetic coding and population management.

The study of asymptotic properties of branching near-critical random processes is one of the many important problems in modern probability theory.

For branching processes, a number of limit theorems with transient phenomena have been proved. However, the estimation of the convergence rate of residual terms in such limit theorems.

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In theorems with transient phenomena for branching random processes starting with a random number of particles, the rate estimation of residual terms and limit theorems also remained unrevealed.

It follows that the main purpose of our study is to investigate asymptotic properties of branching processes close to the critical one and to study limit theorems in transient phenomena for branching processes starting from a random number of particles.

A.N. Kolmogorov, A.A. Dmitriev, B.A. Sevastyanov [1], [2] in their works stated the first results of the theory of branching random processes and obtained. In the monographs [3], [4] one can find information in which the subsequent development of this theory is presented.

As a homogeneous Markov chain with a phase set of states $\{0,1,2,...\}$ and transition probabilities of branching processes of discrete time (Galton-Watson processes) are introduced in monographs [3] (Chapter I, § (1-8), pp. 11 - 14), [4] (Chapter I, Part A, § (1-5)), defined by the branching condition, which is given below by formula (2).

Transient phenomena for the Galton-Watson processes are studied in [3], [5], [10], [11]. Useful and very accurate inequalities for the particle number distributions $n-$ of the generation are proved in [6].

In proving the central limit theorem for non-degenerate Galton-Watson processes we use asymptotic properties of the generalized numerical characteristic of Rotary, which were studied in [9], [10].

Various models of Galton-Watson processes with possible immigration were studied in [11] - [17]. The asymptotic analysis of complex Galton-Watson processes with decomposable components is given in [18]-[22].

An estimate of the convergence rate in the main lemma of the Galton-Watson critical process was first given in [22].

The results obtained in [23]-[36] were used to solve the resulting integral equations and perform asymptotic analysis of complex processes.

This formula

$$Z_0 = 1,\ Z_n = \sum_{j=1}^{Z_{n-1}} \xi_j^{(n)},\ n \geq 1 \tag{1}$$

defines a branching Galton-Watson random process, where $\xi_1, \xi_2, ..., \xi_n, ...$ is a sequence of independent random variables (s. v.) with non-negative and integer values and a general distribution

$$P(\xi_1 = k) = P(Z_1 = k) = p_k,\ k = 0,1,...$$

(see [1], ch.1, §1, pp. 11-13, [2], ch.1, part A, §1, 1-4 pp.).

Received: July 12, 2025

From the definition of a branching Galton-Watson process, equality (1), it follows that this process is a homogeneous Markov chain with phase set of states $\{0,1,2,...,n,...\}$ and transition probabilities

$$p_{ij}(n)=P\left(Z_{n+1}=j/Z_n=i\right)=p_{1j}^{*i}=\sum_{j_1+...+j_i=j}p_{1j_1}(n)p_{1j_2}(n)...p_{1j_i}(n)$$

where $i,j=0,1,2,...,$

$$p_{1j}(n)=P\left(Z_n=j/Z_0=1\right)=P\left(Z_n=j\right),\qquad p_{0j}(n)=p_{1j}^{*0}(n)=\delta_{ij}=\begin{cases}1 & at\ \ i=j,\\ 0 & at\ \ i\neq j.\end{cases}$$

Let $F(x)$ be the derivative function of c. c. ξ_1 i.e.

$$F(x)=Ex^{\xi_1}=\sum_{k=0}^{\infty}P\left(\xi_1=k\right)x^k,\ |x|\leq 1.$$

and let $\nu-$ be an arbitrary non-negative integer s. c. c. with a derivative function

$$G(x)=Ex^{\nu}=\sum_{k=0}^{\infty}P(\nu=k)x^k,\ |x|\leq 1$$

The following is the case:

Proposition. *Let a sequence of s. c. v.*

$$\nu,\xi_1,\xi_2,...,\xi_n,...$$

independent in the aggregate. Then the derivative function of the random sum $S_\nu=\sum_{j=1}^{\nu}\xi_j$ *satisfies the equality*

$$Ex^{S_\nu}=\sum_{k=0}^{\infty}P\left(S_\nu=k\right)x^k=G\left(F(x)\right),\ |x|\leq 1.$$

Using the above proposition, we can establish recurrence relations for the derivative functions

$$F_n(x)=\sum_{k=0}^{\infty}P\left(Z_n=k\right)x^k=\sum_{k=0}^{\infty}P_n(k)x^k,\ n=1,2,...$$

namely, there are equalities

$$F_n(x)=F_{n-1}\left(F(x)\right)=F\left(F_{n-1}(x)\right),\ n\geq 1. \tag{2}$$

The formulas (2) can be easily proved by mathematical induction by considering the proposition and equality (1).

Usually, the following notations are used in the theory of branching processes:

Received: July 12, 2025

$$A = EZ_1 = F^{'}(1), \quad B = EZ_1(Z_1 - 1) = F^{''}(1), . C = EZ_1(Z_1 - 1)(Z_1 - 2) = F^{'''}(1)$$

A Galton-Watson process is called precritical, critical, and supercritical if $0 < A < 1,\ A = 1,\ A > 1$, respectively.

In the critical case $A = 1$, the limiting distribution for Z_n is a particular exponent distribution, while in the non-critical cases $A \neq 1$, these limiting distributions are defined by complex functional equations that have no explicit solutions.

The phenomena arising at $n \to \infty,\ A \to 1$, i.e., asymptotic properties of processes close to critical, will be called transient phenomena. The study of transient phenomena is based on the asymptotic formula for $1 - F_n(x)$ at $n \to \infty,\ A \to 1$. The form of this formula can be defined on the concrete example of branching Galton-Watson processes generated by fractional linear derivative functions of the form

$$F(x) = \frac{ax + b}{cx + d}.$$

As it is noticed in the monograph [1] (Ch.III, §4, p.108), for the derivative function $F(x)$ from formula (2), iterations $F_n(x) = F(F_{n-1}(x))$ are in the form of explicit expression at $n \to \infty,\ A \to 1$

$$r_n(x) = \begin{cases} \dfrac{A^n(1-x)}{1 + \dfrac{B}{2} \cdot \dfrac{A^n - 1}{A - 1}(1-x)}, & at\ A \neq 1, \\ \dfrac{1-x}{1 + \dfrac{Bn}{2}(1-x)}, & at\quad A = 1, \end{cases} \tag{3}$$

Following B.A. Sevastyanov (Theor. ver. and its primes, vol. 4, no. 2 (1959), 121-135), let us define the class of probabilistic derivative functions

$$K(B_0, C_0) = \left\{ F(\cdot), F^{''}(1) = B \geq B_0 > 0,\ F^{'''}(1) = C \leq C_0 < \infty \right\}.$$

The first result about transient phenomena for Galton-Watson processes is the following

Theorem 1. *The equality*

$$1 - F_n(x) = r_n(x)\left(1 + \eta_n(x)\right)$$

where $r_n(x)$ *is defined by formula (1.5), and* $\eta_n(x) \to 0$ *at* $n \to \infty, A \to 1$ *is uniform overall* $F(x) \in K(B_0, C_0)$ *and* $|x| \leq 1$.

Received: July 12, 2025

This theorem, proved by B. A. Sevastyanov is contained in the monograph [1] (Ch. III, §4, p. 106).

In [37] we prove an estimate for the residual term of Theorem 1. Let us study the rate of convergence to zero of an infinitesimal value $\eta_n(A,x)$ uniformly over the class $K(B_0,C_0)$.

We will give the corresponding results in the next theorem.

Theorem 2. *Given* $n\to\infty, A\to 1$ *and* $|1-x|\ge r>0$, *the infinitesimal function* $\eta_n(A,x)$ *in Theorem 1 has the following estimates:*

1°. *If* $\min\left(n,\frac{1}{|A-1|}\right)=n$, *then*

$$\sup_{|x-1|>r>0}|\eta_n(A,x)|=O\left(\frac{\ln g(n,A)}{g(n,A)}\right); \tag{4}$$

2°. *If* $\min\left(n,\frac{1}{|A-1|}\right)=\frac{1}{|A-1|}$, *then*

$$\sup_{|x-1|>r>0}|\eta_n(A,x)|=O(|A-1|)+O\left(\frac{\ln g(n,A)}{g(n,A)}\right); \tag{5}$$

uniformly over the class of derivative functions $F(x)\in K(B_0,C_0)$.

The proof of Theorem 2 is based on the following auxiliary statements.

Lemma 1. *The function* $g(n,A)$ *is increasing for each argument when the second argument is fixed. For all* $n\ge 2$ *and* $A>0$ *there is an inequality*

$$g(n,A)\ge\frac{1}{2}\min\left(n,\frac{1}{|1-A|}\right).$$

This lemma is given in the monograph [1] (Ch.III, §4, p. 104).

2. Key Findings

Now consider a branching random process with discrete time starting with a random number ζ of particles. Assume that the random variable $\zeta-$ is positive and integer with mathematical expectation $E\zeta=m$ and variance $D\zeta=\sigma^2>0$

Let $Z_{n\zeta}-$ be the number of particles by the time n. As easily follows from the definition of the process under consideration, the random variable (c. c.) $Z_{n\zeta}$ can be represented as a sum of independent, singly distributed random variables with a random number of summands:

Received: July 12, 2025

$$Z_{n\zeta}=Z_1^{(n)}+Z_2^{(n)}+\ldots+Z_{\zeta}^{(n)}, \qquad \left(Z_i^{(0)}=1\right)$$

where $Z_i^{(n)}-$ is the number of particles produced $i-$ by a particle over n generations. ($i-$ number of the particle), with s. c. $Z_i^{(n)}-$ lone distributed with Z_n- number of particles by the time n in the Galton-Watson process starting from a single particle.

Let us now introduce the derivative functions of the random variables considered here.

$$\Lambda(s)=\sum_{k=1}^{\infty}P(\zeta=k)s^k=\sum_{k=1}^{\infty}w_k s^k,\ F_n(s)=\sum_{k=0}^{\infty}P(Z_n=k)s^k=\sum_{k=0}^{\infty}P_{nk}s^k,\ |s|\le 1,\ F(s)=F_1(s).$$

How easy it is to see that $\sum_{k=0}^{\infty}P\left(Z_{n\zeta}=k\right)s^k=\Lambda\left(F_n(s)\right)$ and let

$$\overline{Q}_n(s)=1-\Lambda\left(F_n(s)\right),\ \overline{Q}_n=1-\Lambda\left(F_n(0)\right).$$

For the random variable $Z_{n\zeta}$ it is easy to calculate the mathematical expectation and variance, which have in the following form:

$$EZ_{n\zeta}=mA^n,\ \ EZ_{n\zeta}^2=mDZ_n+A^{2n}E\zeta^2,$$

$$DZ_{n\zeta}=\sigma_n^2=\begin{cases} m\dfrac{A^n\left(A^n-1\right)\left(B+A-A^2\right)}{A^2-A}+A^{2n}\sigma^2, & if \quad A>1,\\ mnB+\sigma^2, & if \quad A=1,\\ m\dfrac{A^n\left(1-A^n\right)\left(B+A-A^2\right)}{A-A^2}+A^{2n}\sigma^2, & if \quad A<1.\end{cases}$$

where $A=F'(1),\ B=F''(1),\ C=F'''(1)$.

Similarly, the conditional mathematical expectation of c. c. $Z_{n\zeta}$ is of the form:

$$E\left(Z_{n\zeta}/Z_{n\zeta}>0\right)=\frac{EZ_{n\zeta}}{P\left(Z_{n\zeta}>0\right)}=\frac{mA^n}{1-\Lambda\left(F_n(0)\right)}=\frac{mA^n}{\overline{Q}_n}.$$

Let's put $S_{n\zeta}(x)=P\left\{\dfrac{Z_{n\zeta}}{E\left(Z_{n\zeta}/Z_{n\zeta}>0\right)}<x/Z_{n\zeta}>0\right\}$ and the exemplary distribution

$$S(x)=\begin{cases}1-e^{-\frac{x}{m}}, & if \quad x>0,\\ 0, & if \quad x\le 0.\end{cases}$$

Theorem 3. *If* $A=1,\ 0<B<\infty,\ \sigma^2<\infty$ *, then at* $n\to\infty$

Received: July 12, 2025

$$S_n(x) \to S(x).$$

In transient phenomena in the theory of branching random processes, following B.A. Sevastyanov [2], one introduces a class of derivative functions (a class of processes, satisfying the conditions. A. Sevastyanov [2], introduce a class of derivative functions (class of processes) $K(B_0, C_0)$, satisfying the conditions $K(B_0,C_0)=\left\{F : F''(1)=B \geq B_0 > 0,\ F'''(1)=C \leq C_0 < \infty\right\}.$

Using lemma 1 and theorem 2 we prove the following theorem:

Theorem 4. *Let* $n \to \infty,\ A \to 1$ *and* $|s-1| \geq \varepsilon > 0$ *then the equality*

$$\overline{Q}_n(s) = \bar{r}_n(s)\left(1+\eta_{n\zeta}(s)\right)$$

where $\bar{r}_n(s) = \dfrac{mA^n(1-s)}{1+\dfrac{B}{2}g(n,A)(1-s)}$

and

$$\sup_{|s-1|\geq\varepsilon>0} \left|\eta_{n\zeta}(s)\right| = O\left(\max\left(\frac{\ln g(n,A)}{g(n,A)}, |A-1|\right)\right)$$

uniformly across all producing functions $F(s) \in K(B_0, C_0)$ *and* $|s| \leq 1$

*Proof of Theorem 4***.** Since s. c. c. ζ has mathematical expectation m and variance $\sigma^2 > 0$ then using Taylor expansion we have

$$\Lambda(s) = 1 + m(s-1) + O\left((s-1)^2\right)$$

Hence, replacing s with $F_n(s)$ we get

$$1-\Lambda\left(F_n(s)\right) = m\left(1-F_n(s)\right) + O\left(\left(1-F_n(s)\right)^2\right) \tag{6}$$

or $\overline{Q}_n(s) = mQ_n(s) + O\left(Q_n^2(s)\right)$

From here and from Theorem 1 - 3, the statements of Theorem 4 easily follow. Theorem 4 is proved.

Theorem 5. *Let the conditions of Theorem 4 be satisfied. Then at* $n \to \infty,\ A \to 1$

Received: July 12, 2025

$$\sup_x \left|S_{n\zeta}(x)-S(x)\right| = O\left(\ln mg(n,A)\cdot \max\left(|A-1|, \frac{\ln g(n,A)}{g(n,A)}\right)\right)$$

evenly across all . $F(s)\in K(B_0,C_0)$.

Proof of Theorem 5. It is obvious that the derivative function of conditional probabilities

$$P\left(Z_{n\zeta}=k/Z_{n\zeta}>0\right)=P\left\{Z_1^{(n)}+Z_2^{(n)}+\ldots+Z_\zeta^{(n)}=k/Z_{n\zeta}>0\right\}$$

looks like

$$\sum_{k=1}^{\infty} P\left(Z_{n\zeta}=k/Z_{n\zeta}>0\right)s^k=\frac{\Lambda(F_n(s))-\Lambda(F_n(0))}{1-\Lambda(F_n(0))}.$$

Hence we obtain that the characteristic function of the random variable $\frac{\bar{Q}_n Z_{n\zeta}}{A^n}$, under the condition $Z_{n\zeta}>0$, is equal to

$$\varphi_n(\tau)=\frac{\Lambda\left(F_n\left(e^{i\tau\bar{Q}_nA^{-n}}\right)\right)-\Lambda(F_n(0))}{1-\Lambda(F_n(0))}=1-\frac{1-\Lambda\left(F_n\left(e^{i\tau\bar{Q}_nA^{-n}}\right)\right)}{1-\Lambda(F_n(0))}=1-\frac{1-\Lambda\left(F_n\left(e^{i\tau\bar{Q}_nA^{-n}}\right)\right)}{\bar{Q}_n}.$$

Putting $s=e^{i\tau\bar{Q}_nA^{-n}}$ in equality (6), we easily obtain

$$\frac{1-\Lambda\left(F_n\left(e^{i\tau\bar{Q}_nA^{-n}}\right)\right)}{1-\Lambda(F_n(0))}=m\frac{1-F_n\left(e^{i\tau\bar{Q}_nA^{-n}}\right)}{\bar{Q}_n}+O\left(\bar{Q}_n\left|\frac{1-F_n\left(e^{i\tau\bar{Q}_nA^{-n}}\right)}{\bar{Q}_n}\right|^2\right).$$

It follows easily from Theorem 1 that

$$1-F_n(s)=\frac{A^n(1-s)}{1+\frac{B}{2}\cdot g(n,A)(1-s)}\left(1+\eta_n(s)\right),$$

where $\sup_{|s-1|\geq\varepsilon>0}\left|\eta_n(s)\right|=O\left(\frac{\ln g(n,A)}{g(n,A)}\right)$.

Hence, putting $s=e^{i\tau\bar{Q}_nA^{-n}}$ we have

$$\varphi_n(\tau)=\frac{1}{1-i\tau m}+\frac{i\tau m}{1-i\tau m}\cdot\frac{2}{Bg}-\frac{i\tau m}{(1-i\tau m)^2}\cdot\frac{1+\frac{B}{2}g}{\left(\frac{B}{2}g\right)^2}+O\left(\frac{i\tau m}{1-i\tau m}\cdot\frac{Q_n}{\frac{B}{2}g}\right)-$$

$$-\left(\eta_n(s)+O\left(Q_n\left(1+\eta_n(s)\right)\right)\right)\left(-\frac{i\tau m}{1-i\tau m}\left(1+\frac{B}{2}g\right)+\frac{i\tau m}{(1-i\tau m)^2}\cdot\frac{1+\frac{B}{2}g}{\left(\frac{B}{2}g\right)^2}+O\left(\frac{i\tau m}{1-i\tau m}\cdot\frac{Q_n}{\frac{B}{2}g}\right)\right)$$

Now using Essen's theorem, we have

$$\sup_x\left|S_{n\zeta}(x)-S(x)\right|=\int_{-T}^{T}\left|\frac{\varphi_n(\tau)-\phi(\tau)}{\tau}\right|d\tau+\frac{C}{T}=O\left(\ln m\, g(n,A)\cdot\frac{\ln g(n,A)}{g(n,A)}\right)$$

here $C,\ \delta_0,\ \delta-$ constants and

$$T=\delta_0 E\left(Z_{n\zeta}/Z_{n\zeta}>0\right)=\delta_0\frac{mA^n}{\overline{Q}_n}=\delta_0\frac{mA^n}{mQ_n\left(1+O\left(\frac{Q_n}{m}\right)\right)}=$$

$$=\delta_0\frac{mA^n}{m\dfrac{A^n}{1+\frac{B}{2}g(n,A)}\left(1+O\left(\frac{Q_n}{m}\right)\right)}\sim\delta g(n,A).$$

The second part of Theorem 5 is proved similarly. Theorem 5 is proved.

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