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FIXED POINT THEOREMS AND STABILITY RESULTS FOR NONLINEAR OPERATORS WITH APPLICATIONS TO DIFFERENTIAL AND INTEGRAL EQUATIONS

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Abstract

The theory of fixed point theorems is a core of the study of nonlinear operators, and offers the key to existence and uniqueness proofs of solutions to differential and integral equations. This paper discusses three numerical experiments that exercise the theoretical and practical applications of these theorems through matlab. The first experiment describes how Banach iteration converts to a unique fixed point. The second looks at the asymptotic stability of a linear ordinary differential equation, and it compares the numerical solutions with the exact analytical solution. The third is a picture of Picard successive approximation to an integral equation of volterra, which shows the consistency between numerical and analytical solutions. The experiments all point to the interaction of theory and computation, and show the usefulness of fixed point methods in nonlinear problems and in studying stability. Future applications of the work can include more complex partial and integro-differential equations, engineering, physics, and economics applications.

***Index Terms*—**Fixed Point Theorems, Banach Contraction Principle, Stability Analysis, Nonlinear Operators, Ordinary Differential Equations, Volterra Integral Equations, Picard Iteration, Numerical Methods, Convergence, Integral Equations

I. INTRODUCTION

Fixed Point Theorems represent an elementary branch of nonlinear analysis and functional analysis, and they provide very useful tools in proving that a solution to a broad range of mathematical problems do exist, and that only one of them exists. An operator T has a point xt, a fixed point, then T(x)=x . Such theorems are especially useful when nonlinear operators are involved, as in those cases, direct analytical solutions are frequently hard or impossible to find.

Fixed point theory has a long history of application to ordinary and partial differential equations, integral equations and dynamical systems. When a fixed point exists there is often

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a solution to the underlying equation, so the information about stability can be harnessed to learn about the behaviour of solutions to small perturbations [7].

This paper provides a discussion on the concepts and on the practice concerning fixed point theorems, including the Contraction Principle developed by Banach, the stability of the differential equations and the iterative solution of the Volterra integrals equations. We demonstrate numerically, using MATLAB, how these concepts can be applied to the solving of nonlinear problems, stability analysis, and the comparison of numerical solutions and real solutions. It also notes the close connection between abstract mathematical theory and concrete computational methods, in both existence, uniqueness and stability of solutions to nonlinear problems.

Lanach Fixed Point Theorem (Contraction Mapping Principle)

The Banach Fixed Point Theorem, also known as the Contraction Mapping Principle, is a fundamental result in nonlinear analysis. It states that if (X,d) is a complete metric space and T:X→XT is a contraction mapping — that is, there exists a constant 0< k <10 such that

then T has a unique fixed point x∗∈X satisfying T(x∗)=x∗T. Moreover, for any initial guess x0∈X, the sequence defined iteratively by xn+1=T(xn) converges to x∗x^* at a geometric rate controlled by the contraction constant k [1].

This theorem is of theoretical interest, but also offers a constructive approach to finding approximate solutions to nonlinear equations, differential equations and integral equations. Applications It can be used to establish existence and uniqueness of solutions and to write iterative numerical algorithms which will converge given that the contraction condition is satisfied. The Banach theorem is particularly strong since, unlike other fixed point theorems, it can be used to obtain both existence and uniqueness at once[2].

2 Schauder Fixed Point Theorem

The Schauder Fixed Point Theorem generalizes the fixed point concept to operators that may not be contractions but are continuous and map a convex, closed, and bounded subset of a Banach space into itself. Formally, if C is a convex closed bounded subset of a Banach space X and T:C→C is continuous, then T has at least one fixed point in C [3].

Schauder, unlike Banach, in his theorem does not have a uniqueness allusion, nor does it show a constructive iteration procedure. It is desirable with compact operators, particularly when applied to nonlinear boundary value problems, and to integral equations. It is used in existence proofs when the contraction property fails, thus allowing a researcher to conclude the existence of solutions to more general nonlinear operators.

It is particularly in functional analysis and PDEs and applied mathematics that the theorem of Schauder comes in particularly handy, allowing the formulation of problems in an abstract space and ensuring the existence of a solution without necessarily providing it. This has been frequently used together with other tools, e.g. a priori estimates, to study the stability and qualitative behavior of solutions[4].

II. Brouwer Fixed Point Theorem

The Brouwer Fixed Point Theorem is a classical result in finite-dimensional spaces. It states that any continuous function f mapping a compact convex subset of {R}^n into itself has at

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least one fixed point. In other words, if D⊂Rn, compact, and f:D→Df:, then there exists x∗∈D such that f(x∗)=x∗[5].

The theorem is existential: there is always a fixed point, but does not tell how to compute it. It carries far-reaching consequences in many areas of interest, such as topology, game theory, economics, and differential equations, where an explicit formula may not be known [6].

The theorem by Brouwer is sometimes considered as a finite dimensional analogue to the Schauder theorem, extending the idea of fixed point in a linear or simple space to a more complex continuous mapping. Geometric reasoning is intuitive: it is always possible to map a disk continuously onto itself, so the point fixed is never moved.

Experiments And Scenarios

In this paper, by using matlab three numerical problems are described to show how fix point theorems and stability analysis are applied. Experiment I gives an example of how iteration to a fixed unique point converges with respect to Banach. The second is an overview of the asymptotic stability of an ordinary differential equation and the third is a Picard successive approximation to solve a Volterra integral equation and confirm the numerical and analytical solutions.

A. Banach Fixed Point Iteration

The Banach Fixed Point Theorem is that a contraction mapping TT of a complete metric space has a unique fixed point. It gives also a constructive way to locate this fixed point by iteration:

Objective

Show that it is converging to some specific fixed point.

1. Show the technical use of the Banach theorem.

Method

• Operator: T(x)=cos(x)

• Initial guess: x0=1

• Iteration continues until

Expected Result

• The unique fixed point x 8 has convergence to x at 0.739.

• Iteration process as represented through a convergence plot.

Significance

• Gives a working example of fixed point theory application to nonlinear problems.

• Is a basis of solving more difficult differential or integral equations.

Figure shows the result of Banach Fixed Point scenario, after using MATLAB to solve the equations.

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Figure 1 shows the result of Banach Fixed Point scenario result

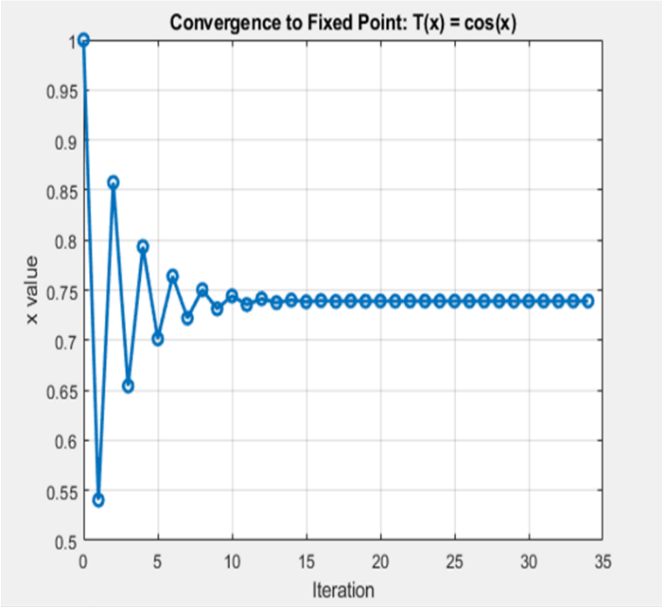

Figure 1 result of Banach Fixed Point scenario.

B. ODE Stability Test

In dynamical systems stability of solutions is a measure of how close to an equilibrium point a solution is. In the ODE dydt= -2y the equilibrium point y=0 is asymptotically stable.

Objective

• Use fixed points to exhibit stability of ODES.

• Compare numerical solutions and analytical solutions.

Method

• Solve dydt=−2y, y(0)=1y(0) = 1

• Numerical approximation Use ode45 solver in MATLAB.

• Graphical solutions of plots are to be plotted.

Expected Result

• Solution approaches zero, which proves the existence of asymptotic stability.

• • Analytical solution is close to numerical solution.

Significance

• Shows how stability concepts apply in dynamical systems which are continuous.

• Confirms that numerical techniques are appropriate to analyze stability.

Figure shows ODE Stability Test scenario results, after using MATLAB to solve the equations.

Figure 2 Figure shows ODE Stability Test scenario result

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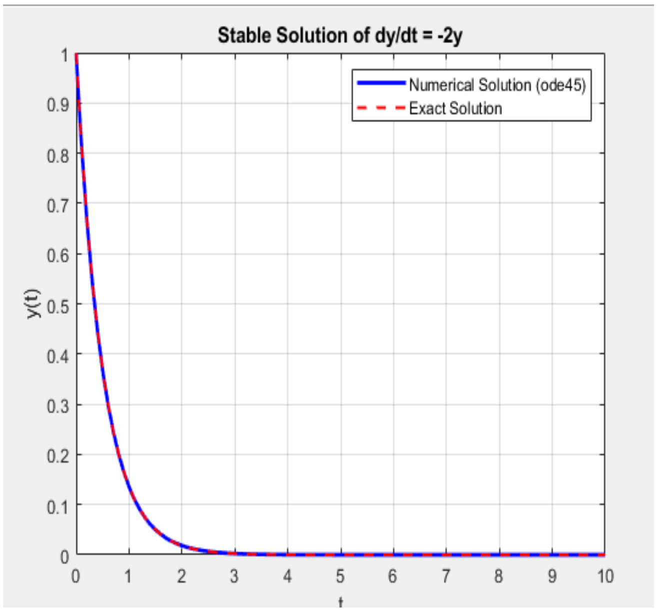

Figure 2 ODE Stability Test scenario result

C. Volterra Integral Equation

Volterra integral equations of the second kind often arise in hereditary systems such as viscoelastic materials or population dynamics.

Equation considered:

The solution corresponds to a fixed point of the operator TT:

Objective

• Picard successive approximation (iterative method): find the numerical solution.

• Compare the numerical solution with the exact solution y(t)=cosh(t)y(t).

Method

• Start with y0(t)=1y_0(t) = 1

• Iterate:

yn+1(t)=1+∫0t(t−s)yn(s) dsy_{n+1}(t) = 1 + \int_0^t (t-s) y_n(s) \, ds

• Use trapezoidal rule of numerical integration.

Expected Result

• Numerical solution converges fast towards the precise solution.

• graph Numerical and analytical results agree very well.

Significance

• Demonstrates how fixed point methods are used in the integral equations.

• Gives a scheme to numerically solve more complicated integral or integro-differential equations.

Figure 3 shows Volterra Integral Equation result, after using MATLAB to solve the equations.

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Figure 3 Volterra Integral Equation result

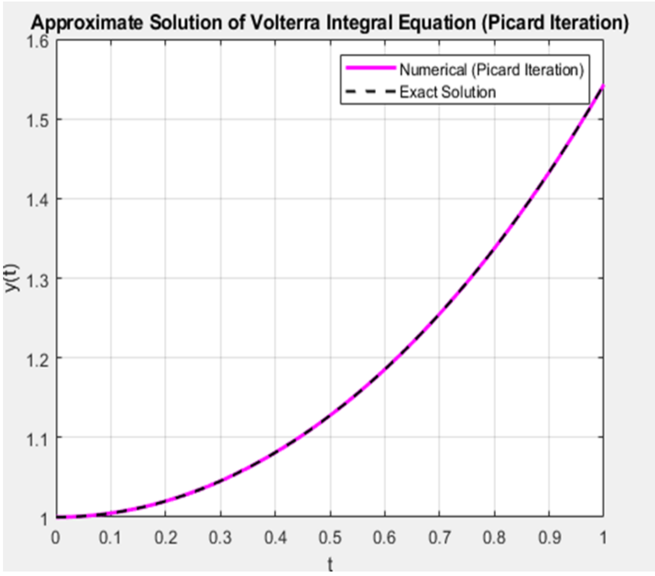

Figure 1 Volterra Integral Equation result

III. SUMMARY AND CONCLUSION

In this work, three numerical experiments have been done to investigate theoretical and practical methods of applying the fixed point theorems and the stability analysis to nonlinear operators. In the context of the first experiment, Banach being an application of the Fixed Point Theory, it was shown that these iterative processes converged to a single fixed point, and this example provides a good illustration of the existence and uniqueness of a simple nonlinear operator. This shows that fixed point theory is useful practical in the development of sound numeric algorithm to solve nonlinear equations.

The second experiment was on the stability of an ordinary differential equation. As the analytical and numerical solutions of the linear ODE $dy/dt=-2y$ showed that the solution asymptotically approaches the equilibrium point $y=0$, it was demonstrated that the concept of asymptotic stability holds true. The experiment brings out the relationship between fixed points and the equilibrium solutions of dynamical system and justifies the effectiveness of numerical solvers in stability analysis.

The third experiment was a Volterra integral equation which was solved using Picard successive approximation. Numical solution quickly approached the perfect analytic solution $y(t) = \cosh(t)y(t)$, an example of how fixed point methods can be used successfully to solve integral equations. The combination of abstract mathematical theory and the method of computation provides a viable example of the validity and consistency of numerical iterative methods.

In general, these experiments highlight the interaction between computation and theory. Existence, uniqueness, and convergence theories are developed based on fixed point theorems, and the MATLAB numerical simulations allow the visualization, validation, and practical implementation of such theories. Its outcomes verify mathematically rigorously the fact that fixed point methods are highly effective in addressing nonlinear differential and integral

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equations.

Future Applications

The present paper shows the convenience of fixed point theorems and the iterative numerical methods in the investigation of the nonlinear operators and the distance equations and the integral equations. In future studies, these methods may be expanded in a number of ways to include more complex and realistic issues. A promising direction is the fix point approach to nonlinear partial differential equations (PDEs), in which the existence, uniqueness and stability of solutions are often difficult to prove analytically. Contraction principle could be implemented to develop iterative numerical methods (i.e. picard-type iterative numerical methods) to solve such PDEs in an efficient manner that converges. The other direction that can be considered is the study of integro-differential equations and delay differential equations that are commonly used in engineering, physics, biology, and economics. A fixed point theorems based stability analysis would be useful in this case both to determine the long term behaviour of the system and to provide an insight into equilibrium solutions. Moreover, iterative computational algorithm (e.g. dynamically adjusting step sizes, high-order integration, parallel computer) must be optimized further so that in future, big problems will be more accurate, and efficient. Finally, the ideas here can be applied to practical dynamical systems, such as population dynamics, control systems, viscoelastic materials, and economic systems, where nonlinearities and memory effects are critical and it is essential to have both strong numerics and strong theory. Not only would such applications prove the mathematical theory, but they would also provide some useful solutions to complex scientific and engineering problems

Acknowledgment

The preferred spelling of the word “acknowledgment” in American English is without an “e” after the “g.” Use the singular heading even if you have many acknowledgments. Avoid expressions such as “One of us (S.B.A.) would like to thank ... .” Instead, write “F. A. Author thanks ... .” **Sponsor and financial support acknowledgments are placed in the unnumbered footnote on the first page, not here.**

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The second paragraph uses the pronoun of the person (he or she) and not the author's last name. It lists military and work experience, including summer and fellowship jobs. Job titles are capitalized. The current job must have a location; previous positions may be listed without one. Information concerning previous publications may be included. Try not to list more than three books or published articles. The format for listing publishers of a book within the biography is: title of book (city, state: publisher name, year) similar to a reference. Current and previous research interests end the paragraph.

The third paragraph begins with the author's title and last name (e.g., Dr. Smith, Prof. Jones, Mr. Kajor, Ms. Hunter). List any memberships in professional societies other than the IAENG. Finally, list any awards and work for IAENG committees and publications. If a photograph is provided, the biography will be indented around it. The photograph is placed at the top left of the biography. Personal hobbies will be deleted from the biography.

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