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Graph-Theoretic Insights into Digraph Complement $\overline{\Gamma(n,2)}$

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Abstract

For each positive integer n, we assign a power digraph modulo n denoted by $\Gamma(n, 2)$ whose vertex set is $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, and for which there is a directed edge from a vertex a to a vertex b if and only if $a^2 \equiv b \ (mod\ n)$, where $a, b \in \mathbb{Z}_n$. We investigate the graph-theoretic properties of the complement of the digraph $\Gamma(n, 2)$, denoted $\overline{\Gamma(n,2)}$. In particular, we study fixed points, isolated points, in-degree, and out-degree of the vertices of the digraph $\overline{\Gamma(n,2)}$.

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1 Introduction

In the last few years, the investigation into the interrelations among Graph theory, Group theory, and Number theory has become an attractive and effective area of study, for example, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13]. In the paper [9], L. Somer *et al.* introduced the quadratic power digraph modulo n denoted by $\Gamma(n, 2)$, whose vertex set is $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and there is a directed arc from the vertex x to the vertex y if and only if $x^2 \equiv y \ (mod\ n)$, where $x, y \in \mathbb{Z}_n$. In the paper [13], S. K. Thakur *et al.* has discussed the complement of the digraph $\Gamma(n, 2)$, denoted by $\overline{\Gamma(n, 2)}$. In this paper, the authors also introduced the Universal directed graph, denoted by $\mathbf{U}_n$ and investigated some results connecting digraphs $\Gamma(n, 2)$ and $\overline{\Gamma(n, 2)}$.

It is important to mention that the study of the digraphs $\overline{\Gamma(n, 2)}$ and $\mathbf{U}_n$ is still open. In this paper, we try to establish some relations connecting the digraphs $\Gamma(n, 2)$, $\overline{\Gamma(n, 2)}$ and $\mathbf{U}_n$. We organize the rest of the paper as follows:

In Section 2, we provide definitions and basic results. In Section 3, we establish results on fixed points and isolated points of the digraph $\overline{\Gamma(n, 2)}$. In Section 4, we present results on the in-degrees and out-degrees of the vertices of the digraph $\overline{\Gamma(n, 2)}$. Finally, in Section 5, we discuss some results related to the digraphs $\Gamma(n, 2)$, $\overline{\Gamma(n, 2)}$ and $\mathbf{U}_n$.

2 Preliminaries

For a positive integer n, we consider a directed graph $\Gamma(n, 2)$ whose vertex set is $\mathbb{Z}_n$ and any two vertices $x, y \in \mathbb{Z}_n$ are connected by exactly one directed arc from x to y if and only if

$$x^2 \equiv y \ (mod\ n)$$

We denote the vertex set and arc set of the digraph $\Gamma(n, 2)$ by $V(\Gamma)(= \mathbb{Z}_n)$ and $A(\Gamma)$ respectively.

The distinct vertices $v_1, v_2, v_3, \ldots, v_t$ in $V(\Gamma)$ will form a cycle of length t if

$$\begin{gathered} {v_1}^2 \equiv v_2 (mod\ n), \\ {v_2}^2 \equiv v_3 (mod\ n), \\ {v_3}^2 \equiv v_4 (mod\ n), \\ \vdots \\ {v_t}^2 \equiv v_1 (mod\ n) \end{gathered}$$

We refer to a cycle of length t as a t- cycle, and a cycle of length 1 as a fixed point (or a self-loop).

A vertex is isolated if it is not connected to any other vertex in $\Gamma(n, 2)$.
In the digraph $\Gamma(n, 2)$, the in-degree of a vertex $v \in V(\Gamma)$, denoted by $d_\Gamma^-(v)$ is the number of directed arcs incident into the vertex v and the out-degree of a vertex v, denoted by $d_\Gamma^+(v)$ is the number of directed arcs incident out of the vertex v. The total degree (or simply degree) of a vertex $v \in V(\Gamma)$, denoted by $d_\Gamma(v)$ is the sum of the out-degree and the in-degree of v i.e., $d_\Gamma(v) = d_\Gamma^+(v) + d_\Gamma^-(v)$.
For an isolated point $v \in V(\Gamma)$, $d_\Gamma^+(v) = d_\Gamma^-(v) = 0$ and for an isolated fixed point $v \in V(\Gamma)$, $d_\Gamma^+(v) = d_\Gamma^-(v) = 1$.
If $d_\Gamma^+(v) = d_\Gamma^-(v)$ for every vertex $v \in V(\Gamma)$, then the digraph $\Gamma(n, 2)$ is said to be an iso-digraph $(mod\ n)$ or balanced digraph $(mod\ n)$ and if $d_\Gamma^+(v) = d_\Gamma^-(v) = k$ for every vertex $v \in V(\Gamma)$, then the digraph $\Gamma(n, 2)$ is said to be a regular digraph of degree k (or k-regular digraph).

Theorem 2.1. *[14] (Handshaking theorem) In the digraph* $\Gamma(n, 2)$*,*

$$\sum_{v \in V(\Gamma)} d_\Gamma^+(v) = \sum_{v \in V(\Gamma)} d_\Gamma^-(v) = |A(\Gamma)|$$

Similarly, in the complement digraph $\overline{\Gamma(n,2)}$, the in-degree of a vertex $v \in V(\overline{\Gamma})$, denoted by $d_{\overline{\Gamma}}^-(v)$ is the number of directed arcs incident into the vertex v and the out-degree of a vertex v, denoted by $d_{\overline{\Gamma}}^+(v)$ is the number of directed arcs incident out of the vertex v. For an isolated point $v \in V(\overline{\Gamma})$, $d_{\overline{\Gamma}}^+(v) = d_{\overline{\Gamma}}^-(v) = 0$ and for an isolated fixed point $v \in V(\overline{\Gamma})$, $d_{\overline{\Gamma}}^+(v) = d_{\overline{\Gamma}}^-(v) = 1$. The total degree (or simply degree) of a vertex $v \in V(\overline{\Gamma})$, denoted by $d_{\overline{\Gamma}}(v)$ is the sum of the out-degree and the in-degree of v i.e., $d_{\overline{\Gamma}}(v) = d_{\overline{\Gamma}}^+(v) + d_{\overline{\Gamma}}^-(v)$. If $d_{\overline{\Gamma}}^+(v) = d_{\overline{\Gamma}}^-(v)$ for every vertex $v \in V(\overline{\Gamma})$, then the digraph $\overline{\Gamma(n,2)}$ is said to be an iso-digraph $(mod\ n)$ or balanced digraph $(mod\ n)$ and if $d_{\overline{\Gamma}}^+(v) = d_{\overline{\Gamma}}^-(v) = k$ for every vertex $v \in V(\overline{\Gamma})$, then the digraph $\overline{\Gamma(n,2)}$ is said to be a regular digraph of degree k (or k-regular digraph).

Digraphs with at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called asymmetric or anti-symmetric digraphs. Digraphs in which for every directed arc (a, b) there is also a directed arc (b, a) are called symmetric digraphs. A complete symmetric digraph is a digraph with exactly one directed arc from every vertex to every other vertex.

An oriented graph is a digraph with no cycle of length two. A tournament is an oriented graph where every pair of distinct vertices are adjacent. In other words, a digraph D with vertex set $\{v_1, v_2, v_3, . . . , v_n\}$ is a tournament if exactly one of the arcs $v_i\, v_j$ or $v_j\, v_i$ is in D for every $i \neq j \in \{1, 2, 3, . . . , n\}$.

Definition 2.1. [13] A universal directed graph (or Universal digraph) is a complete symmetric digraph with self-loops at each vertex. We denote a universal directed graph having n vertices by $\mathbf{U}_n$.

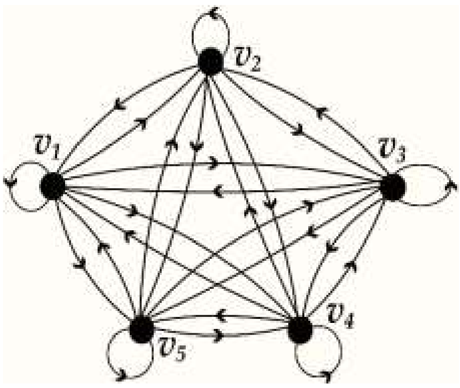

Figure 1: Universal directed graph $\mathbf{U}_5$

Definition 2.1. [13] The complement of the digraph $\Gamma(n, 2)$ denoted by $\overline{\Gamma(n, 2)}$ is defined as the digraph having the same vertex set $V(\Gamma)$ as of $\Gamma(n, 2)$ and there will be a directed arc from x to y in $\overline{\Gamma(n, 2)}$ if and only if $x^2 \not\equiv y \ (mod\ n)$, where $x, y \in V(\Gamma)$.

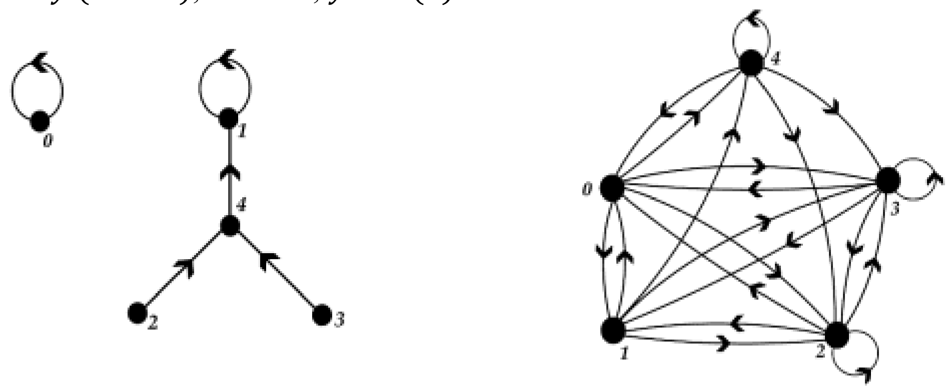

Figure 2: Digraph $\Gamma(5, 2)$ and its complement digraph $\overline{\Gamma(5, 2)}$

From the definition of $\overline{\Gamma(n, 2)}$, it follows that a vertex x in $\overline{\Gamma(n, 2)}$ is a fixed point iff $x^2 \not\equiv x \ (mod\ n)$.

Theorem 2.2. *[11] The number of fixed points of* $\Gamma(n, 2)$ *is* $2^{\omega(n)}$*, where* $\omega(n)$ *is the number of distinct primes dividing n.*

Theorem 2.3. *[13]* $d_{\Gamma}^{-}(v) + d_{\overline{\Gamma}}^{-}(v) = n$, *for every vertex* $v \in V(\Gamma)$.

Theorem 2.4. *[13]* $d_{\Gamma}^{+}(v) + d_{\overline{\Gamma}}^{+}(v) = n$, *for every vertex* $v \in V(\Gamma)$.

Theorem 2.5. *[13]* $d_{\Gamma}(v) + d_{\overline{\Gamma}}(v) = 2n$, *for every vertex* $v \in V(\Gamma)$.

Theorem 2.6. *[13] In the digraph* $\Gamma(n,2)$, $\sum_{i=1}^{n} d_{\Gamma}(v_i) = 2n$; $v_i \in \mathrm{V}(\Gamma)$.

Theorem 2.7. *[13] In the digraph* $\overline{\Gamma(n,2)}$, *the outdegree of each vertex is* $(n-1)$ *i.e.,* $d_{\overline{\Gamma}}^{+}(v) = n-1, for\ every\ vertex\ v \in V(\Gamma)$.

3 Fixed points, Isolated points of $\overline{\Gamma(n,2)}$

In this section, we try to discuss fixed points and isolated points of the complement digraph $\overline{\Gamma(n,2)}$.

Theorem 3.1. *The numbers* 2 *and* $(n-1)$ *are always fixed points of* $\overline{\Gamma(n,2)}$, *for n* > 2.

Proof. A vertex v in $\overline{\Gamma(n,2)}$ is a fixed point iff $v^2 \not\equiv v \ (mod\ n)$.
As $2^2 \not\equiv 2\ (mod\ n)$ so 2 is a fixed point of $\overline{\Gamma(n,2)}$.
Also,

$$\begin{aligned}(n-1)^2 &= n^2 - 2n + 1 \\ &\equiv 1\ (mod\ n) \\ &\not\equiv (n-1)\ (mod\ n)\end{aligned}$$

Hence, 2 and $(n-1)$ are always fixed points in $\overline{\Gamma(n,2)}$ for *n* > 2. □

Theorem 3.2. *A vertex* a *is a fixed point in* $\overline{\Gamma(n,2)}$ *if and only if* $\{n - (a - 1)\}$ *is a fixed point in* $\overline{\Gamma(n,2)}$.

Proof. Suppose, a is a fixed point in $\overline{\Gamma(n,2)}$ then $a^2 \not\equiv a\ (mod\ n)$.
We have,

$$\begin{aligned}\{n - (a - 1)\}^2 &= n^2 - 2n(a-1) + (a-1)^2 \\ &\equiv a^2 - 2a + 1\ (mod\ n)\end{aligned}$$

$$\not\equiv a - 2a + 1 \ (mod\ n)$$

$$\not\equiv -(a-1) \ (mod\ n)$$

$$\not\equiv \{n - (a-1)\} \ (mod\ n)$$

This shows that $\{n - (a - 1)\}$ *is a fixed point in* $\overline{\Gamma(n,2)}$.

Conversely, suppose $\{n - (a - 1)\}$ *is a fixed point in* $\overline{\Gamma(n,2)}$ then

$$\{n - (a - 1)\}^2 \not\equiv \{n - (a-1)\} \ (mod\ n)$$

$$\Rightarrow n^2 - 2n(a-1) + (a-1)^2 \not\equiv \{n - (a-1)\} \ (mod\ n)$$

$$\Rightarrow (a-1)^2 \not\equiv -(a-1) \ (mod\ n)$$

$$\Rightarrow a^2 \not\equiv a \ (mod\ n)$$

This shows that the vertex a is a fixed point in $\overline{\Gamma(n,2)}$. □

Theorem 3.3. *If* $n = p^k$ $(k \geq 1)$, *then the fixed points of* $\overline{\Gamma(n,2)}$ *are* $2, 3, 4, \cdots, (n-1)$, *where* $n > 2$ *and* p *is a prime.*

Proof. Let us consider the congruence,

$$x^2 \equiv x \ (mod\ n)$$

$$\text{i.e., } x^2 \equiv x \ (mod\ p^k)$$

$$\Rightarrow x \equiv 0 \ (mod\ p^k) \text{ or } x \equiv 1 \ (mod\ p^k)$$

This shows that only solutions of $x^2 \equiv x \ (mod\ p^k)$ are 0 and 1 and so $0^2 \equiv 0 \ (mod\ p^k)$ and $1^2 \equiv 1 \ (mod\ p^k)$.

Thus, for any $a \in \{2, 3, 4, . . . , (n-1)\}$, we have $a^2 \not\equiv a \ (mod\ n)$ i.e., $a^2 \not\equiv a \ (mod\ p^k)$. This shows that a is a fixed point in $\overline{\Gamma(n,2)}$ i.e., $2, 3, 4, \cdots, (n-1)$ are fixed points in $\overline{\Gamma(n,2)}$, where $n = p^k$, $k \geq 1$ and p is a prime. □

Theorem 3.4. *If* $n = \prod_{i=1}^{k} {p_i}^{\alpha_i}$, *then* $\overline{\Gamma(n,2)}$ *has* $(n - 2^k)$ *number of fixed points, where* $p_1 < p_2 < . . . < p_k$ *are primes and* $\alpha_i \geq 0$.

Proof. Let $n = \prod_{i=1}^{k} {p_i}^{\alpha_i}$, where $p_1 < p_2 < . . . < p_k$ are primes and $\alpha_i \geq 0$. Then $\omega(n) = k$.

By Theorem 2.2, the number of fixed points in $\Gamma(n,2)$ is $2^{\omega(n)} = 2^k$. So, there are $(n-2^k)$ number of points in $\Gamma(n,2)$ which are not fixed points in $\Gamma(n,2)$ and therefore by the definition of $\overline{\Gamma(n,2)}$ it follows that the points that are not fixed points in $\Gamma(n,2)$ are fixed points in $\overline{\Gamma(n,2)}$ and hence there are $(n-2^k)$ number of fixed points in $\overline{\Gamma(n,2)}$. □

Theorem 3.5. *There is no isolated point in* $\overline{\Gamma(n,2)}$ *, for* $n > 1$.

Proof. If possible, let u be an isolated point in $\overline{\Gamma(n,2)}$, where $n > 1$. Then, by the definition of $\overline{\Gamma(n,2)}$, $u^2 \equiv v \ (mod\ n)$, for all v in $\Gamma(n,2)$ which implies that $d_\Gamma^+(u) > 1$ in $\Gamma(n,2)$, which is not possible as $d_\Gamma^+(u) = 1$, for all u in $\Gamma(n,2)$. Hence, there is no isolated point in $\overline{\Gamma(n,2)}$, for $n > 1$. □

Theorem 3.6. *In* $\overline{\Gamma(n,2)}$ *, a directed edge from the vertex* a *to the vertex* 0 *exists iff a directed edge from the vertex* $(n - a)$ *to the vertex* 0 *exists.*

Proof. Suppose a directed edge from the vertex a to the vertex 0 exist in $\overline{\Gamma(n,2)}$, then $a^2 \not\equiv 0 \ (mod\ n)$.

We have,

$$\begin{aligned}(n-a)^2 &= n^2 - 2na + a^2 \\ &\equiv a^2 \ (mod\ n) \\ &\not\equiv 0 \ (mod\ n)\end{aligned}$$

This shows that a directed edge from the vertex $(n - a)$ to the vertex 0 exists in $\overline{\Gamma(n,2)}$.

Conversely, let a directed edge from the vertex $(n - a)$ to the vertex 0 exists in $\overline{\Gamma(n,2)}$, then

$$\begin{aligned}&(n-a)^2 \not\equiv 0 \ (mod\ n) \\ &\Rightarrow n^2 - 2na + a^2 \not\equiv 0 \ (mod\ n) \\ &\Rightarrow a^2 \not\equiv 0 \ (mod\ n)\end{aligned}$$

This shows that a directed edge from the vertex a to the vertex 0 exists in $\overline{\Gamma(n,2)}$. □

Theorem 3.7. *A directed edge from the vertex a to the vertex b exists in $\overline{\Gamma(n,2)}$ if and only if a directed edge from the vertex $(n-a)$ to the vertex $(b-n)$ exists in $\overline{\Gamma(n,2)}$; $a, b \in \mathbb{Z}_n - \{0\}$.*

Proof. We have,

$$(n-a)^2 \not\equiv (b-n)\ (mod\ n)$$

$$\Leftrightarrow n^2 - 2na + a^2 \not\equiv (b-n)\ (mod\ n)$$

$$\Leftrightarrow a^2 \not\equiv b\ (mod\ n)$$

This shows that a directed edge from the vertex a to the vertex b exists in $\overline{\Gamma(n,2)}$ if and only if a directed edge from the vertex $(n-a)$ to the vertex $(b-n)$ exists in $\overline{\Gamma(n,2)}$; $a, b \in \mathbb{Z}_n - \{0\}$. □

4 In-degree, Out-degree of the vertices of $\overline{\Gamma(n,2)}$

In this section, we try to discuss some results on in-degree and out-degree of the vertices of the complement digraph $\overline{\Gamma(n,2)}$.

Theorem 4.1. *In $\overline{\Gamma(n,2)}$, $\sum_{i=1}^{n} d_{\overline{\Gamma}}^{+}(v_i) = \sum_{i=1}^{n} d_{\overline{\Gamma}}^{-}(v_i) = n^2 - n$.*

Proof. In the digraph $\overline{\Gamma(n,2)}$ by the Handshaking theorem, we have

$$\sum_{i=1}^{n} d_{\overline{\Gamma}}^{+}(v_i) = \sum_{i=1}^{n} d_{\overline{\Gamma}}^{-}(v_i) = |A(\overline{\Gamma})|$$

$$\Rightarrow \sum_{i=1}^{n} d_{\overline{\Gamma}}^{+}(v_i) + \sum_{i=1}^{n} d_{\overline{\Gamma}}^{-}(v_i) = 2k \text{ , where } |A(\overline{\Gamma})| = k$$

$$\Rightarrow \sum_{i=1}^{n}(d_{\overline{\Gamma}}^{+}(v_i) + d_{\overline{\Gamma}}^{-}(v_i)) = 2k$$

$$\Rightarrow \sum_{i=1}^{n} d_{\overline{\Gamma}}(v_i) = 2k$$

$$\Rightarrow \sum_{i=1}^{n}(2n - d_{\Gamma}(v_i)) = 2k \qquad \text{(By Theorem 2.5)}$$

$$\Rightarrow n.2n - \sum_{i=1}^{n} d_{\Gamma}(v_i) = 2k$$

$$\Rightarrow 2n^2 - 2n = 2k \qquad \text{(By Theorem 2.6)}$$

$$\Rightarrow k = n^2 - n$$

$$\Rightarrow \sum_{i=1}^{n} d_{\overline{\Gamma}}^{+}(v_i) = \sum_{i=1}^{n} d_{\overline{\Gamma}}^{-}(v_i) = n^2 - n \qquad \square$$

Theorem 4.2. *In* $\overline{\Gamma(n,2)}$, $\sum_{i=1}^{n} d_{\overline{\Gamma}}(v_i) = 2(n^2 - n)$.

Proof. We have,

$$\begin{aligned} \sum_{i=1}^{n} d_{\overline{\Gamma}}(v_i) &= \sum_{i=1}^{n} (d_{\overline{\Gamma}}^{+}(v_i) + d_{\overline{\Gamma}}^{-}(v_i)) \\ &= \sum_{i=1}^{n} d_{\overline{\Gamma}}^{+}(v_i) + \sum_{i=1}^{n} d_{\overline{\Gamma}}^{-}(v_i) \\ &= (n^2 - n) + (n^2 - n) \quad \text{(By Theorem 4.1)} \\ &= 2(n^2 - n) \qquad \square \end{aligned}$$

Theorem 4.3. *The number of odd in-degree vertices in* $\overline{\Gamma(n,2)}$ *is even.*

Proof. Let E and O denote the set of even in-degree and odd in-degree vertices in $\overline{\Gamma(n,2)}$ respectively.
We have,

$$\sum_{v_i \in V(\overline{\Gamma})} d_{\overline{\Gamma}}^{-}(v_i) = n^2 - n \qquad \text{(By Theorem 4.1)}$$

$$\Rightarrow \sum_{v_i \in E} d_{\overline{\Gamma}}^{-}(v_i) + \sum_{v_i \in O} d_{\overline{\Gamma}}^{-}(v_i) = n^2 - n$$

$$\Rightarrow \sum_{v_i \in O} d_{\overline{\Gamma}}^{-}(v_i) = (n^2 - n) - \sum_{v_i \in E} d_{\overline{\Gamma}}^{-}(v_i) \qquad (4.1)$$

Clearly, $(n^2 - n)$ and $\sum_{v_i \in E} d_{\overline{\Gamma}}^{-}(v_i)$ are even, and hence from (4.1), we can conclude that $\sum_{v_i \in O} d_{\overline{\Gamma}}^{-}(v_i)$ is even, i.e., the number of odd in-degree vertices in $\overline{\Gamma(n,2)}$ is even. $\square$

Theorem 4.4. *If* $\overline{\Gamma(n,2)}$ *is a digraph with an even number of vertices, then* $\overline{\Gamma(n,2)}$ *has an even number of vertices with even in-degree*

Proof. Let us consider the digraph $\overline{\Gamma(n,2)}$ with n vertices $v_1, v_2, \dots, v_n$; where n is even.
For the sake of generality, let us assume that $v_1, v_2, \dots, v_t$ be the t number of vertices with even in-degree and the remaining $(n - t)$ vertices $v_{t+1}, v_{t+2}, \dots, v_n$ with odd in-degree. By Theorem 4.3, we can conclude that $(n - t)$ is even, which implies that t is also even as n is even. Hence the result. $\square$

Theorem 4.5. *If* $\overline{\Gamma(n,2)}$ *is a digraph with an odd number of vertices, then* $\overline{\Gamma(n,2)}$ *has an odd number of vertices with even in-degree.*

Proof. Let us consider the digraph $\overline{\Gamma(n,2)}$ with n vertices $v_1, v_2, \ldots, v_n$; where n is odd.
For the sake of generality, let us assume that $v_1, v_2, \ldots, v_t$ be the t number of vertices with even in-degree and the remaining $(n - t)$ vertices $v_{t+1}, v_{t+2}, \ldots, v_n$ with odd in-degree. By Theorem 4.3, we can conclude that $(n - t)$ is even, which implies that t is odd as n is odd. Hence the result. □

5 The digraphs $\Gamma(n, 2)$, $\overline{\Gamma(n,2)}$ and $\mathbf{U}_n$

In this section, we try to discuss some results on the digraphs $\Gamma(n, 2)$, $\overline{\Gamma(n,2)}$ and $\mathbf{U}_n$.

Theorem 5.1. *The digraph* $\Gamma(n,2)$ *is not a tournament.*

Proof. For $n = 1$, we have $0^2 \equiv 0 \ (mod\ n)$, which means that 0 is the fixed point in this case, and hence $\Gamma(n,2)$ is not a tournament for $n = 1$.
Let us assume that $n > 1$.
By the definition of $\Gamma(n,2)$ it is clear that $\Gamma(n,2)$ is a disconnected graph, so there exist at least two components, say C_1 and C_2 in $\Gamma(n,2)$. Let $a \in V(C_1)$ and $b \in V(C_2)$, where $V(C_1)$ and $V(C_2)$ are vertex sets of C_1 and C_2 respectively. Then clearly there is no directed edge between the vertices a and b and hence $\Gamma(n,2)$ is not a tournament. □

Theorem 5.2. *The digraph* $\overline{\Gamma(n,2)}$ *is not a tournament, for* $n > 1$.

Proof. As the digraph $\Gamma(n,2)$ is a disconnected graph, for $n > 1$, so there exist at least two components, say C_1 and C_2 in $\Gamma(n,2)$ such that there is no directed arc between the vertices of C_1 and C_2.
Let $a \in V(C_1)$ and $b \in V(C_2)$, where $V(C_1)$ and $V(C_2)$ are vertex sets of C_1 and C_2 respectively. Clearly, $a, b \in V(\Gamma)$. Since, $V(\Gamma) = V(\bar{\Gamma})$ so $a, b \in V(\bar{\Gamma})$ and by the definition of $\overline{\Gamma(n,2)}$, it follows that *arc* (a, b) and *arc* (b, a) both are in $A(\bar{\Gamma})$ and hence the digraph $\overline{\Gamma(n,2)}$ is not a tournament, for $n > 1$. □

Theorem 5.3. *In a Universal directed graph* **U**$_n$ *with n-vertices, the number of directed arcs is* n^2.

Proof. By definition, **U**$_n$ is a complete symmetric digraph with self-loops at each vertex.
In a complete asymmetric digraph, there is exactly one directed arc between every pair of vertices. So, the number of directed arcs in a complete asymmetric digraph is equal to the number of pairs of vertices, and the number of pairs of vertices that can be chosen from n −vertices is $n_{C_2} = \frac{n(n-1)}{2}$. Thus, a complete asymmetric digraph with n −vertices have exactly $\frac{n(n-1)}{2}$ directed arcs. But, in a complete symmetric digraph, there exist two directed arcs with opposite directions between every pair of vertices. So, the number of directed arcs in a complete symmetric digraph with $n-$ vertices is $2 \times \frac{n(n-1)}{2} = n(n-1)$.
Moreover, in **U**$_n$, there are also self-loops at each of the n vertices, which contribute n number of directed arcs to the digraph **U**$_n$. Hence, the total number of directed arcs in **U**$_n$ is $n(n-1) + n = n^2$. □

Theorem 5.4. **U**$_n$ *is a* $n-$ *regular digraph.*

Proof. In a complete symmetric digraph K_r with r −vertices, each vertex has a directed arc towards every other vertex and a directed arc away from every other vertex. Therefore, each vertex in K_r has both an in-degree and an out-degree equal to $(r-1)$. But **U**$_n$ is a complete symmetric digraph with $n-$ vertices having a self-loop at each vertex. Therefore, each vertex in **U**$_n$ has both an in-degree and an out-degree equal to $(n-1)+1$ (as self-loop edges contribute 1 to both in-degree and out-degree). So, we have

$$d^+_{\mathbf{U}_n}(v) = (n-1) + 1 = n$$
$$\text{and, } d^-_{\mathbf{U}_n}(v) = (n-1) + 1 = n, \forall v \in V(\mathbf{U}_n).$$

Thus, we get
$$d^+_{\mathbf{U}_n}(v) = d^-_{\mathbf{U}_n}(v) = n, \forall v \in V(\mathbf{U}_n).$$
Hence, **U**$_n$ is a $n-$ regular digraph. □

Corollary 5.1. **U**$_n$ *is a balanced digraph.*

Theorem 5.5. *The complement digraph* $\overline{\mathbf{U}_\mathrm{n}}$ *of the Universal digraph* $\mathbf{U}_\mathrm{n}$ *is a digraph with* n *−isolated vertices.*

Proof. The Universal digraph $\mathbf{U}_\mathrm{n}$ with n −vertices have a directed arc from every vertex to every other vertex, with a self-loop at each vertex, and $|A(\mathbf{U}_\mathrm{n})| = n^2$.

The complement digraph $\overline{\mathbf{U}_\mathrm{n}}$ has a directed arc from the vertex v_i to the vertex v_j if and only if there is no directed arc from the vertex v_i to the vertex v_j in $\mathbf{U}_\mathrm{n}$ and adding a self-loop at each of the vertices if there is no self-loop at that vertex in $\mathbf{U}_\mathrm{n}$, where $v_i, v_j \in V(\mathbf{U}_\mathrm{n})$. Therefore, the number of directed arcs in $\overline{\mathbf{U}_\mathrm{n}}$ is

$$n^2 - |A(\mathbf{U}_\mathrm{n})| = n^2 - n^2 = 0$$

This shows that $\overline{\mathbf{U}_\mathrm{n}}$ has no directed arcs, which means that there is no directed arc between any pair of vertices in $\overline{\mathbf{U}_\mathrm{n}}$. Therefore, every vertex in $\overline{\mathbf{U}_\mathrm{n}}$ is isolated and hence $\overline{\mathbf{U}_\mathrm{n}}$ is a digraph with n −isolated vertices. □

Theorem 5.6. $\Gamma(n,2) \cup \overline{\Gamma(n,2)} = \mathbf{U}_\mathrm{n}$.

Proof. To prove this, we need to show that every possible directed arc in $\mathbf{U}_\mathrm{n}$ is either in the digraph $\Gamma(n,2)$ or in the digraph $\overline{\Gamma(n,2)}$ and conversely, every directed arc in the digraph $\Gamma(n,2)$ or in the digraph $\overline{\Gamma(n,2)}$ is also in the digraph $\mathbf{U}_\mathrm{n}$.
Clearly, the vertex set of the digraphs $\Gamma(n,2) \cup \overline{\Gamma(n,2)}$ and $\mathbf{U}_\mathrm{n}$ is the same set, namely the set $\mathbb{Z}_n$.
Let $v_i, v_j \in \mathbb{Z}_n$ (including $i = j$). The complement digraph $\overline{\Gamma(n,2)}$ has a directed arc from the vertex v_i to the vertex v_j if and only if there is no directed arc from the vertex v_i to the vertex v_j in the digraph $\Gamma(n,2)$.

Now, let us consider the Universal digraph $\mathbf{U}_\mathrm{n}$. In $\mathbf{U}_\mathrm{n}$, there is a directed arc from every vertex to every other vertex, including a self-loop at each vertex. Therefore, for any pair of vertices v_i and v_j in $V(\mathbf{U}_\mathrm{n})$, there is a directed arc from the vertex v_i to the vertex v_j and a directed arc from the vertex v_j to the vertex v_i including a self-loop at each vertex v_i.

Again, for every pair of vertices v_i and v_j in $V(\Gamma(n,2) \cup \overline{\Gamma(n,2)})$, a directed arc from the vertex v_i to the vertex v_j can be either in the digraph $\Gamma(n,2)$ or in the digraph $\overline{\Gamma(n,2)}$, which means that there is a directed arc from the vertex v_i to the vertex v_j in $\Gamma(n,2)$ then there is no directed arc from the vertex v_i to the vertex v_j in $\overline{\Gamma(n,2)}$, and vice versa. Therefore, for every pair of vertices v_i and v_j in $V(\mathbf{U}_n)$, either the directed arc from the vertex v_i to the vertex v_j is in the digraph $\Gamma(n,2)$ or the directed arc from the vertex v_i to the vertex v_j is in the digraph $\overline{\Gamma(n,2)}$ which implies that every directed arc in the digraph $\mathbf{U}_n$ is in the digraph $\Gamma(n,2) \cup \overline{\Gamma(n,2)}$.

Conversely, each directed arc (including self-loops) in the digraph $\Gamma(n,2)$ or in the digraph $\overline{\Gamma(n,2)}$ is also in the digraph $\mathbf{U}_n$ because the digraph $\overline{\Gamma(n,2)}$ is the complement of the digraph $\Gamma(n,2)$, and together $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$ cover all possible directed arcs (including self-loops) between the vertices of the digraph $\mathbf{U}_n$. Hence, $\Gamma(n,2) \cup \overline{\Gamma(n,2)} = \mathbf{U}_n$. □

Theorem 5.7. $\Gamma(n,2) \cap \overline{\Gamma(n,2)} = N_n$, *where* N_n *denotes the null graph with* $n -$ *vertices.*

Proof. Clearly, the vertex set of the digraphs $\Gamma(n,2) \cap \overline{\Gamma(n,2)}$ and N_n is the same set, namely the set $\mathbb{Z}_n$.

We now consider the digraph $\Gamma(n,2) \cap \overline{\Gamma(n,2)}$, which represents the common directed arcs between the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$. A directed arc from the vertex v_i to the vertex v_j can be in $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$ simultaneously, only if there is a directed arc from the vertex v_i to the vertex v_j in $\Gamma(n,2)$ and no directed arc from the vertex v_i to the vertex v_j in $\overline{\Gamma(n,2)}$ and vice versa, where $v_i, v_j \in \mathbb{Z}_n$.

If possible, suppose that there exists a directed arc from the vertex v_i to the vertex v_j in $\Gamma(n,2)$ and a directed arc from the vertex v_i to the vertex v_j in $\overline{\Gamma(n,2)}$ simultaneously. This implies that there is a directed arc from the vertex v_i to the vertex v_j in $\Gamma(n,2)$ and there is no directed arc from the vertex v_i to the vertex v_j in $\Gamma(n,2)$, which is a contradiction. Similarly, if there is a directed arc from the vertex v_i to the vertex v_j in $\overline{\Gamma(n,2)}$ then we get a contradiction again. These contradictions imply that there is no common directed arc between the digraphs $\Gamma(n,2)$ and $\overline{\Gamma(n,2)}$. Hence, $\Gamma(n,2) \cap \overline{\Gamma(n,2)} = N_n$. □

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