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MULTI VAGUE ORDER OF AN ELEMENT IN A GROUP

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Abstract

In this paper, we introduce the concept of Multi vague order of an element in a group with dimension **k** which extends the classical notion of vague order. We establish various properties and characterizations of Multi vague groups based on this order.
Further, we study the concepts of Multi vague-p group, Multi vague torsion. For instance, we show that $GV_A = \{x \in G \mid t_{iA}(x) = t_{iA}(e) \ \& \ f_{iA}(x) = f_{iA}(e), i = 1, 2, ..., k.\}$ is a normal subgroup of G, and that A is a Multi vague-p

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group if and only if the quotient group $\frac{G}{GV_A}$ is a p-group. Moreover, if A is a Multi vague group of an abelian group G, then A is Multi vague torsion if and only if $\frac{G}{GV_A}$ is a torsion group.

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Key Words and Phrases: Multi vague set, Multi vague group, Multi vague normal group, Multi vague order of an element, Multi vague-p group, Multi vague torsion.

1 Introduction

The study of fuzzy and vague algebraic structures has become an important area in modern algebra due to its potential applications in areas involving uncertainty, such as decision making, pattern recognition, and computational intelligence. The concept of vague order of an element in a group, which generalizes the classical notion of element order, has attracted considerable attention. Early developments in fuzzy subgroup theory by Bhattacharya [14] and Das [15] established key frameworks for handling uncertainty in group theory. Subsequently, N.Ramakrishna [11], [12], [13] & [19] expanded this framework by introducing Multi vague groups and Multi sets, allowing for a richer structure capturing multiple dimensions of vagueness.

In this paper, we extend the classical vague order of an element in a group with dimension **k**, providing a Multi dimensional approach to analyzing elements under vagueness. This concept naturally generalizes the vague order and enables finer classifications of group elements. We rigorously investigate the properties and characterizations of Multi vague groups based on this order. Furthermore, we examine Multi vague p-groups, revealing new insights into their algebraic behaviour.

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Our work builds upon and generalizes previous research by Biswas [16],[17], Rosenfeld [1], Sebastian and Ramakrishnan [22], and others who contributed to the theory of fuzzy and vague groups. Demici's [8] approach is based on crisp set and fuzzy operation. We follow Ranjit Biswas's appraoch for our development of vague algebra. By advancing the theoretical framework of Multi vague orders, this paper aims to open new avenues for applying vague algebraic structures in both theoretical investigations and practical scenarios involving uncertainty and imprecise information.

2 Preliminaries

Definition 1. A Vague set A in the universe of discourse G is a pair (t_A, f_A) where $t_A : G \to [0,1]$, $f_A : G \to [0,1]$, are the mappings such that $t_A(x) + f_A(x) \leq 1$, for all $x \in G$. The functions $t_A(x)$ *and* $f_A(x)$ are called true and false membership functions respectively.

Definition 2. Let G be a non-empty set. A vague set $A = (t_A, f_A)$ where $t_A(x) = (t_{1A}(x), t_{2A}(x), ..., t_{kA}(x))$ and $f_A(x) = (f_{1A}(x), f_{2A}(x), ..., f_{kA}(x))$ and $t_{iA} : G \to [0,1]$, $f_{iA} : G \to [0,1]$, are mappings such that $t_{iA}(x) + f_{iA}(x) \leq 1$, for all $x \in G$, for i=1,2,3,.........k,is called Multi vague set of G with dimension **k**. Here $t_{1A}(x) \geq t_{2A}(x) \geq ......t_{kA}(x)$, for all $x \in G$.

Note: We arranged the true membership sequence is decreasing order, then the corresponding false membership sequence need not be in decreasing or increasing order.

Example 3. *Consider a group* $G = \{1, \omega, \omega^2\}$ *with usual multiplication.*
$A = \{\langle 1, (0.6, 0.4, 0.3, 0.2), (0.3, 0.5, 0.4, 0.4)\rangle,$
$\langle \omega, (0.6, 0.3, 0.2, 0.1), (0.2, 0.4, 0.3, 0.2)\rangle,$
$\langle \omega^2, (0.5, 0.4, 0.3, 0.2), (0.3, 0.4, 0.2, 0.6)\rangle\}$ *is a Multi vague set.*

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Definition 4. A Multi vague set $A = (t_A, f_A)$ of dimension **k** of a group G is said to be a Multi vague group of G if, for all $x, y \in G$, 1) $t_A(xy) \geq t_A(x) \wedge t_A(y)$ and $t_A(x^{-1}) = t_A(x)$ and 2) $f_A(xy) \leq f_A(x) \vee f_A(y)$ and $f_A(x^{-1}) = f_A(x)$

i.e., 1) $t_{iA}(xy) \geq t_{iA}(x) \wedge t_{iA}(y)$ and $t_{iA}(x^{-1}) = t_{iA}(x)$
2) $f_{iA}(xy) \leq f_{iA}(x) \vee f_{iA}(y)$ and $f_{iA}(x^{-1}) = f_{iA}(x)$ for all i=1,2,3....k.

Equivalently, we define Multi vague group as, for all $x, y \in G$, $t_{iA}(xy^{-1}) \geq t_{iA}(x) \wedge t_{iA}(y)$, and $f_{iA}(xy^{-1}) \leq f_{iA}(x) \vee f_{iA}(y)$, and i=1,2,.....k.

Theorem 5. *Let A be a Multi vague group of a group G then* $t_{iA}(xy^{-1}) = t_{iA}(e)\}_{i=1}^{k} and \quad f_{iA}(xy^{-1}) = f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ *and* $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$.

Theorem 6. *If A and B are Multi vague groups of a group G then* $A \bigcap B$ *is also a Multi vague group.*

Definition 7. Let $A = (t_A, f_A)$ be a Multi vague set of a non-empty set G. where $t_A(x) = (t_{1A}(x), t_{2A}(x), ..., t_{kA}(x))$ and $f_A(x) = (f_{1A}(x), f_{2A}(x), ..., f_{kA}(x))$ and $\alpha = (\alpha_1, \alpha_2, ...\alpha_k)$, $\beta = (\beta_1, \beta_2, ..., \beta_k)$ where $0 \leq \alpha_i + \beta_i \leq 1$, for i=1,2,....,k., then (α, β) - cut of A is defined as $A_{[\alpha,\beta]} = \{x \mid x \in G, t_{iA}(x) \geq \alpha, f_{iA}(x) \leq \beta\}$. It is a crisp Multi set of G.

Definition 8. Let A be Multi vague group of a group G of dimension **k** and $\alpha = (\alpha_1, \alpha_2, ...\alpha_k) \quad 0 \leq \alpha_i \leq 1, i = 1, 2...., k.$,
then the true α cut of A is a crisp Multi set defined as
$T_{\alpha_i} = \{x \mid x \in G, \{t_{iA}(x) \leq \alpha_{A_i}\}_{i=1}^{k}\}$.

Definition 9. Let A be Multi vague group of a group G of dimension **k** and $\alpha = (\alpha_1, \alpha_2, ...\alpha_k) \quad 0 \leq \alpha_i \leq 1, i = 1, 2...., k.$,
then the false α cut of A is a crisp Multi set defined as
$F_{\alpha_i} = \{x \mid x \in G, \{f_{iA}(x) \geq \alpha_{A_i}\}_{i=1}^{k}\}$.

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Theorem 10. *Let A be a Multi vague group of G. Then A is a Multi vague group of G iff for every $\alpha \in [0,1]^k$, the non-empty sets TA_α and FA_α are crisp subgroups of G.*

Definition 11. Let G be a non-empty set. A vague group A is a pair (t_A, f_A) of dimension **k** of a group G is said to be a Multi vague normal group of G, $\forall x, g \in G$.

$t_A(x) \leq \wedge t_A(gxg^{-1})$ and $f_A(x) \geq \vee f_A(gxg^{-1})$

i.e., $t_{iA}(x) \leq \wedge t_{iA}(gxg^{-1})\}_{i=1}^{k}$ and $f_{iA}(x) \geq \vee f_{iA}(gxg^{-1})\}_{i=1}^{k}$

Example 12. *Let $G = \{1, -1, i, -i\}$ be a group. and A be a Multi vague group of G with dimension 4.*

$A = \{\langle 1, (0.7, 0.4, 0.3, 0.2), (0.3, 0.5, 0.4, 0.3)\rangle,$

$\langle -1, (0.6, 0.3, 0.2, 0.1), (0.3, 0.6, 0.5, 0.4)\rangle\}$

Clearly A, is a Multi vague normal group.

Theorem 13. *If G is a commutative group, and A be a Multi vague group of G with dimension **k**, then A is a Multi vague normal group of G.*

Definition 14. Let A be a vague group of a group G. For a prime p, A is said to be vague p-group if $VO_A(x)$ is a power of p, $\forall x \in G$.

Definition 15. Let A be a vague group of an abelian group G. A is called vague torsion if $VO_A(x)$ is finite for all $x \in G$.

Example 16. *Let $G = \{e, x, y, z\}, x^2 = y^2 = z^2 = e$ be the Klein's 4- group. Define a vague group A defined by $v_A(z) = v_A(e) = I_0 = [a_0, b_0]$, $v_A(x) = v_A(y) = I_1 = [a_1, b_1]$. where $I_0 > I_1$. Thus A is a vague torsion.*

Definition 1 7. I n a g roup G . L et $a \in G$, t hen t he fuzzy order of an $x \in G$ is the smallest positive integer n such that $\mu(x^n) = \mu(e)$. where μ is the fuzzy membership function e is the identity element in G. It is denoted as FO(x) = n.

If no such n exists, the fuzzy order of x is said to be infinite.

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Example 18. *Let* $G = (Z_4, +)$ *and define*
$\mu(0) = 1, \mu(1) = 0.7, \mu(2) = 0.8, \mu(3) = 0.7$
where 0 is the identity element in Z_4
Here $\mu(0) = \mu(0) \Rightarrow fuzzy\ order\ of\ (0) = 0 \Rightarrow FO(0) = 1$
$\mu(1 +_4 1 +_4 1 +_4 1) = \mu(0) \Rightarrow fuzzy\ order\ of\ (1) = 4 \Rightarrow FO(1) = 4$
$\mu(2 +_4 2) = \mu(0) \Rightarrow fuzzy\ order\ of\ (2) = 2 \Rightarrow FO(2) = 2$
$\mu(3 +_4 3 +_4 3 +_4 3) = \mu(0) \Rightarrow fuzzy\ order\ of\ (3) = 4 \Rightarrow FO(3) = 4$

Definition 1 9. Let A b e a vague group of a group G.
Given $x \in G$, if there exists a least positive integer n such that $V_A(x^n) = V_A(e)$, then n is called the vague order of x, w.r.to A.
We denote it as $VO_A(x)$ i.e., $VO_A(x) = n$
where $A = (t_A, f_A), t_A(x) + f_A(x) \leq 1\ \forall x \in G$
$V_A(x^n) = (t_A(x^n), f_A(x^n)) = (t_A(e), f_A(e))$

Example 20. *Let* $G = (Z_4, +)$ *be the group.*
Now defined a vague group
$A = \{(0, 1, 0), (1, 0.7, 0.2), (2, 0.8, 0.1), (3, 0.7, 0.2)\}$
Here $t_A(0) = 1,\ f_A(0) = 0$
$t_A(1) = 0.7,\ f_A(1) = 0.2$
$t_A(2) = 0.8,\ f_A(2) = 0.1$
$t_A(3) = 0.7,\ f_A(0) = 0.2$

Here 0 is the identity element of Z_4.
$t_A(e) = t_A(0) = 1, f_A(e) = f_A(0) = 0$

Now $0 = 0 \Rightarrow VO(0) = 1$
$1 +_4 1 +_4 1 +_4 1 = 0 \Rightarrow VO(1) = 4$
$2 +_4 2 = 0 \Rightarrow VO(2) = 2$
$3 +_4 3 +_4 3 +_4 3 = 0 \Rightarrow VO(3) = 4$

3 Multi Vague Order of an element

Now, we introduce the concept of Multi vague order of an element.

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Definition 21. Let G be a group and A is a Multi vague group of group G with dimension **k**, then $A = (t_A, f_A)$ where $t_A = (t_{1A}, t_{2A}, ..., t_{kA})$, $f_A = (f_{1A}, f_{2A}, ..., f_{kA})$ with $t_{1A} \geq t_{2A} \geq ..... \geq t_{kA}$ and $t_{iA} + f_{iA} \leq 1, \forall i = 1, 2, ..., k.$, then Multi vague order of an element $x \in G$ is defined as the smallest positive integer n such that $t_A(x^n) = t_A(e)$ & $f_A(x^n) = f_A(e)$
i.e., $t_{iA}(x^n) = t_{iA}(e)\}_{i=1}^k$ & $f_{iA}(x^n) = f_{iA}(e)\}_{i=1}^k$.
It is denoted by $MVO_A(x) = n$.
If no such n exists, the Multi vague order of x is infinite.

Example 22. *Let the group* $G = (Z_4, +_4)$ *and*
$Z_4 = \{0, 1, 2, 3\}$
$A = \{\langle 0, (1, 0.9, 0.8, 0.7), (0, 0, 0, 0)\rangle$
$\langle 1, (0.7, 0.6, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle$
$\langle 2, (0.8, 0.7, 0.6, 0.5), (0.1, 0.2, 0.3, 0.4)\rangle$
$\langle 3, (0.7, 0.6, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle\}$
Clearly A is a Multi vague group of G with dimension 4.

In Z_4, *0 is the identity element.*
(i) $0 = 0 \Rightarrow t_{iA}(0) = t_{iA}(0)\}_{i=1}^k$ & $f_{iA}(0) = f_{iA}(0)\}_{i=1}^k$
$\Rightarrow MVO_A(0) = 1$
(ii) $1 +_4 1 +_4 1 +_4 1 = 0 \Rightarrow t_{iA}(1^4) = t_{iA}(0)\}_{i=1}^k$ &
$f_{iA}(1^4) = f_{iA}(0)\}_{i=1}^k \Rightarrow MVO_A(1) = 4$
(iii) $2 +_4 2 = 0 \Rightarrow t_{iA}(2^2) = t_{iA}(0)\}_{i=1}^k$ & $f_{iA}(2^2) = f_{iA}(0)\}_{i=1}^k$
$\Rightarrow MVO_A(2) = 2$
(iv) $3 +_4 3 +_4 3 +_4 3 = 0 \Rightarrow t_{iA}(3^4) = t_{iA}(0)\}_{i=1}^k$&
$f_{iA}(3^4) = f_{iA}(0)\}_{i=1}^k \Rightarrow MVO_A(3) = 4$

Example 23. *Let* $G = \{e, x, y, z\}, x^2 = y^2 = z^2 = e$,
be the Klein's-4 group.
Define a Multi vague group of A with dimension 4, defined as
$t_A(z) = t_A(e) = (a_0, b_0, c_0, d_0)$ *and* $f_A(z) = f_A(e) = (e_0, f_0, g_0, h_0)$
$t_A(x) = t_A(y) = (a_1, b_1, c_1, d_1)$ *and* $f_A(x) = f_A(y) = (e_1, f_1, g_1, h_1)$
where $a_0 > a_1, b_0 > b_1, c_0 > c_1, d_0 > d_1$ *and* $e_0 < e_1, f_0 < f_1,$

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$g_0 < g_1, h_0 < h_1$.
$a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1, e_0, e_1, f_0, f_1, g_0, g_1, h_0, h_1 \in [0, 1]$ *and*
$a_i + e_i \leq 1, b_i + f_i \leq 1, c_i + g_i \leq 1, d_i + h_i \leq 1, \forall i = 1, 2, 3, 4.$
Clearly G *is a group, by Cayley's composition table.*

Now we have $t_{iA}(x^2) = t_{iA}(e)\}_{i=1}^{4}$ *and* $f_{iA}(x^2) = f_{iA}(e)\}_{i=1}^{4}$
$\Rightarrow MVO_A(x) = 2$,
$t_{iA}(y^2) = t_{iA}(e)\}_{i=1}^{4}$ *and* $f_{iA}(y^2) = f_{iA}(e)\}_{i=1}^{4} \Rightarrow MVO_A(y) = 2$,
$t_{iA}(z) = t_{iA}(e)\}_{i=1}^{4}$ *and* $f_{iA}(z) = f_{iA}(e)\}_{i=1}^{4} \Rightarrow MVO_A(z) = 1$,
$t_{iA}(e) = t_{iA}(e)\}_{i=1}^{4}$ *and* $f_{iA}(e) = f_{iA}(e)\}_{i=1}^{4} \Rightarrow MVO_A(e) = 1$.
$\therefore\ MVO_A(e) = 1, MVO_A(x) = 2, MVO_A(y) = 2, MVO_A(z) = 1$

Example 24. *Let* $G = \{1, -1, i, -i\}$ *be a group of fourth roots of unity.*
Define a vague group A *of* G *with dimension 4 be*
$A = \{\langle 1, (1, 0.9, 0.8, 0.7), (0, 0, 0, 0)\rangle$
$\langle -1, (0.8, 0.7, 0.6, 0.5), (0.1, 0.1, 0.3, 0.3)\rangle$
$\langle i, (0.7, 0.6, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle$
$\langle -i, (0.7, 0.6, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle\}$
then clearly A *is multi vague group with dimension 4.*

$t_A(1) = (1, 0.9, 0.8, 0.7) = t_A(e)$ & $f_A(1) = (0, 0, 0, 0) = f_A(e)$
$\Rightarrow MVO_A(1) = 1$
$t_A(-1)^2 = t_A(1) = (1, 0.9, 0.8, 0.7)$
and $f_A(-1)^2 = f_A(1) = (0, 0, 0, 0) \Rightarrow MVO_A(-1) = 2$
$t_A(i^4) = t_A(1) = (1, 0.9, 0.8, 0.7)$ & $f_A(i^4) = f_A(1) = (0, 0, 0, 0)$
$\Rightarrow MVO_A(i) = 4$
$t_A((-i)^4) = t_A(1) = (1, 0.9, 0.8, 0.7)$ *and*
$f_A((-i)^4) = f_A(1) = (0, 0, 0, 0)$
$\Rightarrow MVO_A(-i) = 4$
$\therefore MVO_A(1) = 1, MVO_A(-1) = 2, MVO_A(i) = 4, MVO_A(-i) = 4$

Example 25. *Let* $G = \{1, -1, i, -i\}$ *be a group of fourth roots of unity.*
Define a vague group A *of* G *with dimension 4 be*
$A = \{\langle 1, (1, 0.9, 0.8, 0.7), (0, 0, 0, 0)\rangle$
$\langle -1, (1, 0.9, 0.8, 0.7), (0, 0, 0, 0)\rangle$
$\langle i, (0.6, 0.5, 0.4, 0.3), (0.2, 0.3, 0.3, 0.4)\rangle$

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$\langle -i, (0.6, 0.5, 0.4, 0.3), (0.2, 0.3, 0.3, 0.4)\rangle\}$
then clearly A is multi vague group with dimension 4.

Here 1 is the identity element of G.
$t_A(1) = (1, 0.9, 0.8, 0.7) = t_A(e)$ & $f_A(1) = (0, 0, 0, 0) = f_A(e)$
$\Rightarrow MVO_A(1) = 1$
$t_A(-1) = (1, 0.9, 0.8, 0.7) = t_A(1)$ & $f_A(-1) = (0, 0, 0, 0) = f_A(1)$
$\Rightarrow MVO_A(-1) = 1$
$t_A(i^2) = t_A(-1) = t_A(1) = (1, 0.9, 0.8, 0.7)$
and $f_A(i^2) = f_A(-1) = f_A(1) = (0, 0, 0, 0) \Rightarrow MVO_A(i) = 2$
$t_A((-i)^2) = t_A(-1) = t_A(1) = (1, 0.9, 0.8, 0.7)$
and $f_A((-i)^2) = f_A(-1) = f_A(1) = (0, 0, 0, 0) \Rightarrow MVO_A(-i) = 2$
$\therefore MVO_A(1) = 1, MVO_A(-1) = 1, MVO_A(i) = 2, MVO_A(-i) = 2$

Remark 26. *We note that $O(x)$ and $MVO_A(x)$ are not same in general.*
In example 3.5 we have $MVO_A(z) = 1$, whereas $O(z) = 2$
In example 3.6 we have $MVO_A(-1) = 1$, whereas $O(-1) = 2$

Theorem 27. *Let A be Multi vague group of a group G with dimension **k**. If $t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^k$ and $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^k$, for some $x \in G$, then $MVO_A(x)$ divides m.*
*Proof: Given A is a Multi vague group of a group G with dimension **k**. for all $x, y \in G$,*

1. $t_{iA}(xy) \geq t_{iA}(x) \wedge t_{iA}(y)\}_{i=1}^k$ and $t_{iA}(x^{-1}) = t_{iA}(x)\}_{i=1}^k$
2. $f_{iA}(xy) \leq f_{iA}(x) \vee f_{iA}(y)\}_{i=1}^k$ and $f_{iA}(x^{-1}) = f_{iA}(x)\}_{i=1}^k$

Let $t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^k$ and $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^k$ *for some* $x \in G$
Let $MVO_A(x) = n$.
$\Rightarrow n$ *is the least positive integer such that*
$t_{iA}(x^n) = t_{iA}(e)\}_{i=1}^k$ *and* $f_{iA}(x^n) = f_{iA}(e)\}_{i=1}^k \Rightarrow n \leq m$.
therefore by Euclidean Algorithm $m = nq + r$, *where* $0 \leq r < n$
So $t_A(x^r) = t_{iA}(x^r)\}_{i=1}^k = t_{iA}(x^{m-nq})\}_{i=1}^k = t_{iA}(x^m(x^n)^{-q}))\}_{i=1}^k$
$\geq min\{t_{iA}(x^m), t_{iA}((x^n)^{-q})\}_{i=1}^k \geq min\{t_{iA}(x^m), t_{iA}(x^n)\}_{i=1}^k$

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$= min\{t_{iA}(e), t_{iA}(e)\}\}_{i=1}^{k} = t_{iA}(e)\}_{i=1}^{k} = t_A(e).$
$\therefore\ t_A(x^r) \geq t_A(e)$ *and* $t_A(e) \geq t_A(x^r)$
$\Rightarrow t_A(x^r) = t_A(e) \to (1)$
similarly we can prove $f_A(x^r) = f_{iA}(x^r)\}_{i=1}^{k} = f_{iA}(x^{m-nq})\}_{i=1}^{k}$
$= f_{iA}(x^m(x^n)^{-q}))\}_{i=1}^{k}$
$\leq max\{f_{iA}(x^m), f_{iA}((x^n)^{-q})\}_{i=1}^{k} \leq max\{f_{iA}(x^m), f_{iA}(x^n)\}_{i=1}^{k}$
$= max\{f_{iA}(e), f_{iA}(e)\}\}_{i=1}^{k} = f_{iA}(e)\}_{i=1}^{k} = f_A(e).$
$\therefore\ f_A(x^r) \leq f_A(e)$ *and* $f_A(e) \leq f_A(x^r)$
$\Rightarrow f_A(x^r) = f_A(e) \to (2)$
from (1) & (2) $t_A(x^r) = t_A(e)$ & $f_A(x^r) = f_A(e)$
contradiction to n is the least positive integer n.
$\Rightarrow r = 0 \Rightarrow m = nq \Rightarrow n|m$
$\therefore\ MVO_A(x)$ *divides m.*

Corollary 28. $MVO_A(x)$ *is always a divisor of* $O(x)$.
Proof: Let $O(x) = n, MVO_A(x) = m$
Since $O(x) = n \Rightarrow n$ *is the least positive integer such that* $x^n = e$
$\Rightarrow t_{iA}(x^n) = t_{iA}(e)\}_{i=1}^{k}$ *and* $f_{iA}(x^n) = f_{iA}(e)\}_{i=1}^{k}$
But $MVO_A(x) = m$
$\Rightarrow t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^{k}$ *and* $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^{k}$
$\because$ *m is the least positive integer by the above theorem*
$m|n \Rightarrow MVO_A(x)|O(x)$
It completes the proof.

Theorem 29. *Let A be Multi vague group of a group G with dimension* $\boldsymbol{k}$. *If x has finite Multi vague order in G, then* $MVO_A(x^m) = \frac{MVO_A(x)}{(MVO_A(x),m)}$, *for all positive integer m.*
Proof: Given A is a Multi vague group with dimension $\boldsymbol{k}$.
$A = (t_A, f_A)$ *where* $t_A(x) = (t_{1A}(x), t_{2A}(x), ..., t_{kA}(x))$, $f_A(x) = (f_{1A}(x), f_{2A}(x), ..., f_{kA}(x))$ *and* $t_{iA} : G \to [0,1], f_{iA} : G \to [0,1]$, *are mappings such that* $t_{iA}(x) + f_{iA}(x) \leq 1$, *for all* $x \in G$, *for i=1,2,3,...........k., is called Multi vague set of G with dimension* $\boldsymbol{k}$.
Here $t_{1A}(x) \geq t_{2A}(x) \geq ......t_{kA}(x)$, *for all* $x \in G$.
Let $MVO_A(x) = n \Rightarrow t_{iA}(x^n) = t_{iA}(e)\}_{i=1}^{k}$ *and*
$f_{iA}(x^n) = f_{iA}(e)\}_{i=1}^{k}$ *then* x^m *is also has finite Multi vague order.*

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Let $MVO_A(x^m) = k$.
Let$(m,n) = d \Rightarrow n = dp, m = dq$
$(p,q) = 1, p, q$ *are relatively primes.*
$t_{iA}(x^m)^k = t_{iA}(e)\}_{i=1}^k$ and $f_{iA}(x^m)^k = f_{iA}(e)\}_{i=1}^k$
$t_{iA}(x^m k) = t_{iA}(e)\}_{i=1}^k$ and $f_{iA}(x^m k) = f_{iA}(e)\}_{i=1}^k$
$MVO_A(x) = n \Rightarrow n|mk$.
by the theorem
[*Let A be Multi vague group of a group G with dimension* ***k***.
If $t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^k$ and $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^k$, *for some* $x \in G$,
then $MVO_A(x)$ *divides m.*]
$\Rightarrow dp|dqp \Rightarrow p|qk$
$\because\ (p,q) = 1 \Rightarrow p|k \to (3)$
again consider $t_A((x^{dq})^p) = t_{iA}(x^{dq})^p\}_{i=1}^k = t_{iA}(x^{dpq})\}_{i=1}^k = t_{iA}(x^{nq})\}_{i=1}^k$
$= t_{iA}(x^q)^n\}_{i=1}^k = t_{iA}(e)\}_{i=1}^k = t_A(e)$
similarly $f_A((x^{dq})^p) = f_{iA}(x^{dq})^p\}_{i=1}^k = f_{iA}(x^{dpq})\}_{i=1}^k = f_{iA}(x^{nq})\}_{i=1}^k$
$= f_{iA}(x^q)^n\}_{i=1}^k = f_{iA}(e)\}_{i=1}^k = f_A(e)$
$\therefore t_A((x^{dq})^p) = t_A(e)\ \&\ f_A((x^{dq})^p) = f_A(e)$
but $MVO_A(x^m) = k \Rightarrow k|p \to (4)$
from (3) and (4) $k = p$ $(\because\ n = dp \Rightarrow k = \frac{n}{d})$
$\Rightarrow MVO_A(x^m) = \frac{MVO_A(x)}{MVO_A(x,m)}$

Theorem 30. *If G is finite group, then* $GV_A = \{e\}$ *iff* $MVO_A(x) = O(x)$, *for every* $x \in G$. *where*
$GV_A = \{x \in G \mid t_{iA}(x) = t_{iA}(e)\}_{i=1}^k\ \&\ f_{iA}(x) = f_{iA}(e)\}_{i=1}^k\}$
Proof: Case:I Suppose $GV_A = \{e\}$
where $GV_A = \{x \in G \mid t_{iA}(x) = t_{iA}(e)\}_{i=1}^k\ \&\ f_{iA}(x) = f_{iA}(e)\}_{i=1}^k\}$
by corollary $MVO_A(x)$ *is always a divisor of* $O(x)$
$\Rightarrow MVO_A(x)|O(x) \to (5)$.
Let $MVO_A(x) = m, O(x) = n$.
$\because MVO_A(x) = m \Rightarrow t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^k\ \&\ f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^k\}$
$\Rightarrow x^m \in GV_A$
but $GV_A = \{e\} \Rightarrow x^m = e$
since $O(x) = n \Rightarrow n|m \Rightarrow O(x)|\ MVO_A(x) \to (6)$
from (5) and (6) $MVO_A(x) = O(x)$

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Case:II Conversely suppose $MVO_A(x) = O(x), x \in G$
claim: $GV_A = \{e\}$
Let $x \in GV_A \Rightarrow t_{iA}(x) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(e)\}_{i=1}^{k}\}$
$MVO_A(x) = 1, \because MVO_A(x) = O(x) \Rightarrow O(x) = 1 \Rightarrow x = e$
$\therefore GV_A = \{e\}$

Theorem 31. *Let A be Multi vague group of a group G with dimension **k**. and let* $MVO_A(x) = n, where\ x \in G$. *If m is an integer such that (m, n) = 1, then* $t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^{k}$ *and* $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^{k}$
Proof: Since (n, m)= 1, ⇒ there exists integer s and t ∋ ns + mt =1.
So, $t_A(x) = t_A(x^1) = t_{iA}(x^1)\}_{i=1}^{k} = t_{iA}(x^{ns+mt})\}_{i=1}^{k} = t_{iA}(x^{ns}x^{mt})\}_{i=1}^{k}$
$= t_{iA}(x^{n^s}x^{m^t})\}_{i=1}^{k} \geq min\{t_{iA}(x^n), t_{iA}(x^m)\}_{i=1}^{k} = min\{t_{iA}(e), t_{iA}(x^m)\}_{i=1}^{k}$
$= t_{iA}(x^m)\}_{i=1}^{k} \geq t_{iA}(x)\}_{i=1}^{k} = t_A(x).$
$\because\ t_A(x) \geq t_A(x) \Rightarrow t_{iA}(x^m) = t_{iA}(x)\}_{i=1}^{k}$

Also consider $f_A(x) = f_A(x^1) = f_{iA}(x^1)\}_{i=1}^{k} = f_{iA}(x^{ns+mt})\}_{i=1}^{k}$
$= f_{iA}(x^{ns}x^{mt})_{i=1}^{k}$
$= f_{iA}(x^{n^s}x^{m^t})_{i=1}^{k} \leq max\{f_{iA}(x^n), f_{iA}(x^m)\}_{i=1}^{k} = max\{f_{iA}(e), f_{iA}(x^m)\}_{i=1}^{k}$
$= f_{iA}(x^m)\}_{i=1}^{k} \leq t_{iA}(x)\}_{i=1}^{k} = f_A(x).$
$\because\ f_A(x) \leq f_A(x) \Rightarrow f_{iA}(x^m) = f_{iA}(x)\}_{i=1}^{k}$
$\because\ t_{iA}(x^m) = t_{iA}(x)\}_{i=1}^{k}$ & $f_{iA}(x^m) = f_{iA}(x)\}_{i=1}^{k}$
$\Rightarrow t_A(x^m) = t_A(x)$ & $f_A(x^m) = f_A(x)$

In what follows below, we use two conditions given below by (I) and (II) respectively.
Given A is a Multi vague group of a group G with dimension **k**. $A = (t_A, f_A)$ where $t_A(x) = (t_{1A}(x), t_{2A}(x), ..., t_{kA}(x))$, $f_A(x) = (f_{1A}(x), f_{2A}(x), ..., f_{kA}(x))$ and $t_{iA} : G \to [0,1], f_{iA} : G \to [0,1]$, are mappings such that $t_{iA}(x) + f_{iA}(x) \leq 1$, for all $x \in G$, for i=1,2,3,...........k.,is called Multi vague set of G with dimension **k**. Here $t_{1A}(x) \geq t_{2A}(x) \geq ......t_{kA}(x)$, for all $x \in G$.

1 $t_{iA}(xy) \geq t_{iA}(x) \wedge t_{iA}(y)\}_{i=1}^{k}$ and $t_{iA}(x^{-1}) = t_{iA}(x)\}_{i=1}^{k}$

2 $f_{iA}(xy) \leq f_{iA}(x) \vee f_{iA}(y)\}_{i=1}^{k}$and $f_{iA}(x^{-1}) = f_{iA}(x)\}_{i=1}^{k}$

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Condition I:
$O(x) = O(y) \Rightarrow t_A(x) = t_A(y)$ & $f_A(x) = f_A(y)$
i.e., $O(x) = O(y) \Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$

Condition II:
$MVO_A(x) = MVO_A(y) \Rightarrow t_A(x) = t_A(y)$ & $f_A(x) = f_A(y)$
i.e., $MVO_A(x) = MVO_A(y) \Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ &
$f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$

Theorem 32. *Let Given A is a Multi vague group of a group G with dimension* $\boldsymbol{k}$*, with condition (I). If* $O(x) = O(y)$*, for some* $x, y \in G$ *then* $MVO_A(x) = MVO_A(y)$
Proof: Let $O(x) = O(y)$
$\Rightarrow O(x^n) = O(y^n), \forall n$ $(\because x = y \Rightarrow x^n = y^n)$ *by condition (I)*
$\Rightarrow t_{iA}(x^n) = t_{iA}(y^n)\}_{i=1}^{k}$ & $f_{iA}(x^n) = f_{iA}(y^n)\}_{i=1}^{k} \rightarrow (7)$
Let $MVO_A(x) = m$
$\Rightarrow t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow t_{iA}(y^m) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^m) = f_{iA}(e)\}_{i=1}^{k}(from(7))$
$\Rightarrow MVO_A(y) = m$
$\therefore$ $MVO_A(x) = MVO_A(y)$

We recall the definition of a p-group.

Definition 33. For a prime p, a group G is said to be a p-group if the order of x is a power of p $\forall x \in G$.

Theorem 34. *Let A be a Multi vague group with dimension* $\boldsymbol{k}$ *of a finite group-p group G, with condition I. If* $MVO_A(x) = MVO_A(y) > 1$*, for some* $x, y \in G$ *then* $O(x) = O(x)$*.*
Proof: Given G is a finite p-group.
A be a Multi vague group with dimension $\boldsymbol{k}$
Let $MVO_A(x) = MVO_A(y) = p^t$ *where p is a prime,* $t \geq 1$
since $MVO_A(x) = p^t$*, is a divisor of O(x) and O(y)*
$\therefore$ *Let* $O(x) = p^{t+a}$ *and* $O(y) = p^{t+b}$ *where a, b are non-negative integers.*
we may assume that $a \geq b$

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$since\ \ O(x^{p^{a-b}}) = \frac{O(x)}{(p^{a-b},p^{t+a})}$
$\frac{p^{t+a}}{(p^{a-b},p^{t+a-b+b})} = \frac{t+a}{p^{a-b}(1,p^{t+b})} = \frac{t+a}{p^{a-b}(1)} = \frac{t+a}{p^{a-b}} = p^{t+b} = O(y)$

$similarly\ \ MVO_A(x^{p^{a-b}}) = \frac{MVO_A(x)}{(MVO_A(x),p^{a-b})}$
$p^t = \frac{p^t}{(p^t,p^{a-b}}$
$\Rightarrow 1 = (p^t, p^{a-b})$
$\Rightarrow a-b = 0[\because t \geq 1]$
$\Rightarrow a = b$
$O(x) = p^{t+a} = p^{t+b} = O(y)$
Thus $O(x) = O(y)$.

Theorem 35. *Let G be a finite group, then G is cyclic iff every Multi vague group of G satisfies condition (I).*
Proof: Let G is a finite cyclic group. by the theorem in [4], every Multi vague of G satisfies the condition (I),
i.e., if $O(x) = O(y) \Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$

Let A is a Multi vague group of a group G with dimension $\boldsymbol{k}$.
$A = (t_A, f_A)$ *where* $t_A(x) = (t_{1A}(x), t_{2A}(x), ..., t_{kA}(x))$,
$f_A(x) = (f_{1A}(x), f_{2A}(x), ..., f_{kA}(x))$ *and* $t_{iA} : G \to [0,1]$,
$f_{iA} : G \to [0,1]$, *are mappings such that* $t_{iA}(x) + f_{iA}(x) \leq 1$,
for all $x \in G$, *for i=1,2,3,...........k., is called Multi vague set of G with dimension* $\boldsymbol{k}$. *Here* $t_{1A}(x) \geq t_{2A}(x) \geq ......t_{kA}(x)$, *for all* $x \in G$. *Here* $t'_{iA}s, f'_{iA}s$ *are fuzzy subsets of G.*

1 $t_{iA}(xy) \geq t_{iA}(x) \wedge t_{iA}(y)\}_{i=1}^{k}$ and $t_{iA}(x^{-1}) = t_{iA}(x)\}_{i=1}^{k}$

2 $f_{iA}(xy) \leq f_{iA}(x) \vee f_{iA}(y)\}_{i=1}^{k}$ *and* $f_{iA}(x^{-1}) = f_{iA}(x)\}_{i=1}^{k}$

Let $O(x) = O(y), \forall x, y \in G$
$\Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$
$\therefore t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$

Conversely assume every Multi vague group of G satisfies the condition (I)
Here $(t_{iA})'s, (f_{iA})'s$ *are fuzzy subsets of G*

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$\Rightarrow A = (t_{iA}\}_{i=1}^{k}, f_{iA}\}_{i=1}^{k})$
$\Rightarrow$ *A is a Multi vague group with dimension* $\boldsymbol{k}$.
by the hypothesis A satisfies condition (I)
so, t_A, f_A *satisfies condition (I)*
$\Rightarrow$ *G is cyclic.*

Theorem 36. *Let G be a finite group, If Multi vague group A of G with dimension* $\boldsymbol{k}$ *satisfies the condition (I) and* $O(x)|O(y)$, $\forall x, y \in G$, *then* $t_A(x) \geq t_A(y)$ *and* $f_A(x) \leq f_A(y)$.
i.e., $t_{iA}(x) \geq t_{iA}(y)\}_{i=1}^{k}$ *and* $f_{iA}(x) \leq f_{iA}(y)\}_{i=1}^{k}$.

Proof: Let A be a Multi vague group of a group G with dimension $\boldsymbol{k}$ *and* $O(x)|O(y)$
Here $A = (t_A, f_A)$ *where* $t_A(x) = (t_{1A}(x), t_{2A}(x), ..., t_{kA}(x))$, $f_A(x) = (f_{1A}(x), f_{2A}(x), ..., f_{kA}(x))$ *and* $t_{iA} : G \rightarrow [0, 1]$, $f_{iA} : G \rightarrow [0, 1]$, *are mappings such that* $t_{iA}(x) + f_{iA}(x) \leq 1$, *for all* $x \in G$, *for i=1,2,3,...........k.Here* $t_{1A}(x) \geq t_{2A}(x) \geq ......t_{kA}(x)$, *for all* $x \in G$
$\because$ *A is a Multi vague group.*
$O(x)|O(y) \Rightarrow O(\langle x \rangle)|O(\langle y \rangle)$
since $O(\langle y \rangle)$ *is cyclic subgroup, it follows that* $O(\langle y \rangle)$ *has subgroup H such that* $O(H) = O(O(\langle x \rangle))$
since H is cyclic group, then $H = O(\langle y_1 \rangle)$ *for some* $y_1 \in O(\langle y_1 \rangle)$
$\Rightarrow y_1 = y^m$, *for some positive integer m.*
Now $O(x) = O(y_1) \Rightarrow t_A(x) = t_A(y_1)\}_{i=1}^{k}$ & $f_A(x) = f_A(y_1)\}_{i=1}^{k}$
i.e., $t_{iA}(x) = t_{iA}(y_1)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y_1)\}_{i=1}^{k}$

Now consider $t_{iA}(x)\}_{i=1}^{k} = t_{iA}(y_1)\}_{i=1}^{k} = t_{iA}(y^m)\}_{i=1}^{k}$
$(\because \ y_1 = y^m) \geq t_{iA}(y)\}_{i=1}^{k}$
$t_{iA}(x)\}_{i=1}^{k} \geq t_{iA}(y)\}_{i=1}^{k} \Rightarrow t_A(x) \geq t_A(y) \rightarrow (8)$
similarly $f_{iA}(x)\}_{i=1}^{k} = f_{iA}(y_1)\}_{i=1}^{k} = f_{iA}(y^m)\}_{i=1}^{k}$
$(\because \ y_1 = y^m) \leq f_{iA}(y)\}_{i=1}^{k}$
$f_{iA}(x)\}_{i=1}^{k} \leq f_{iA}(y)\}_{i=1}^{k} \Rightarrow f_A(x) \leq f_A(y) \rightarrow (9)$
from (8) & (9) $\Rightarrow t_A(x) \geq t_A(y)$ & $f_A(x) \leq f_A(y)$
It completes the proof.

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Theorem 37. *Let H_1 and H_2 be subgroups of a group G. If $x \in H_1 - H_2$ and $y \in H_2 - H_1$ such that $O(x) = O(y)$, then there exists a Multi vague group A of group G with dimension $\boldsymbol{k}$ such that A has neither condition (I) nor condition(II).*

Proof: Given G is a group.
H_1, H_2 are subgroups of G.
choose real numbers $(a_1, a_2, ..., a_k), (c_1, c_2, ..., c_k), (d_1, d_2, ..., d_k)$, $(e_1, e_2, ..., e_k)$ & $(f_1, f_2, ..., f_k)$, such that $a_i(x) + b_i(x) \leq 1$, $c_i(x) + d_i(x) \leq 1, e_i(x) + f_i(x) \leq 1$
and $(a_1 > a_2 > ...... > a_k), (c_1 > c_2 > ...... > c_k)$ and $(e_1 > e_2 > ...... > e_k)$ $a_i's > c_i's > e_i's$ and $b_i's < d_i's < f_i's$ for i= 1, 2,....k.

$A = \{< e, (a_1, a_2, ..., a_k), (b_1, b_2, ..., b_k) >,$
$< x, (c_1, c_2, ..., c_k), (d_1, d_2, ..., d_k) >,$
$< y, (e_1, e_2, ..., e_k), (f_1, f_2, ..., f_k) > .\}$
is clearly a Multi vague group with dimension $\boldsymbol{k}$. and $O(x) = O(y)$
$t_A(x) = t_{iA}(x)\}_{i=1}^{k} = (c_1, c_2, ..., c_k)$ and
$f_A(x) = f_{iA}(x)\}_{i=1}^{k} = (d_1, d_2, ..., d_k)$
$t_A(y) = t_{iA}(y)\}_{i=1}^{k} = (e_1, e_2, ..., e_k)$ and
$f_A(y) = f_{iA}(y)\}_{i=1}^{k} = (f_1, f_2, ..., f_k)$
But $t_{iA}(x) \neq t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) \neq f_{iA}(y)\}_{i=1}^{k}$
$\therefore$ A doesn't satisfy the condition I.
$O(x) = O(y) \Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$
Let O(x) = k = O(y), then we have $t_{iA}(x^k) = t_{iA}(e)\}_{i=1}^{k}$ and $f_{iA}(x^k) = f_{iA}(e)\}_{i=1}^{k}$
Now let $t_{iA}(x^s) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x^s) = f_{iA}(e)\}_{i=1}^{k} \Rightarrow x^s = e$
by the definition of Multi vague order least positive $k \Rightarrow k|s$
since $k = O(x)$
$\therefore$ $MVO_A(x) = O(x)$
similarly $MVO_A(y) = O(y) \Rightarrow MVO_A(x) = MVO_A(y)$
but $t_{iA}(x) \neq t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) \neq f_{iA}(y)\}_{i=1}^{k}$
then A does not satisfies condition II
i.e., $[MVO_A(x) = MVO_A(y) \Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$
& $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}]$

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Theorem 38. *Let A be a Multi vague group of a group G, then GV_A is normal in G iff $MVO_A(x) = MVO_A(y^{-1}xy), \forall x, y \in G$.*
Proof: Let A is a Multi vague group of a group G with dimension $\boldsymbol{k}$. $A = (t_A, f_A)$ where $t_A(x) = (t_{1A}(x), t_{2A}(x), ..., t_{kA}(x))$, $f_A(x) = (f_{1A}(x), f_{2A}(x), ..., f_{kA}(x))$ and $t_{iA} : G \to [0,1]$, $f_{iA} : G \to [0,1]$, are mappings such that $t_{iA}(x) + f_{iA}(x) \leq 1$, for all $x \in X$, for i=1,2,3,...........k, is called Multi vague set of G with dimension $\boldsymbol{k}$. Here $t_{1A}(x) \geq t_{2A}(x) \geq ......t_{kA}(x)$, for all $x \in G$. Here $(t_{iA})'s, (f_{iA})'s$ are fuzzy subsets of G.

1 $t_{iA}(xy) \geq t_{iA}(x) \wedge t_{iA}(y)\}_{i=1}^{k}$ and $t_{iA}(x^{-1}) = t_{iA}(x)\}_{i=1}^{k}$

2 $f_{iA}(xy) \leq f_{iA}(x) \vee f_{iA}(y)\}_{i=1}^{k}$ and $f_{iA}(x^{-1}) = f_{iA}(x)\}_{i=1}^{k}$

Case I: GV_A *be a normal subgroup of* G.
where $GV_A = \{x \in G \mid t_{iA}(x) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(e)\}_{i=1}^{k}$
If $MVO_A(x) = \infty$, *then there is no 'n'* $\ni t_{iA}(x^n) = t_{iA}(e)\}_{i=1}^{k}$
& $f_{iA}(x^n) = f_{iA}(e)\}_{i=1}^{k} \Rightarrow x^n \notin GV_A$.
$\because\ x^n \notin GV_A \Rightarrow y^{-1}x^n y \notin GV_A,\ \forall y \in G$
($\because GV_A$ *is a normal subgroup of G.*)
$\Rightarrow t_{iA}(y^{-1}x^n y) \neq t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}x^n y) \neq f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow t_{iA}(y^{-1}xy)^n \neq t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}xy)^n \neq f_{iA}(e)\}_{i=1}^{k}$
$[consider(y^{-1}xy)^2 = (y^{-1}xy)(y^{-1}xy) = y^{-1}x^2y$
$(y^{-1}xy)^3 = (y^{-1}xy)^2(y^{-1}xy) = y^{-1}x^2y(y^{-1}xy) = y^{-1}x^3y$
similarly $(y^{-1}xy)^n = y^{-1}x^n y]$
$\therefore MVO_A(y^{-1}xy) = \infty$
Hence $MVO_A(x) = MVO_A(y^{-1}xy)$

Case:II If $MVO_A(x) = n\ (< \infty)$
i.e., $t_{iA}(x^n) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x^n) = f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow x^n \in GV_A$
$\Rightarrow y^{-1}x^n y \in GV_A(\because\ GV_A$ *normal subgroup of* $G, \forall y \in G)$
$\Rightarrow t_{iA}(y^{-1}x^n y) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}x^n y) = f_{iA}(e)\}_{i=1}^{k}$
$MVO_A(y^{-1}x^n y)|n$
$MVO_A(y^{-1}x^n y) \mid MVO_A(x) \to (10)$
let $MVO_A(y^{-1}x^n y) = s$

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$\Rightarrow t_{iA}(y^{-1}xy)^s = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}xy)^s = f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow t_{iA}(y^{-1}x^s y) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}x^s y) = f_{iA}(e)\}_{i=1}^{k}$
$y^{-1}x^s y \in GV_A \Rightarrow x^s \in GV_A$
$\Rightarrow t_{iA}(x^s) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x^s) = f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow MVO_A(x) \mid s$
$\Rightarrow MVO_A(x) \mid MVO_A(y^{-1}xy) \rightarrow (11)$
from (10) & (11) $\Rightarrow MVO_A(x) = MVO_A(y^{-1}xy)$

Conversely, suppose let $x \in GV_A, y \in G$
$\Rightarrow MVO_A(x) = MVO_A(y^{-1}xy)$
$\Rightarrow t_{iA}(y^{-1}xy) = t_{iA}(x)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}xy) = f_{iA}(x)\}_{i=1}^{k}$
but $t_{iA}(x) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow t_{iA}(y^{-1}xy) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}xy) = f_{iA}(e)\}_{i=1}^{k}$
$y^{-1}xy \in GV_A, \forall y \in G.$
Hence GV_A *is a normal subgroup of G.*

Theorem 39. *Let A be a Multi vague group of a group G with dimension* $\boldsymbol{k}$. *If A has the condition (II) , and* GV_A *is normal in G, then A is a multi vague normal group of G.*

Proof: Given A is a Multi vague group of a group G with dimension $\boldsymbol{k}$. *and* $MVO_A(x) = MVO_A(y)$
$\Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}$
Here $GV_A = \{x \in G \mid t_{iA}(x) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(e)\}_{i=1}^{k}$
and GV_A *is a normal subgroup of G.*
$\Rightarrow MVO_A(x) = MVO_A(y^{-1}xy)$ *(by the above theorem)*
$\Rightarrow t_{iA}(y^{-1}xy) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(y^{-1}xy) = f_{iA}(e)\}_{i=1}^{k}$
(from condition (II))
$\Rightarrow$ *A is a Multi vague normal group of G with dimension* $\boldsymbol{k}$.

The following example shows that A is Multi vague normal group of G, but A does not satisfy condition (II).

Example 40. *Define a Multi vague normal group with dimension 4 of Klein's-4 group,* $G = \{e, a, b, ab\}$,
where $a^2 = b^2 = (ab)^2 = e$ *by*

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$A = \{\langle e, (p_0, q_0, r_0, s_0), (x_0, y_0, z_0, w_0)\rangle,$
$\langle a, (p_1, q_1, r_1, s_1), (x_1, y_1, z_1, w_1)\rangle,$
$\langle b, (p_2, q_2, r_2, s_2), (x_2, y_2, z_2, w_2)\rangle,$
$\langle ab, (p_2, q_2, r_2, s_2), (x_2, y_2, z_2, w_2)\rangle\}$
Here $p_i \geq q_i \geq r_i \geq s_i$ *and* $p_i + x_i \leq 1, q_i + y_i \leq 1,$
$r_i + z_i \leq 1$ & $s_i + w_i \leq 1$ *and i= 1, 2,.....,k.*

Consider $t_A(ab) = (p_2, q_2, r_2, s_2)$
$min\{t_A(a), t_A(b)\} = min\{(p_1, q_1, r_1, s_1), (p_2, q_2, r_2, s_2)\} = (p_2, q_2, r_2, s_2)$
$\therefore\ t_{iA}(ab) \geq min\{t_{iA}(a), t_{iA}(b)\}\}_{i=1}^{4}$ *Also consider* $f_A(ab) = (x_2, y_2, z_2, w_2)$
$max\{f_A(a), f_A(b)\} = max\{(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2)\} = (x_2, y_2, z_2, w_2)$
$\therefore\ f_{iA}(ab) \leq min\{f_{iA}(a), f_{iA}(b)\}\}_{i=1}^{4}$

$\because\ a^2 = b^2 = (ab)^2 = e \Rightarrow a^{-1} = a, b^{-1} = b,$
$(ab)^{-1} = (ab)$ & $e^{-1} = e.$
$\Rightarrow t_{iA}(a^{-1}) = t_{iA}(a)\}\}_{i=1}^{4}$ & $f_{iA}(a^{1}) = f_{iA}(a)\}\}_{i=1}^{4},$
$t_{iA}(b^{-1}) = t_{iA}(b)\}\}_{i=1}^{4}$ & $f_{iA}(b^{1}) = f_{iA}(b)\}\}_{i=1}^{4},$
$t_{iA}((ab)^{-1}) = t_{iA}(ab)\}\}_{i=1}^{4}$ & $f_{iA}((ab)^{1}) = f_{iA}(ab)\}\}_{i=1}^{4},$
$t_{iA}(e^{-1}) = t_{iA}(e)\}\}_{i=1}^{4}$ & $f_{iA}(e^{1}) = f_{iA}(e)\}\}_{i=1}^{4}.$
Hence A is a Multi vague group with dimension 4.

Also $t_{iA}(ab) = t_{iA}(ba)\}_{i=1}^{4}$ & $f_{iA}(ab) = f_{iA}(ba)\}_{i=1}^{4},$
$\Rightarrow$ *A is Multi vague normal group of a group G with dimension 4.*
and $MVO_A(a) = 2$ *and* $MVO_A(b) = 2 \Rightarrow MVO_A(a) = MVO_A(b)$
but $t_A(a) = (p_1, q_1, r_1, s_1)$ & $f_A(a) = (x_1, y_1, z_1, w_1)$
$t_A(b) = (p_2, q_2, r_2, s_2)$ & $f_A(b) = (x_2, y_2, z_2, w_2)$
but $t_A(a) \neq t_A(b)$ & $f_A(a) \neq f_A(b)$
$\therefore$ *A does not satisfy the condition (II).*
(i.e., $MVO_A(x) = MVO_A(y) \Rightarrow t_{iA}(x) = t_{iA}(y)\}_{i=1}^{k}$ &
$f_{iA}(x) = f_{iA}(y)\}_{i=1}^{k}.$*)*

Note : The following is an example of a Multi vague group A which satisfies the condition (II) but A is not a Multi vague normal group of a group G and GV_A is not normal subgroup of G.

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Example 41. *Define a Multi vague group A with dimension* $\boldsymbol{k}$ *of the dihedral group.* $D_3\{e, a, b, a^2, ab, ba\}$ *where*
$a^3 = b^2 = e, ba = a^2b$
$t_A(x) = (x_1, x_2, ..., x_k)$ & $f_A(x) = (y_1, y_2, ..., y_k)$ *if* $x \in (b)$
$t_A(x) = (x_1, x_2, ..., x_k)$ & $f_A(x) = (y_1, y_2, ..., y_k)$ *otherwise*
$x_i \geq z_i,\ y_i \leq w_i$ *for i=1,2......,k.*
$x_i + y_i \leq 1$ & $z_i + w_i \leq 1$ *for i=1, 2,....,k.*
clearly

1 $t_{iA}(xy) \geq min(t_{iA}(x), t_{iA}(y))\}_{i=1}^{k}$ and $t_{iA}(x^{-1}) = t_{iA}(x)\}_{i=1}^{k}$

2 $f_{iA}(xy) \leq max(f_{iA}(x), f_{iA}(y))\}_{i=1}^{k}$ and $f_{iA}(x^{-1}) = f_{iA}(x)\}_{i=1}^{k}$

$\Rightarrow$ *A is a Multi vague group with dimension k.*
Also $MVO_A(a) = 3(\because\ a^3 = e)$
$MVO_A(b) = 1(\because\ b^2 = e)$
$MVO_A(e) = 1$
$MVO_A(a^2) = 3\ (\because\ (a^2)^3 = (a^3)^2 = e^2 = e)$
$MVO_A(ab) = 2(\because\ (ab)^2 = a(ba)b = a(a^2b)b = a^3b^2 = b^2 = e)$
$MvO_A(ba) = 1$
$\because\ MVO_A(b) = MVO_A(ba) = 1$
$\because$ *A satisfies condition (II).*
$\Rightarrow t_{iA}(b) = t_{iA}(ba)\}_{i=1}^{k}$ & $f_{iA}(b) = f_{iA}(ba)\}_{i=1}^{k}$.
Now consider $t_A(a.ab) = t_{iA}(a^2)\}_{i=1}^{k} = t_{iA}(ba)\}_{i=1}^{k}$
$t_A(a.ab) = t_{iA}(a^2)\}_{i=1}^{k} = t_{iA}(ba)\}_{i=1}^{k}$
$t_A(ab.a) = t_{iA}(a.ba)\}_{i=1}^{k} = t_{iA}(a.a^2b)\}_{i=1}^{k} = t_{iA}(a^3b)\}_{i=1}^{k}$
$= t_{iA}(eb)\}_{i=1}^{k} = t_{iA}(b)\}_{i=1}^{k}$
$\therefore\ t_{iA}(a.ab) = t_{iA}(ba)\}_{i=1}^{k} \& f_{iA}(a.ab) = f_{iA}(ba)\}_{i=1}^{k}$
and $t_{iA}(ab.a) = t_{iA}(b)\}_{i=1}^{k} \& f_{iA}(ab.a) = f_{iA}(b)\}_{i=1}^{k}$
$\because\ t_{iA}(ba) \neq t_{iA}(b)\}_{i=1}^{k} \& f_{iA}(ba) \neq f_{iA}(b)\}_{i=1}^{k}$
$\Rightarrow t_{iA}(a.ab) \neq t_{iA}(ab.a)\}_{i=1}^{k} \& f_{iA}(a.ab) \neq f_{iA}(ab.a)\}_{i=1}^{k}$
$\Rightarrow$ *A is not a Multi vague group of* D_3.
Also $GV_A = \{e, b\}, \because\ a \in G, b \in GV_A$
$\therefore aba^{-1} = aba^2\ (\because\ a^3 = e \Rightarrow a.a^2 = e \Rightarrow a^{-1} = a^2)$
$= abaa = a(ba)a = a(a^2b)a)(\because ba = a^2b)$
$= a^3ba = eba = ba \notin GV_A$.
$\Rightarrow GV_A$ *is not a normal subgroup of G.*

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Theorem 42. *If a Multi vague group of a group G with dimension **k** satisfies condition(I),then A is Multi vague normal in G.*
*Proof: Given A is a Multi vague group of a group G with dimension **k** satisfies condition (I), i.e., $x, y \in G$ then $O(x) = O(y^{-1}xy)$*
$\Rightarrow t_{iA}(x) = t_{iA}(y^{-1}xy)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(y^{-1}xy)\}_{i=1}^{k}$
(from condition (I))
$\Rightarrow$ *A is a Multi vague normal group of G.*

For the next theorem we use the following theorem which is a well known theorem in "Group Theory" by B. Huppert.
"Let G be a non abelian finite group in which all subgroups are normal. Then $G = Q \times C \times E$, where Q is the quaternion group of order 8, C is an abelian group in which every element is of odd order and E is an abelian group of exponent '2' or '1'."

Theorem 43. *Let G be a finite group. If either condition (I) or condition (II) holds then G is cyclic.*
*(i) every Multi vague group A of G with dimension **k** satisfies condition (I).*
(ii) Every vague group A of G satisfies condition (II) and GV_A is normal in G.
Proof :Given G be a finite group.
*A be a Multi vague group of G with dimension **k***
If A satisfies (I), then G is cyclic.
(by the theorem, Let G be finite group. Then G is cyclic iff every Multi vague group of G satisfies condition (I))
*If A satisfies condition(II), then A is a Multi vague normal group of G. (from the theorem, Let A be Multi vague group of a group G, with dimension **k**. If A has the condition (II), and GV_A is normal in G, then A Multi vague normal group of G.)*

$\therefore$ TA_α, FA_α *are normal. Thus subgroups of G are normal and hence G is an abelian group (in view of the above theorem)*

Now, if possible G is not cyclic.

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Let $x \in G \Rightarrow \langle x \rangle \neq G$. *Then* $\exists\, y \in G, \ni y \notin \langle x \rangle$.
Let $O(x) = m$ *and* $O(y) = n$
Let $x_1 = x^{\frac{m}{(m,n)}}$ $y_1 = y^{\frac{n}{(m,n)}}$
Then $O(x_1) = (m,n) and O(x_1) = (m,n)$
since $x_1 \in \langle x_1 \rangle - \langle y_1 \rangle and y_1 \in \langle y_1 \rangle - \langle x_1 \rangle with O(x_1) = O(y_1)$
by the theorem (Let H_1 *and* H_2 *be subgroups of a group G.*
If $x \in H_1 - H_2$ *and* $y \in H_2 - H_1$ *such that* $O(x) = O(y)$, *then there exists a Multi vague group A of group G with dimension* ***k*** *such that A has neither condition (I) nor condition(II)).* $\exists$ *a Multi vague group A of G, then satisfies neither condition (I) nor (II).*
This is a contradiction.
Hence G is cyclic.

J.G. Kim introduced the concept of fuzzy p-subgroup and fuzzy p^* subgroup, and studied some characterizations. Now we introduce the following J. G. Kim notions, which are generalizations of fuzzy p-group, fuzzy P^*- subgroups.

Definition 4 4. Let A be a Multi vague group of a group G. For a prime p, A is said to be Multi vague p-group if $MVO_A(x)$ is a power of p, $\forall x \in G$.

Definition 4 5. Let A be a Multi vague group of an abelian group G. A is called Multi vague torsion if $MVO_A(x)$ is finite for all $x \in G$.

Example 46. *Let* $G = \{e, x, y, z\}, x^2 = y^2 = z^2 = e$ *be the Klein's 4- group.*
Define a Multi vague group A with dimension - 4, by
$A = \{\langle e, (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3)\rangle$
$\langle x, (c_0, c_1, c_2, c_3), (d_0, d_1, d_2, d_3)\rangle$
$\langle y, (c_0, c_1, c_2, c_3), (d_0, d_1, d_2, d_3)\rangle$
$\langle z, (a_0, a_1, a_2, a_3), (x_1, y_1, z_1, w_1)\rangle\}$
Here $a_0 \geq a_1 \geq a_2 \geq a_3 \& c_0 \geq c_1 \geq c_2 \geq c_3$ *and* $a_i + b_i \leq 1$,
$c_i + d_i \leq 1$ *and i= 0, 1, 2 and 3*
Here $MVO_A(e) = 1$

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$MVO_A(x) = 2(\because\ t_{iA}(x^2)\}_{i=1}^4 = t_{iA}(e)\}_{i=1}^4 \& f_{iA}(x^2)\}_{i=1}^4 = f_{iA}(e)\}_{i=1}^4)$
$MVO_A(y) = 2(\because\ t_{iA}(x^2)\}_{i=1}^4 = t_{iA}(e)\}_{i=1}^4 \& f_{iA}(x^2)\}_{i=1}^4 = f_{iA}(e)\}_{i=1}^4)$
$MVO_A(z) = 1(\because\ t_{iA}(x)\}_{i=1}^4 = t_{iA}(e)\}_{i=1}^4 \& f_{iA}(x)\}_{i=1}^4 = f_{iA}(e)\}_{i=1}^4)$
$\because$ *MVO_A of every element is finite.*
$\Rightarrow$ *A is Multi vague torsion.*

Definition 47. Let A be a Multi vague group of G, with dimension **k** and 'p' is a prime. A is said to be a Multi vague p^* group, if for any $x \in G, min\{n \in \mathbb{N} \mid t_{iA}(x) \leq t_{iA}(x^p)^n\}_{i=1}^k$ & $f_{iA}(x) \geq f_{iA}(x^p)^n\}_{i=1}^k\}$ is a power of p, whenever this minimum exists.

(Recall the example
Define a vague group A of $\mathbb{Z}$ as

$$V_A(x) = \begin{cases} I_1\ if\ x = 0 \\ I_2\ if\ x \in 3Z - \{0\} \\ I_3, otherwise, \end{cases}$$

where $I_1 \geq I_2 \geq I_3, I_1 = [a_1, b_1], I_2 = [a_2, b_2], I_3 = [a_3, b_3]$, clearly A is vague 3^* group of Z.)

Example 48. *Define a Multi vague group A of $\mathbb{Z}$, with dimension **k** as* $A = (t_A(x), f_A(x))$

$$(t_A(x), f_A(x)) = \begin{cases} ((a_1, a_2, ..., a_k), (b_1, b_2, ..., b_k))\ if\ x = 0 \\ ((c_1, c_2, ..., c_k), (d_1, d_2, ..., d))\ if\ x \in 3Z - \{0\} \\ ((e_1, e_2, ..., e_k), (f_1, f_2, ..., f_k)), otherwise, \end{cases}$$

Here $a_i + b_i \leq 1, b_i + c_i \leq 1, e_i + f_i \leq 1$
$a_1 \geq a_2 \geq .. \geq a_k,\ c_1 \geq c_2 \geq .... \geq c_k$ *and* $e_1 \geq e_2 \geq .. \geq e_k$
$\forall i = 1, 2, ..., k$, *clearly A is Multi vague 3^* group of $\mathbb{Z}$.*

Case I: Suppose $x = 0 \Rightarrow x^2 = 0\ \&\ x^3 = 0$
$\therefore t_{iA}(x) = t_{iA}(0) = a_i\}_{i=1}^k\ \&\ f_{iA}(x) = f_{iA}(0) = b_i\}_{i=1}^k$
i.e., $t_A(x) = t_A(0) = (a_1, a_2, ..., a_k)\ \&\ f_A(0) = (b_1, b_2, ..., b_k)$
$x^2 = 0 \Rightarrow t_A(x^{3^1}) = t_A(0) = (a_1, a_2, ..., a_k)$ *and*
$f_A(x^{3^1}) = f_A(0) = (b_1, b_2, ..., b_k)$
$x^2 = 0 \Rightarrow t_A(x^{3^n}) = t_A(x) = t_A(0) = (a_1, a_2, ..., a_k)$
and $f_A(x^{3^n}) = f_A(0) = (b_1, b_2, ..., b_k)$

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Now $min\{n \in \mathbb{N} \mid t_{iA}(x) \leq t_{iA}(x^p)^n\}_{i=1}^k \;\&\; f_{iA}(x) \geq f_{iA}(x^p)^n\}_{i=1}^k\}$
$\Rightarrow min\{3^1, 3^2, ..., 3^n\} = 3$

Case II: Suppose $x \neq 0, x \in 3\mathbb{Z}$
$\therefore t_{iA}(x) = t_{iA}(3\mathbb{Z}) = c_i\}_{i=1}^k \;\&\; f_{iA}(x) = f_{iA}(3\mathbb{Z}) = d_i\}_{i=1}^k$
i.e., $t_A(x) = t_A(3\mathbb{Z}) = (c_1, c_2, ..., c_k) \;\&\; f_A(3\mathbb{Z}) = (d_1, d_2, ..., d_k)$
$t_A(x^{3^1}) = t_A(3x) = (c_1, c_2, ..., c_k) \;\&\; f_A(x^{3^1}) = f_A(3x) = (d_1, d_2, ..., d_k)$
$t_A(x^{3^n}) = t_A(3^n x) = (c_1, c_2, ..., c_k)$
and $f_A(x^{3^n}) = f_A(3^n x) = (d_1, d_2, ..., d_k)$

Now $min\{n \in \mathbb{N} \mid t_{iA}(x) \leq t_{iA}(x^p)^n\}_{i=1}^k \;\&\; f_{iA}(x) \geq f_{iA}(x^p)^n\}_{i=1}^k\}$
$\therefore min\{3^1, 3^2, ..., 3^n\} = 3$

Case III: Suppose $x \neq 0, x \neq 3\mathbb{Z}$
$\therefore t_{iA}(x) = e_i\}_{i=1}^k \;\&\; f_{iA}(x) = f_i\}_{i=1}^k$
$t_A(x^{3^1}) = t_A(3.x) = (c_1, c_2, ..., c_k) \;\&\; f_A(x^{3^1}) = f_A(3.x) = (d_1, d_2, ..., d_k)$
$t_A(x^{3^n}) = t_A(3^n x) = (c_1, c_2, ..., c_k)$
and $f_A(x^{3^n}) = f_A(3^n x) = (d_1, d_2, ..., d_k)$
$\because e_i > c_i and f_i < d_i \forall i = 1, 2, , ..., k.$
$\Rightarrow t_{iA}(x) > tiA((x^3)^n) \& f_{iA}(x) < fiA((x^3)^n)$
Now $min\{n \in \mathbb{N} \mid t_{iA}(x) \leq t_{iA}(x^p)^n\}_{i=1}^k \;\&\; f_{iA}(x) \geq f_{iA}(x^p)^n\}_{i=1}^k\}$
$\therefore min\{3^1, 3^2, ..., 3^n\} = 3$
$\therefore$ *A is the Multi vague* 3^* *group.*

(We recall the theorem
Let A be a vague group of G such that GV_A is a normal S.G of G.
Then A is a vague p-group $\Leftrightarrow \frac{G}{GV_A}$ is a p-group.)

Theorem 49. *Let A be a Multi vague group of G such that GV_A is a normal Subgroup of G with dimension **k**. Then A is a Multi vague p- group $\Leftrightarrow \frac{G}{GV_A}$ is a p-group.*
*Proof: Let A be a Multi vague p-group with dimension **k**.*
i.e, $MVO_A(x) = p^r$, *where p is a prime and* $r \in \mathbb{Z}$
$\Rightarrow t_{iA}(x^{p^r}) = t_{iA}(e)\}_{i=1}^k \;\&\; f_{iA}(x^{p^r}) = f_{iA}(e)\}_{i=1}^k$
$\Rightarrow x^{p^r} \in GV_A$
$\Rightarrow (xGV_A)p^r = GV_A = eGV_A$
$\Rightarrow$ *order of every element in* $\frac{G}{GV_A}$ *is prime.*

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$\Rightarrow \frac{G}{GV_A}$ *is a p-group.*
where $GV_A = \{x \in G | t_{iA}(x) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x) = f_{iA}(e)\}_{i=1}^{k}\}$

Conversely, $\frac{G}{GV_A}$ *is a p- group.*
$(xGV_A)p^r = GV_A$, *for,* $x \in G \Rightarrow x^{p^r} \in GV_A$
$\Rightarrow t_{iA}(x^{p^r}) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x^{p^r}) = f_{iA}(e)\}_{i=1}^{k}$
$\Rightarrow MVO_A(x)$ *is a power of p.*
Thus A is a Multi vague p - group.

Theorem 50. *Let A be a Multi vague group of abelian group G with dimension* $\boldsymbol{k}$, *then A is Multi vague torsion iff* $\frac{G}{GV_A}$ *is torsion.*
Proof: Given A is Multi vague torsion for $x \in G, MVO_A(x)$ *is finite.* $\Rightarrow t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^{k}$
Thus $x^m \in GV_A \Rightarrow (xGV_A)^m = GV_A$ *where* $xGV_A \in \frac{G}{GV_A}$
Thus $\frac{G}{GV_A}$ *is torsion*
$\Rightarrow O(xGV_A)$ *is finite, for* $xGV_A \in \frac{G}{GV_A}$
Let $O(xGV_A) = k \Rightarrow (xGV_A)^k = GV_A \Rightarrow x^k \in GV_A$
$\Rightarrow t_{iA}(x^m) = t_{iA}(e)\}_{i=1}^{k}$ & $f_{iA}(x^m) = f_{iA}(e)\}_{i=1}^{k}$
Thus $MVO_A(x) = m$ *is finite.*
$\therefore$ *A is Multi vague torsion.*

Theorem 51. *Let B be a Multi vague* p^* *group of a group G, with dimension* $\boldsymbol{k}$. *Let n, m, t be positive integers such that* $n = p^t m$ *and* $(p, m) = 1$,*then* $t_{iB}(x^n) = t_{iB}(x^{p^t})\}_{i=1}^{k}$ *and* $f_{iB}(x^n) = f_{iA}(x^{p^t})\}_{i=1}^{k} \forall x \in G$.
Proof: Let B be a Multi vague p^* *group of a group G, with dimension* $\boldsymbol{k}$, $B = (t_B, f_B)$ *where* $t_B(x) = (t_{1B}(x), t_{2B}(x), ..., t_{kB}(x))$ *and* $f_A(x) = (f_{1B}(x), f_{2B}(x), ..., f_{kB}(x))$
$t_{iB} : G \to [0, 1], f_{iB} : G \to [0, 1]$, *are mappings such that*
$t_{iB}(x) + f_{iB}(x) \leq 1$, *for all* $x \in G$, *for i=1,2,3,...........k.*
Here $t_{1B}(x) \geq t_{2B}(x) \geq ......t_{kB}(x)$, *for all* $x \in G$.
since $n = p^t m$, *and* $(p, m) = 1$
$= t_B(x^n) = t_{iB}(x^n)\}_{i=1}^{k} = t_{iB}(x^{p^t m})\}_{i=1}^{k}$
$= t_{iB}(x^{p^t}.x^{p^t}.......x^{p^t}\ m\ times)\}_{i=1}^{k}$

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$\geq min\{t_{iB}(x^{p^t}, t_{iB}(x^{p^t}, ..., t_{iB}(x^{p^t})\}_{i=1}^k$
$= t_{iB}(x^{p^t})\}_{i=1}^k = t_B(x^{p^t})$
$\Rightarrow t_B(x^n) \geq t_B(x^{p^t})$
$\therefore t_B(x^{p^t}) \leq t_B(x^n) \rightarrow (12)$
If possible, let $t_B(x^{p^t} < t_B(x^n)$
i.e., $t_{iB}(x^{p^t} < t_{iB}(x^n)\}_{i=1}^k$
Let $y = x^{p^t}$ *then* $t_{iB}(y) < t_{iB}(y^m)\}_{i=1}^k$
since B is a Multi vague p^* *group, there is a natural number 's'*
such that $p^s = min\{l \in \mathbb{N} \mid t_{iB}(y) < t_{iB}(y^t)\}_{i=1}^k\} \leq m$
so, $t_{iB}(y^j) = t_{iB}(y)$ *for* $j = 1, 2, ..., p^{s-1}$
By Euclidean algorithm, $\exists$ *integers q and r such that* $m = p^s q + r$
and $0 \leq r \leq p^s$, *Here* $r \neq 0$, *since* $(p, m) = 1$
so, $t_{iB}(y) < t_{iB}(y^{p^s}) < t_{iB}(y^{ps})\}_{i=1}^k$
and $t_{iB}(y) = t_{iB}(y^r)\}_{i=1}^k$
Thus $t_{iB}(y^r) = t_{iB}(y^{ps}.y^r) = t_{iB}(y^r) = t_{iB}(y)\}_{i=1}^k$
$t_{iB}(y^m) = t_{iB}(y)\}_{i=1}^k$
This is a contradiction.
$\therefore t_B(x^{p^t}) \leq t_B(x^n) \rightarrow (13)$
$\therefore from(12)\ \&\ (13)\ t_B(x^n) = t_B(x^{p^t})$
i.e., $t_{iB}(x^n) = t_{iB}(x^{p^t})\}_{i=1}^k$

similarly we can prove $f_B(x^n) = f_B(x^{p^t})$
since $n = p^t m$, *and* $(p, m) = 1$
$= f_B(x^n) = f_{iB}(x^n)\}_{i=1}^k = fiB(x^{p^t m})\}_{i=1}^k$
$= f_{iB}(x^{p^t}.x^{p^t}.......x^{p^t}\ m\ times)\}_{i=1}^k$
$\leq max\{f_{iB}(x^{p^t}, f_{iB}(x^{p^t}, ..., f_{iB}(x^{p^t})\}_{i=1}^k$
$= f_{iB}(x^{p^t})\}_{i=1}^k = f_B(x^{p^t})$
$\Rightarrow f_B(x^n) \leq f_B(x^{p^t})$
$\therefore f_B(x^{p^t}) \geq f_B(x^n) \rightarrow (14)$
If possible, let $f_B(x^{p^t} > f_B(x^n)$
i.e., $f_{iB}(x^{p^t} > f_{iB}(x^n)\}_{i=1}^k$
Let $y = x^{p^t}$ *then* $f_{iB}(y) > f_{iB}(y^m)\}_{i=1}^k$
since B is a Multi vague p^* *group, there is a natural number 's'*
such that $p^s = max\{l \in \mathbb{N} \mid f_{iB}(y) < f_{iB}(y^t)\}_{i=1}^k\} \geq m$
so, $f_{iB}(y^j) = f_{iB}(y)$ *for* $j = 1, 2, ..., p^{s-1}$

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By Euclidean algorithm, $\exists$ *integers q and r such that* $m = p^s q + r$ *and* $0 \le r \le p^s$, *Here* $r \neq 0$, *since* $(p, m) = 1$
so, $f_{iB}(y) > f_{iB}(y^{p^s}) > f_{iB}(y^{ps})\}_{i=1}^{k}$
and $f_{iB}(y) = f_{iB}(y^r)\}_{i=1}^{k}$
Thus $f_{iB}(y^r) = f_{iB}(y^{ps}.y^r) = f_{iB}(y^r) = f_{iB}(y)\}_{i=1}^{k}$
$f_{iB}(y^m) = f_{iB}(y)\}_{i=1}^{k}$
This is a contradiction.
$\therefore f_B(x^{p^t}) \ge f_B(x^n) \to (15)$
$\therefore from(14)$ & (15) $f_B(x^n) = f_B(x^{p^t})$
i.e., $f_{iB}(x^n) = f_{iB}(x^{p^t})\}_{i=1}^{k}$
$\therefore t_{iB}(x^n) = t_{iB}(x^{p^t})\}_{i=1}^{k}$ & $f_{iB}(x^n) = f_{iB}(x^{p^t})\}_{i=1}^{k}$.
$\Rightarrow t_B(x^{p^t}) = t_A(x^n)$ & $f_B(x^{p^t}) = f_A(x^n)$

Theorem 52. *Every Multi vague p - group of a group G with dimension* $\boldsymbol{k}$ *is a Multi vague* p^* *group of G with dimension* $\boldsymbol{k}$.
Proof: Let A be a Multi vague p group of G with dimension $\boldsymbol{k}$
Let $x \in G$, *and let* $n = p^t.m$ *be positive integers such that (p, m) = 1,* $t_{iA}(x) < t_{iA}(x^n)\}_{i=1}^{k}$ *and* $f_{iA}(x) > f_{iA}(x^n)\}_{i=1}^{k}$
Now, let $y = x^{p^t}$, *then* $MVO_A(y)$ *is a power of p.*
so $(MVO_A(y), m) = 1$
Thus $t_{iA}(x^{p^t}) = t_{iA}(y) = t_{iA}(y^m) = t_{iA}(x^n)\}_{i=1}^{k}$
and $f_{iA}(x^{p^t}) = f_{iA}(y) = f_{iA}(y^m) = f_{iA}(x^n)\}_{i=1}^{k}$
Thus $t_{iA}(x^{p^t}) = t_{iA}(x^n)\}_{i=1}^{k}$ & $f_{iA}(x^{p^t}) = f_{iA}(x^n)\}_{i=1}^{k}$
$\Rightarrow t_A(x^{p^t}) = t_A(x^n)$ & $f_A(x^{p^t}) = f_A(x^n)$
$\therefore$ *A is a Multi vague* p^* *group of G.*

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