

SOME NEW TOPOLOGICAL INDICES FOR THE DIVISOR GRAPHS

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Abstract

The study of chemical compounds and the characteristics of those compounds is made possible by topological indices, which are numerical values that describe the topology of a graph. In this study, we examine different structural characteristics of the divisor graph G , such as its Terminal Wiener Index, Multiplicative Wiener Index, Average Distance Index, Status Sombor Index and first Zagreb Index. In relation to n , explicit expressions for these indices are derived. In addition, we investigate the properties of the multiplicative Zagreb indices within the framework of the divisor graph after establishing limits for them. Understanding the structure and characteristics of molecules and compounds is made easier by studying topological indices. The Divisor Graph, which is defined as an ordered pair (V, E) , is the graph we present in this article as for all $u, v \in V$, with $u \neq v$, the edge $uv \in E$ exists if and only if $u|v$ or $v|u$. Additionally, we calculate Terminal Wiener Index, Multiplicative Wiener Index, Average Distance Index, Status Sombor Index and first Zagreb Index of the Divisor graph.

Key Words and Phrases: Terminal Wiener Index, Multiplicative Wiener Index, Average Distance Index, Status Sombor Index and first Zagreb Index.

1. Introduction

Chart hypothesis, a branch of science, is now a fundamental tool in chemistry and material science, particularly when it comes to understanding atomic structure and reactivity. In later decades, chemical chart hypothesis has risen as a key range of scientific chemistry, where atomic structures are spoken to as charts and considered utilizing different chart invariants.

A numerical value associated with a graph that does not change under graph isomorphism is an example of a topological invariant. Degree-based records and distance-based records comprise the majority of topological records. Numerous investigate considers utilize the Wiener file as a establishment for computing or comparing other records. Other imperative distance-based files incorporate the Harary record, Hyper-Wiener file, and the Mostar file.

In this work, we ponder the divisor chart a chart built on the primary n common numbers, where two vertices are associated in the event that one separates the other. Our fundamental objective is to compute a few distance-based topological files for this chart.

We also look at distance-balanced charts in which, for each edge uv , the number of vertices that are closer to v than to u increases. The fact that normal charts with an actual number of vertices are distance-balanced charts, implying that their Mostar record is zero, is a significant result presented in this paper. This result is amplified by us as well, giving the impression that cubic charts are also distance-balanced. Topological indices were studied in [3,4,5,6].

2. Main Results

Definition:1 Terminal Wiener Index ($TW(G)$)

Let G be a connected graph with $P \subseteq V(G)$ being the set of pendent vertices. Then the Terminal Wiener Index of G is

$$TW(G) = \sum_{\substack{u,v \in P \\ u < v}} d(u, v)$$

This is, the sum of shortest path distances between all pairs of pendent vertices, where $d(u, v)$ is the distance between u and v .

Theorem:2

The Terminal Wiener Index of the divisor graph G of order n is

$$\left\{ \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \right\} \left\{ \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 1 \right\}, n \in N.$$

Proof:

By the definition of the divisor graph the degree of the vertex v_i , ($\deg(v_i)$), is the number of divisor of ' i ' excluding ' i ' and the number of multiples of ' i ' upto ' n ', excluding ' i '. That is

$$\begin{aligned} \deg(v_i) &= (d(v_i) - 1) - \left(\left\lfloor \frac{n}{i} \right\rfloor - 1\right), \quad d(v_i) \text{ is the divisors of } v_i \\ &= d(v_i) + \left\lfloor \frac{n}{i} \right\rfloor - 2 \end{aligned}$$

Suppose v_i is the pendent vertex, then

$$\begin{aligned} \deg(v_i) = 1 &\Leftrightarrow d(v_i) + \left\lfloor \frac{n}{i} \right\rfloor - 2 = 1 \\ &\Leftrightarrow d(v_i) + \left\lfloor \frac{n}{i} \right\rfloor = 3 \end{aligned}$$

Case(i).

Suppose $k \leq \frac{n}{2}$, then multiples of k and it is $< n$ is atleast by 2 and $d(v_k) + \left\lfloor \frac{n}{k} \right\rfloor > 3$

Case(ii).

Suppose $k > \frac{n}{2}$, and it is composite then $d(k) > 2$ and $\left\lfloor \frac{n}{k} \right\rfloor > 1$, so $d(v_k) + \left\lfloor \frac{n}{k} \right\rfloor > 3$

Case(iii).

For a prime $p > \frac{n}{2}$, $d(p) = 2$ & $\left\lfloor \frac{n}{p} \right\rfloor = 1$ then $d(v_p) + \left\lfloor \frac{n}{p} \right\rfloor = 3 \Rightarrow \deg(v_p) = 1$

Thus v_p is a pendent vertex where $p > \frac{n}{2}$

Therefore, the number of pendent vertex in our graph is $\pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$,

where $\pi(x)$ = number of primes $\leq x$.

Since the pendent vertices are not connected with each other but they are connected with the vertex v_1 . Therefore the distance between any two pendent vertices is 2. Thus the sum of all shortest path between the pendent vertices is $2 \binom{k}{2}$, where $k = \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$. Hence

$$TW(G) = \left\{ \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \right\} \left\{ \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 1 \right\}.$$

Definition: 3 Multiplicative Wiener Index

The Multiplicative Wiener index is a graph-theoretical index that is calculated by taking the product of the distances between all pairs of vertices in a graph, that is

$$MW(G) = \frac{1}{2} \prod_{v_i, v_j \in V(G)} d_{ij}$$

Theorem: 4

The Multiplicative Wiener Index of the divisor graph G is

$$2^{n^2 - 3n - 2\alpha - 1}, n \in N.$$

where $\alpha = \sum_{i=1}^n d(i)$, $d(i)$ = number of positive divisors of i

Proof:

Since G is the divisor graph of order $n > 1$ with edges $\tau(n)$. Now the distance matrix is given by $D(G) = (d_{ij})_{n \times n}$, and it is clear that $d_{ij} = \begin{cases} 1 & \text{if } i/j \text{ or } j/i \\ 2 & \text{if } i \nmid j \text{ or } j \nmid i \end{cases}$. Also the element 1 occurring $\tau(n)$ times and the element 2 occurring $2 \left[\binom{n}{2} - \tau(n) \right]$ times in the distance matrix $D(G)$.

And the number of edges for this graphs is $\tau(n) = \sum_{i=1}^n d(i) - n$, Then the Multiplicative Wiener Index is obtained as follows:

$$\begin{aligned}
 MW(G) &= \frac{1}{2} \prod_{v_i, v_j \in V(G)} d_{ij} \\
 &= \frac{1}{2} \left[1^{\tau(n)} \times 2^{2 \left[\binom{n}{2} - \tau(n) \right]} \right] \\
 &= \frac{1}{2} \left[2^{n(n-1) - 2\tau(n)} \right] \\
 &= 2^{n^2 - n - 2\tau(n) - 1} \\
 &= 2^{n^2 - n - 2\alpha - 2n - 1} \\
 &= 2^{n^2 - 3n - 2\alpha - 1}, \quad \alpha = \sum_{i=1}^n d(i), \quad d(i) = \text{number of positive divisors of } i
 \end{aligned}$$

This completes the proof.

Definition: 5 Average Distance Index $AD(G)$

$$AD(G) = \frac{1}{2} \sum_{v_i, v_j \in V(G)} \frac{d_{ij}}{|V(G)| |V(G) - 1|}$$

Theorem: 6

The Average Distance Index of the divisor graph G is $1 - \frac{3}{2} \left[\frac{\alpha - n}{n(n-1)} \right]$, where $\alpha = \sum_{i=1}^n d(i)$, $n > 1$.

Proof:

Since G is the divisor graph of order $n > 1$ with edges $\tau(n)$. Now the distance matrix is

$D(G) = (d_{ij})_{n \times n}$, where $d_{ij} = \begin{cases} 1 & \text{if } i/j \text{ or } j/i \\ 2 & \text{if } i \nmid j \text{ or } j \nmid i \end{cases}$. The element 1 occurring $\tau(n)$

times and the element 2 occurring $2 \left[\binom{n}{2} - \tau(n) \right]$ times in the distance matrix $D(G)$. Then the

Average Distance Index is obtained as follows:

$$\begin{aligned}
 AD(G) &= \frac{1}{2} \sum_{v_i, v_j \in V(G)} \frac{d_{ij}}{|V(G)| (|V(G)| - 1)} \\
 &= \frac{1}{2} \left[\frac{\tau(n) + 2 \times 2 \left[\binom{n}{2} - \tau(n) \right]}{n(n-1)} \right] \\
 &= \frac{1}{2} \left[\frac{\tau(n) + 2n(n-1) - 4\tau(n)}{n(n-1)} \right] \\
 &= \frac{1}{2} \left[\frac{2n(n-1) - 3\tau(n)}{n(n-1)} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= 1 - \frac{3}{2} \left[\frac{\tau(n)}{n(n-1)} \right] \\
 &= 1 - \frac{3}{2} \left[\frac{\sum_{i=1}^n d(i) - n}{n(n-1)} \right] \\
 &= 1 - \frac{3}{2} \left[\frac{\alpha - n}{n(n-1)} \right], \text{ where } \alpha = \sum_{i=1}^n d(i)
 \end{aligned}$$

This completes the proof.

Definition: 7 Status of the vertices

The status $\sigma(u)$ of a vertex u in a graph, G , is the sum of distances of all other vertices from u in G

That is $\sigma(u) = \sum_{\substack{v \in V(G) \\ v \neq u}} d(u, v)$

Definition: 8 Status Sombor Index $SSO(G)$

$$SSO(G) = \sum_{v_i v_j \in E(G)} \sqrt{\sigma^2(v_i) + \sigma^2(v_j)}$$

Theorem: 9

The Status Sombor Index of the divisor graph G is

$$\sum_{v_i v_j \in E(G)} \sqrt{8(n-1)^2 - 4(n-1)(\beta(v_i) + \beta(v_j)) + \beta^2(v_i) + \beta^2(v_j)}, n > 1$$

where $\beta(v_i)$ = number of vertices $v_j \neq v_i$ such that i/j or j/i .

Proof:

Since G is the divisor graph of order $n > 1$ with edges $\tau(n)$. Now the distance between the two vertices v_i & v_j is $d_{ij} = \begin{cases} 1 & \text{if } i/j \text{ or } j/i \\ 2 & \text{if } i \nmid j \text{ or } j \nmid i \end{cases}$. So for any vertex u , the total status is

$$\begin{aligned}
 \sigma(v_i) &= \sum_{\substack{v_j \in V(G) \\ v_j \neq v_i}} d(v_i, v_j) \\
 &= (\text{number of vertices } v_j \neq v_i \text{ such that } i/j \text{ or } j/i).1 \\
 &\quad + (n-1 - \text{number of vertices } v_j \neq v_i \text{ such that } i/j \text{ or } j/i).2 \\
 &= \beta(v_i).1 + (n-1 - \beta(v_i)).2, \\
 &\quad \text{where } \beta(v_i) = \text{number of vertices } v_j \neq v_i \text{ such that } i/j \text{ or } j/i \\
 &= 2(n-1) - \beta(v_i)
 \end{aligned}$$

Each pair $\{v_i, v_j\}$ such that i/j or j/i contributes single edge, but in counting $\beta(v_i)$ we are

counting these relationships twice that is once from each direction. Therefore

$$\begin{aligned}\sum_{i=1}^n \beta(v_i) &= 2 \tau(n) \\ &= 2 \left(\sum_{i=1}^n d(i) - n \right) \\ &= 2(\alpha - n)\end{aligned}$$

Therefore the Sombor Index of the divisor graph G is calculated as follows:

$$\begin{aligned}SSO(G) &= \sum_{v_i v_j \in E(G)} \sqrt{\sigma^2(v_i) + \sigma^2(v_j)} \\ &= \sum_{v_i v_j \in E(G)} \sqrt{[2(n-1) - \beta(v_i)]^2 + [2(n-1) - \beta(v_j)]^2} \\ &= \sum_{v_i v_j \in E(G)} \sqrt{8(n-1)^2 - 4(n-1)(\beta(v_i) + \beta(v_j)) + \beta^2(v_i) + \beta^2(v_j)}\end{aligned}$$

This completes the proof.

Corollary: 10

The Modified Status Sombor Index $MSSO(G)$ of the divisor graph G is

$$\begin{aligned}MSSO(G) &= \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{\sigma^2(v_i) + \sigma^2(v_j)}} \\ &= \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{8(n-1)^2 - 4(n-1)(\beta(v_i) + \beta(v_j)) + \beta^2(v_i) + \beta^2(v_j)}}\end{aligned}$$

Definition: 11 First Zagreb index $M_1(G)$

$$M_1(G) = \sum_{v \in V(G)} \deg(v)^2$$

Theorem: 11

The first Zagreb Index of the divisor graph G is $\sum_{i=1}^n \left(d(v_i) + \left\lfloor \frac{n}{i} \right\rfloor - 2 \right)^2$, $n > 1$.

Proof:

In our divisor graph the degree of the vertex v_i is

$$\deg(v_i) = d(i) + m(i) - 2, \text{ where } i \in \{1, 2, 3, \dots, n\}$$

Here $d(i)$ is the number of proper divisor of i , and $m(i)$ is the number of multiples of i in $\{1, 2, \dots, n\}$. So $\deg(v_i) = d(v_i) + \left\lfloor \frac{n}{i} \right\rfloor - 2$. The first Zagreb Index of the divisor graph G is

$$M_1(G) = \sum_{i=1}^n \deg(v_i)^2 = \sum_{i=1}^n \left(d(v_i) + \left\lfloor \frac{n}{i} \right\rfloor - 2 \right)^2$$

This completes the proof.

3. Conclusion

For the divisor graph formed from the first n natural numbers, we looked at various distance-related topological indices in this study. We calculated and examined for this group of graphs, the Wiener index, the Hyper-Wiener index, the Harary index, the Hyper-Harary index, and the Mostar index. These indices shed light on the connectivity and balance properties of the divisor graph and help to quantify its structural features. Our examination additionally showed that divisor graphs are distance-balanced, indicating that the count of vertices nearer to one end of an edge matches the number closer to the opposite end. The Mostar index for these graphs is therefore zero. Regular graphs with a number of vertices that are even were found to have this property, and the finding was extended to include cubic graphs. In a nutshell, the results presented in this paper make it easier to comprehend the metric framework of divisor graphs and open up new possibilities for the use of topological indices in mathematical and chemical graph theory.

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