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OPTIMIZATION OF MIN-MAX PROBLEMS WITH NON-DIFFERENTIABLE FUNCTIONS

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Abstract

The paper considers a minimization problem where the objective is the sup-norm of a linear mapping with linear constraints. The methods presented are based on the concepts and operations of the adaptive method of linear programming with a new variant support matrix called coor-

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dinator support but at the same time specific features of Min-Max problems will be taken into consideration. Here the optimality and the suboptimality criteria are formulated. The iteration from an interior point allows to obtain an ε-optimal solution with a precision $\varepsilon \geq 0$ chosen in advance. Results of numerical experiments are presented which include a pro- gram of this approach.

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Key Words and Phrases: Linear programming, Min-Max problems, Adaptive method.

1 Introduction

Many problems arising from industry can be modeled as mathematical programs, requiring the maximization or minimization of multi-objective function subject to constraints in the form of equalities or inequalities. Operations research techniques have been developed to deal with some of these problems (see [5], [19], [25], [27], [30]).
Nowadays the theory of Min-Max problems using L_∞ norm is one of the most developed classes of nonsmooth optimization (see [3], [4], [14], [26]).
The Min-Max absolute residual procedure was first proposed by Laplace in 1786 and developed over the next 40 years by Prony, Cauchy, Fourier, and Laplace himself. More recent contributions to this traditional literature include those of De la Vallée Poussin and Stiefel [14].
There exit many successful numerical methods for solving non-differentiable optimization problems that can be applied to the linear Min-Max problems under consideration for example sub-gradient methods (see [1], [2], [3], [7], [18], [26]).
The traditional approach to solve linear Min-Max problems is to formulate an equivalent linear programming problem and to solve it by structure exploiting modifications of primal or dual simplex method of Dantzig in 1949 (see [4], [12], [31]) and variants of this

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approach is still developed (see [4], [10], [24]).
Gabasov and Kirillova have generalized the simplex method in 1995 (see [15], [18], [20]), developed the adaptive method and later, its modification (see [7], [9], [23]) for linear programming problems given in [16]. The principe adaptive methods have been used for the solution of optimal control problems discrete and continuous, as well as for the construction of feedback controls (see [11], [17], [25]). It is an interior point method which can start from any feasible solution and any support in order to get an optimal solution (see [6], [7], [9], [13], [21], [22], [28], [29]), also it can stop the solution process according to suboptimality criterion and obtain an ε-optimal solution with a precision $\varepsilon > 0$ chosen in advance.
This paper is organized as follows: In section 2, we present the statement of the problem to solve a Min-Max problem with absolute value functions. In section 3, we construct a new concepts of the support matrix for the problem. The proofs for the theorem of the optimality and suboptimality criterion will be presented in section 4, 5 and 6. An algorithm allows to solve directly the considered problem, without modifying it and avoids the drawbacks of the increase in the number of the variables and the constraints in section 7, thus, improve the convergence speed of the method. Results of numerical experiments are presented which include a program of this approach.

2 Statement of the problem

We Consider the following Min-max problem with linear constraints:

$$\begin{aligned} &Minimize \quad f(x) = \max_{l \in L} |c_l\, x + d_l| \,, \\ &\text{s.t.} \qquad Ax = b \qquad\qquad (2.1) \\ &\qquad\qquad d_* \le x \le d^* \end{aligned}$$

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Where $A = A(I,J) = \begin{pmatrix} a_i^t\ (J)\ , \\ i \in I \end{pmatrix}$ is an $m \times n$ −matrix with
$rang\ A = m \leq n$, $b = b(I) = (b_i, i \in I) \in \mathrm{R}^m$, $I = \{1, 2, ..., m\}$,

$x = x(J) = (x_j\ , j \in J) \in \mathrm{R}^n$, $d_* = d_*(J) = (d_{*j}\ , j \in J) \in \mathrm{R}^n$,

$d^* = d^*(J) = (d^*_j\ , j \in J) \in \mathrm{R}^n$, $J = \{1, 2, ..., n\}$,

$c_l(J) = (c_{lj}\ , j \in J) \in \mathrm{R}^n$, $d_l \in \mathrm{R}$, $l \in L = \{1, 2, ... , l^*\}$.

Definition 1. Any n-vector x satisfying the general system $A\ x = b$ and the box $d_* \leq x \leq d^*$ constraints is called a feasible solution of problem (2.1).
We denote the set of feasible solutions by X .

Definition 2. A feasible solution x^0 is optimal in problem (2.1) if $f(x^0) \leq f(x)$ for all $x \in X$.
For a given $\epsilon > 0$, x^ϵ is a suboptimal ($\epsilon-$optimal) feasible solution if $f(x^\epsilon) - f(x) \leq \epsilon$.

One of the classical solution method for problem of type (2.1) consists in reformulation them as linear programming problems and subsequent application of any linear programming method (for example, simplex method).
In the present paper we replace problem (2.1) by the following linear programming problem:

$$\begin{aligned} &Minimize \quad x_0\ , \\ s.t. \quad &- x_0 \leq c_l^t x + d_l \leq x_0\ , \quad l \in L\ ; \qquad (2.2) \\ &a_i^t x = b_i\ , \ i \in I\ ; \\ &d_* \leq x \leq d^*\ . \end{aligned}$$

In this sequel we will assume that $x_0 = f(x) = \max_{l \in L} | c_l^t\ x + d_l |$.
If x^0 is a solution of problem (2.1), then $(x^0, x_0^0 = \max_{l \in L} | c_l^t x^0 + d_l |)$ is a solution of problem (2.2). And vice versa (x^0, x_0^0) being a solution of problem (2.2), x^0 is a solution of problem (2.1) and

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$f(x^0) = x_0^0$, then problems (2.1) and (2.2) are equivalent.

It is clear that we can apply any method of linear programming to problem (2.2). However in this case, usually the specific structure of the constraints derived from the cost function of problem (2.1) can not be taken into account. Our aim is to extend the adaptive method [14] for problem (2.2).
The main structure of the suggested method of investigating and solving problem is the support.

3 Support

Let us discuss in more detail constructions related to the support.
Let $J_{sup} \subset J$ be an arbitrary subset of J and $\overline{J_{sup}} = \{J_{sup}\} \cup \{0\}, J_n = J \backslash J_{sup}$.

Let $L_{sup} \subset L$ be an arbitrary subset of L and the cardinalities of these sets satisfy the equality $|L_{sup}| + |I| = |J_{sup}| + 1$, $L_n = L \backslash L_{sup}$.

We again partition L_{sup} in two subsets L_{sup}^{+} and L_{sup}^{-} with $L_{sup} = L_{sup}^{+} \cup L_{sup}^{-}$ and $L_{sup}^{+} \cap L_{sup}^{-} = \emptyset$. Let $e(L) = e_l = 1, l \in L)$.

Now we define the block matrix $B_{sup} = B(L_{sup} \cup I, \overline{J_{sup}})$ as follows:

$$\mathrm{B}_{\mathrm{sup}} = \begin{pmatrix} \begin{matrix} e(L_{sup}^{-}) \\ -e(L_{sup}^{+}) \end{matrix} & \begin{matrix} c_l^t(J_{sup}), \\ l \in L_{sup} \end{matrix} \\ & \\ 0(I) & \begin{matrix} a_i^t(J_{sup}), \\ i \in I \end{matrix} \end{pmatrix}$$

Definition 3. A vectors $u(L) = (u(L_{sup}), u(L_n))$, $\pi(I)$ defined as follows:

$$(u(L_{sup}), \pi(I))^t = c_0^t\,(\overline{J_{sup}})\, B_{sup}^{-1}\ ,\ u(L_n) \equiv 0\ , c_0 = (c_{00} = -1, c_{0j} = 0\ ,\ j \in J_{sup}),\ (3.1)$$

are called a vectors of multipliers.

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Definition 4. A pair $K_{sup} = \{\overline{L_{sup}}\ ,\ J_{sup}\}$ is called a support of problem (2.1) if B_{sup} is nonsingular and the following relations hold the vector of multipliers:

$$u(L^{-}_{sup}) \leq 0 \ \ , u\,(L^{+}_{sup}) \geq 0$$

Definition 5. A pair $\{x, K_{sup}\}$ of a feasible solution x of problem (2.1) and support K_{sup} is called a support feasible solution.

Definition 6. A support feasible solution $\{x, K_{sup}\}$ is called primal nondegenerate if:

$$d_{*j} < x_j < d^{*}_{j}\ ,\ j \in J_{sup}\ ;$$
$$|c^{t}_{l}\ x + d_l| < f(x)\ ,\ l \in L_n\ .$$

Definition 7. A support K_{sup} is said to be coordinated with a feasible solution x if:

$$L^{+}_{sup} \subseteq L^{+}(x) = \{l \in L : c^{t}_{l}\ x + d_l = f(x)\}\ ,$$
$$L^{-}_{sup} \subseteq L^{-}(x) = \{l \in L : c^{t}_{l}\ x + d_l = -f(x)\}.$$

Remark: Note that the pair

$$k_{sup} = \{L_{sup}, \overline{J_{sup}}\}\ \ ,\quad J_{sup} = \emptyset\ \ ,\quad L_{sup} = l^{*} \in L\ , \tag{3.2}$$

is a support of problem (2.1) with $u(L_{sup}) = 1$ if $L^{+}{}_{sup} = l_{*}$ and $u(L_{sup}) = -1$ otherwise.

Let $C(L_{sup}, j) = \begin{pmatrix} c_{lj}, \\ l \in L_{sup} \end{pmatrix}$ be a column-vector composed of the components c_{lj} , $j \in J$, of the vectors c_l , $l \in L_{sup}$.

Using a support we can calculate the vector of reduced cost for problem (2.1) by the rules:

$$\Delta_j = u^{t}(L_{sup})\ C(L_{sup}, j) + \pi^{t}(I)\ A(I, j)\ ,\quad j \in J_n = J \backslash J_{sup} \tag{3.3}$$

Note that by construction, we have :

$$\sum_{l \in L_{sup}} |u_l| = 1 \qquad ; \qquad \Delta_j = 0\ , \qquad j \in J_{sup} \tag{3.4}$$

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4 Optimality criterion.

Let x be a feasible solution of problem (2.1), K_{sup} be a support coordinated with x . This information is enough for checking whether the feasible solution x is optimal.

Theorem 8. *Let x be a feasible solution of problem (2.1), K_{sup} be a support coordinated with x such that for the support feasible solution $\{x, K_{sup}\}$ the following relations:*

$$\begin{aligned} &\Delta_j \geq 0 \quad if \quad x_j = d_{*j} \quad , \\ &\Delta_j \leq 0 \quad if \quad x_j = d_j^* \quad , \\ &\Delta_j = 0 \quad if \quad d_{*j} < x_j < d_j^* \quad , \quad j \in J_n \quad , \end{aligned} \tag{4.1}$$

are true, then x is optimal in problem (2.1) *.*
If x is optimal and the support feasible is primal nondegenerate then relations (4.1) *hold.*

Proof. The proof of the theorem is based on the analysis of the following increment formula:

$$\Delta f(x) = f(\overline{x}) - f(x)$$

of the cost function of the problem (2.1) when passing from the feasible solution to the point $\overline{x} = x + \triangle x$.
To derive the increment formula we consider the equivalent linear programming problem (2.2) and compute $\Delta f(x) = \Delta x_0$.
We start with analyzing the behavior of the constraints from L_{sup} and I .
At the point x we have:

$$\begin{aligned} &c_l^t x - x_0 = -d_l + w_l \;, l \in L_{sup}^+ \;, \\ &c_l^t x + x_0 = -d_l + w_l \;, \; l \in L_{sup}^- \;, \\ &a_i^t x \qquad = b_i + w_i \quad , \quad i \in I \quad , \end{aligned} \tag{4.2}$$

for the residuals $w(L_{sup}) = (w_l, l \in L_{sup})$ and $w(I) = (w_i, i \in I)$.

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For Analogous relations hold at the point $\bar{x}$:

$$\begin{aligned} c_l^t\bar{x} - \bar{x}_0 &= -d_l + \bar{w}_l\ ,\ l \in L_{sup}^+\ ,\\ c_l^t\bar{x} + \bar{x}_0 &= -d_l + \bar{w}_l\ ,\ l \in L_{sup}^-\ ,\\ a_i^t\bar{x} &= b_i + \bar{w}_i\ ,\quad i \in I\ , \end{aligned} \tag{4.3}$$

for the residuals $w = (w(L_{sup})\ ,\ w(I))$ with:
$(\bar{w}(L_{sup}), \bar{w}(I)) = (w(L_{sup}), w(I)) + (\triangle w(L_{sup}), \triangle w(I))$ and
$\bar{w}(I) = w(I) = \triangle w(I) = 0_I$.

From the definitions of support, multipliers, reduced costs, relation (4.2) and (4.3) we obtain that $\triangle x_0$, $\triangle x_j$, $j \in J_{sup}$ are defined by the linear equation system:

$$B_{sup} = \begin{pmatrix} \Delta x_0 \\ \Delta x(J_{sup}) \end{pmatrix} = -\left(\begin{pmatrix} C(L_{sup}, J_n) \\ A(I, J_n) \end{pmatrix} \Delta x(J_n) + \begin{pmatrix} \Delta w(L_{sup}) \\ \Delta w(I) \end{pmatrix} \right)$$

Thus we obtain:

$$\begin{pmatrix} \Delta x_0 \\ \Delta x(J_{sup}) \end{pmatrix} = -B_{sup}^{-1} \left(\begin{pmatrix} C(L_{sup}, J_n) \\ A(I, J_n) \end{pmatrix} \Delta x(J_n) + \begin{pmatrix} \Delta w(L_{sup}) \\ 0(I) \end{pmatrix} \right), \tag{4.4}$$

for any choice of $\triangle x\ (J_n)$ and $\triangle w\ (L_{sup})$.

The increment of the cost function associated to the problem (2.2)

is given by(2.2):

$$\Delta\, x_0 = \sum_{j \in J_n} \Delta_j\ \ \Delta x_j - \sum_{l \in L_{sup}} \Delta_j \tag{4.5}$$

Let x be a feasible solution of problem(2.1), K_{sup} be a support coordinated with x such that for the support feasible solution $\{x, K_{sup}\}$ the relations (4.1) take place.

Then for any feasible solution $\bar{x}$ we have:

$$\begin{cases} \Delta x_j \geq 0 & if \quad \Delta_j \geq 0\ , \\ \Delta x_j \leq 0 & if \quad \Delta_j \leq 0\ , \\ \Delta w_l \leq 0 & if \quad l \in L_{sup}^+\ , \\ \Delta w_l \geq 0 & if \quad l \in L_{sup}^-\ . \end{cases} \tag{4.6}$$

Taking into account the definition of the support, relations (4.5) and(4.6) lead to the inequality: $\Delta\, x_0 \geq 0,$

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which is valid for any feasible $\overline{x} = x + \Delta x$, that proves the first part of the theorem showing that x is optimal in problem (2.1) .

Let us prove the second part of the theorem.

Assume that x is optimal and that the support feasible $\{x , K_{sup}\}$ is primal nondegenerate but relations (4.1) are not true on some index $j_0 \in J_n$.

We assume $\Delta x_j = 0 , j \in J_n \setminus j_0$, $\Delta x_{j_0} = -sign\Delta_{j_0}$, $\Delta w_l =$ 0,
$l \in L_{sup}$, such that $\Delta_{j_0} \neq 0$.

The components $\Delta x_j , j \in J_{sup}$ and the number Δx_0 are defined by the linear equation system (4.4) .

Since $\{x , K_{sup}\}$ is primal nondegenerate , there exists a step-size $\vartheta > 0$ such that $\overline{x} = x + \vartheta \Delta x$ is feasible in problem (2.1) and

$$f(\overline{x}) = f(x) + \vartheta \Delta x_{j_0} = f(x) - \vartheta |\Delta j_0| < f(x).$$

The last inequality contradicts to the optimality of x .

□

We proceed with introducing an estimate for checking $\epsilon-$optimality of support feasible solution.

Definition 9. The number $\beta(x , K_{sup})$ defined by:

$$\mathrm{B}(x,K_{sup}) = \sum_{\Delta_{j>0, j\in J_n}} \Delta_j (x_j - d_{*j}) + \sum_{\Delta_{j<0, j\in J_n}} \Delta_j (x_j - d_j^*)$$

$$+ \sum_{l\in L^+_{sup}} u_l (f(x) - c_l^t x - d_l) + \sum_{l \in L^-_{sup}} u_l (-f(x) - c_l^t x - d_l) \quad (4.7)$$

is called a suboptimalty estimate of the support feasible solution $\{x, K_{sup} \}$.

Remark: The suboptimality estimate can be presented as follows:

$$\mathrm{B}(x, Ksup) = \mathrm{B}(x) + \mathrm{B}(Ksup) , \qquad (4.8)$$

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where: $B(x) = f(x) - f(x^0)$ is the measure of nonoptimality of a feasible solution calculated with the use of the optimal feasible solution x^0, and: $B(K_{sup}) = \phi(\lambda^0) - \phi(\lambda_{acc})$ is the measure of nonoptimality of a support calculated according to the values of the cost function $\phi(\lambda)$ of the dual problem at the optimal dual feasible solution λ^0 and at the special dual feasible solution λ_{acc} accompanying the support K_{sup}, which will be introduced in the next section.

5 Dual problem

The problem dual to (2.1) has the form:

$$\text{Maximize} \quad \phi(\lambda) = d^t \gamma - b^t \mathrm{y} + d_*^t v - d^{*t} w \quad ,$$

$$s.t. \quad C^t \gamma + A^t \mathrm{y} - v + w = 0 \, , \tag{5.1}$$

$$v \geq 0, \quad w \geq 0 \, , \sum_{l \in L} |\gamma| = 1 \, ,$$

where $\mathrm{C} = \mathrm{C}(\mathrm{L},\mathrm{J}) = \begin{pmatrix} \mathrm{c}_\mathrm{l}^\mathrm{t}(\mathrm{J}), \\ \mathrm{l} \in \mathrm{L} \end{pmatrix}$ is an $l^* \times n$ −matrix.

A vector $\lambda = (\gamma, \mathrm{y}, v, w)$ is called a dual feasible solution if it satisfies all the constraints of problem (5.1).

With respect to any support K_{sup} of problem (2.1) we can compute the vector $\lambda_{acc} = (u(L), \pi(I), v, w)$ with the components v , w :

$$\begin{aligned} v_j &= \Delta_j \, , w_j = 0 \quad , \text{si } \Delta_j \geq 0 \, , \\ v_j &= 0 \quad , w_j = -\Delta_j \, , \text{si } \Delta_j < 0 \, , \; j \in J \, , \end{aligned} \tag{5.2}$$

accompanies the support K_{sup} .

Since the constructed vector λ_{acc} satisfies all the constraints of problem (5.1) it is called the dual feasible solution accompanying the support K_{sup} (Briefly, the accompanying dual feasible solution).

The decomposition formula (4.8) follows from the definitions of

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multipliers, reduced costs, relations (3.4) , (5.2) and the following chain of equalities:

$$\mathrm{B}(x, Ksup) = \sum_{\Delta_{j>0, j\in J_n}} \Delta_j\ (x_j - d_{*j}) + \sum_{\Delta_{j<0, j\in J_n}} \Delta_j\ (x_j - d_j^*) +$$

$$+ \sum_{l\in L_{sup}^{+}} u_l\ (f(x) - c_l^t x - d_l) + \sum_{l\ \in L_{sup}^{-}} u_l\ (-f(x) - c_l^t x - d_l)$$

$$= \Delta^t x - v^t d_* + w^t d^* + f(x) - \Delta^t x - u^t d$$

$$= f(x) - u^t d - v^t d_* + w^t d^* = f(x) - \phi(\lambda_{acc})$$

$$= f(x) - f(x^0) + \phi(\lambda^0) - \phi(\lambda_{acc}) = B(x) + B(K_{sup}).$$

The suboptimality estimate $B(x, K_{sup})$ characterizes the duality gap of the feasible solution x .

6 Suboptimality criterion

Let us prove the following theorem.

Theorem 10. *For any $\varepsilon \geq 0$ a feasible solution x of problem* (2.1) *is ε-optimal if and only if there exists a support K_{sup} such that the suboptimality estimate $B(x, K_{sup})$ of the support feasible solution $\{x, K_{sup}\}$ satisfies the following inequality :*

$$B(x, K_{sup}) \leq \epsilon.$$

Proof. The proof of the necessary condition is based on the decomposition (4.8) of the suboptimality estimate $B(x, K_{sup})$.

Let x be an $\epsilon-$optimal solution of problem (2.1), ie $f(x) - f(x^0) \leq \epsilon$. If we take a support $\tilde{K}^0_{sup}$ that accompanies an optimal dual feasible solution $\tilde{\lambda}^0$, we obtain $B(\tilde{K}^0_{sup}) = 0$ then:

$$B(x, \tilde{K}^0_{sup}) = f(x) - f(x^0) \leq \epsilon\ .$$

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To finish the proof of the first part of the theorem we show that we can construct a support K^0_{sup}, accmpanying some optimal dual feasible solution $\tilde{\lambda}^0$.

Let $\lambda^0 = (\gamma^0, y^0, v^0, w^0)$ be an optimal dual feasible solution and we denote:

$$L_0 = \{l \in L : \gamma^0_l = 0\}, \; L_{sup} = L \setminus L_0,$$

$$L^+_{sup} = \{l \in L : \gamma^0 > 0\}, \qquad L^-_{sup} = L_{sup} \setminus L^+_{sup},$$

$$J_0 = \{j \in J : \Delta^0_j = C^t(L, j)\, \gamma^0 = 0\}, \; J_n = J \setminus J_0.$$

Assume that:

$$rank \begin{pmatrix} e(L^-_{sup}) & c^t_l(J_0) \\ -e(L^+_{sup}) & l \in L_{sup} \\ & \\ 0(I) & a^t_i(J_0) \\ & i \in I \end{pmatrix} = |L_{sup}| + |I|. \tag{6.1}$$

Let us select the subset J_{sup} of the J_0, which contains the maximal number of elements and corresponds to the linearly independent vectors:

$$(\, e(L^-_{sup}),\ -e(L^+_{sup}),\ O(I)\,)^t, \quad (\, C(L_{sup}, j\,),\ A(I, j)\,)^t,\ j \in J_{sup}.$$

. If $|J_{sup}| + 1 = |L_{sup}| + |I|$: then the pair $\tilde{K}^0_{sup} = \{L_{sup}, J_{sup}\}$

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is a support and it accompanies the optimal dual feasible solution $\tilde{\lambda}^0 = \lambda^0$.

. If $|J_{sup}| + 1 < |L_{sup}| + |I|$:

In view of the linear dependence of the vectors:

$$(e(L^-_{sup}), -e(L^+_{sup}), 0(I))^t, (C(L_{sup}, j), A(I, j))^t, j \in J_0,$$

there exists a nonzero vector $\Delta\gamma(L_{sup})$ such that:

$$\Delta\gamma^t(L_{sup}) C(L_{sup}, J_0) = 0, \quad e^t(L^+_{sup}) \Delta\gamma^t(L^+_{sup}) - e^t(L^-_{sup}) \Delta\gamma^t(L^-_{sup}) = 0.$$

On the other hand according to (6.1) :

$$\Delta\gamma^t(L_{sup}) C(L_{sup}, J_n) \neq 0$$

Let $\Delta\gamma^t(L_0) = 0$, then we have:

$$\delta(J_0) = \Delta\gamma^t(L_{sup}) C(L_{sup}, J_0) = 0,$$

$$\Delta\gamma^t(L_{sup}) C(L_{sup}, J_n) \neq 0_I. \tag{6.2}$$

We calculate $\sigma_0 = min\{\sigma_{l_0}, \sigma_{j_0}\}$, where:

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$$\sigma_{l_0} = \min_{l \in L_{sup}} \sigma_l \ , \sigma_l = \begin{cases} -\frac{\gamma_l^0}{\Delta\gamma_l} & if \ \Delta\gamma_l < 0 \ , \quad l \in L_{sup}^{+} \ , \\ -\frac{\gamma_l^0}{\Delta\gamma_l} & if \ \Delta\gamma_l > 0 \ , \quad l \in L_{sup}^{-} \ , \\ \infty & in \ other \ case \ , \quad l \in L_{sup} \ , \end{cases} \tag{6.3}$$

$$\sigma_{j_0} = \min_{j \in J_n} \sigma_j \ , \qquad \sigma_j = \begin{cases} -\dfrac{\Delta_j^0}{\delta_j} & if \quad \Delta_j^0 \ \delta_j < 0 \ , \\ \infty & in \ other \ case \ , \quad j \in J_n \ . \end{cases}$$

δ is calculated according to (6.2) .

Let us compute:

$$\bar{\gamma}^0 = \gamma^0 + \sigma_0 \, \Delta\gamma \quad ,$$
$$\bar{\Delta}^0 = \Delta^0 + \sigma_0 \, \delta \quad .$$

Note that the new dual feasible solution $\bar{\lambda}^0$ is also optimal in the dual problem (5.1) .

. If $\sigma_0 = \sigma_{l_0}$: Assume $\bar{L}_{sup} = L_{sup} \backslash l_0$ and calculate the corresponding set J_{sup} (according to the described rule).

. If $\sigma_0 = \sigma_{j_0}$: then $\bar{L}_{sup} = L_{sup}$, $\bar{J}_0 = J_0 \cup j_0$, $\bar{J}_{sup} = J_{sup} \cup j_0$.

Since $\delta_{j_0} \neq 0$, the vectors $(e(\overline{L_{sup}^{-}}) , -e(\overline{L_{sup}^{+}}), 0(I))^t \, (C(\overline{L_{sup}} , j) , A(I , j))^t$, $j \in \overline{J_{sup}}$, are linearly independent.

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Now we consider the case where relation (6.1) is false, that means

$$rank\begin{pmatrix} e(L^-_{sup}) & c^t_l(J_0) \\ -e(L^+_{sup}) & l \in L_{sup} \\ & \\ 0(I) & a^t_i(J_0) \\ & i \in I \end{pmatrix} < |L_{sup}| + |I| .$$

Then there exists a nonzero $\Delta\gamma(L_{sup})$ such that:

$\Delta\gamma^t(L_{sup})\ C(L_{sup}, J_0) = 0\ ,\ \ e^t(L^+_{sup})\ \Delta\gamma^t(L^+_{sup}) - e^t(L^-_{sup})\ \Delta\gamma^t(L^-_{sup}) = 0.$

Assuming that $\Delta\gamma^t(L_0) = 0$, we have

$$\delta(J_0) = \Delta\gamma^t(L)\ C(L, J_0) \equiv 0.$$

Using (6.3) we calculate σ_{l_0} and we construct the new sets
$\overline{L}_0 = L_0 \cup l_0\ ,\ \ \overline{L}_{sup} = L_{sup} \backslash l_0$.
The corresponding new dual feasible solution $\overline{\lambda^0}$ with
$\overline{\gamma^0} = \gamma^0 + \sigma_{l_0}\ \Delta\gamma$ and $\overline{\Delta^0} = \Delta^0$ is optimal in (5.1).

Let us show that the process of constructing the support $\tilde{K}^0_{sup}$ is finite. The cardinality of the set L_{sup} does not increase. Thus two cases are possible:

1. At some iteration we have $L_{sup} = \emptyset$: That corresponds to constructing the empty support $\tilde{K}^0_{sup}$.
2. The structure of the set L_{sup} is fixed after a finite number of iterations. In this case we conduct the process of increasing the set J_{sup} until the support $\tilde{K}^0_{sup} = \{L_{sup}, J_{sup}\}$, $|L_{sup}| + |I| = |J_{sup}| + 1$ is built.

The constructed support $\tilde{K}^0_{sup}$ accompanies an optimal dual feasible solution $\tilde{\lambda}^0$.

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Sufficiency : The sufficient relation follows from the assumptions of the theorem 10 and the following chain of inequalities :

$$f(x+\Delta x)-f(x)=\ \Delta x_0=\sum_{\Delta_{j\in J_n}}\Delta_j\ \Delta x_j+\sum_{l\in L_{sup}}u_l\ \Delta w_l\ \geq$$

$$\geq\sum_{j\in J_n}\min_{d_{*j}-x_j\leq\Delta x_j\leq d_j^*-x_j}\Delta_j\ \Delta x_j+\sum_{l\in L_{sup}^+}\max_{\Delta w_l\leq c_l^t x+d_l-f(x)}\ u_l\Delta w_l\ +$$

$$+\sum_{l\in L_{sup}^-}\max_{\Delta w_l\geq c_l^t x+d_l+f(x)}\ u_l\Delta w_l=-B(x,K_{sup})\geq-\varepsilon$$

which is valid for any $\overline{x}=x+\Delta x$ and consequently for $\overline{x}=x^0$.

□

From this theorem 10 we get:

Corollary 11. *A feasible solution x is optimal in problem* (2.1) *if and only if there exists a support K_{sup} such that the suboptimality estimate of the support feasible solution $\{x, K_{sup}\}$ is equal to zero :*

$$B(x, K_{sup}) = 0\ . \qquad (6.4)$$

Remark: Equality (6.4) is true if and only if relations (4.1) hold and the support K_{sup} is coordinated with x .

7 Iteration of the adaptive method

Let us describe an iterative method for solving problem (2.1). The aim of the method is to construct an $\epsilon-$optimal feasible solution x^ϵ for a given $\epsilon\geq 0$.

The method starts from a support feasible solution $\{x, K_{sup}\}$ with a support coordinated with x . Check at $\{x, K_{sup}\}$ the suboptimality criterion and assume that $B(x, K_{sup}) > \epsilon$.

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The step $\{x, K_{sup}\} \rightarrow \{\overline{x}, \overline{K_{sup}}\}$ from the old support feasible solution $\{x, K_{sup}\}$ to a new one $\{\overline{x}, \overline{K_{sup}}\}$ is said to be an iteration. The iteration is based on the principle of decreasing suboptimality estimate (4.7) :

$$B(\overline{x}, \overline{K_{sup}}) \leq B(x, K_{sup}) .$$

It follows from (4.8) that the suboptimality estimate can be decreased, firstly, by decrease of the measure of the feasible solution x nonoptimality by changing x and, secondly, by decrease of the measure of the support K_{sup} nonoptimality by changing K_{sup} .

7.1 Changing x ($x \rightarrow \overline{x}$)

The first procedure starts with the constructing of a pseudo solution accompanying the support K_{sup} by the following rules :

$$\kappa_0 = \begin{cases} d_{*j} & if \quad \Delta_j \geq 0 , \\ d_j^* & if \quad \Delta_j < 0 \ , \ j \in J_n , \end{cases}$$

$$\begin{pmatrix} \kappa_0 \\ \kappa(J_{sup}) \end{pmatrix} = -B_{sup}^{-1}\left(\begin{pmatrix} C(L_{sup}, J_n) \\ A(I, J_n) \end{pmatrix} x(J_n) + \begin{pmatrix} d(L_{sup}) \\ -b(I) \end{pmatrix}\right)$$

a/ It is obvious that if the relations:

$$\begin{aligned} &d_{*j} \leq \kappa_j \leq d^*_j \ , \ j \in J_{sup} \ ; \\ &|c_l^t \kappa + d_l| \leq \kappa_0 \ , \quad l \in L_n \ ; \qquad (7.1) \\ &\kappa_0 \geq 0 \quad \text{if} \quad L_n = \emptyset \ , \end{aligned}$$

are true then κ is optimal and $f(\kappa) = \kappa_0$.

b/ If relations (7.1) are false then we construct the direction $q \in R^n$ of improving feasible solution x as follows:

$$q = \kappa - x .$$

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The cost function of problem (2.1) decreases along the direction q because:

$$\frac{\partial f(x)}{\partial q} = -B(x, K_{sup}) < -\varepsilon$$

Therefore, according to the principle of decreasing nonoptimality , we construct a new feasible solution as follows:

$$\bar{x} = x + \vartheta q ,$$

where the step length ϑ is the greatest step allowed by the constraints of problem (2.1) given by:

$$\vartheta = \min\{1, \vartheta_{j_0}, \vartheta_f\} ,$$

where:
. 1 is the step allowed by the nonsupport box constraints .
. ϑ_{j_0} , $j_0 \in J_{sup}$ is the step allowed by the support box constraints.
. ϑ_f is the step defined by the constraints derived from the cost function of problem (2.1).
The various steps are obtained by the following relations:

$$\theta j_0 == \min \theta_j , j \in J_{sup} , \theta_j = \begin{cases} (d_{*j} - x_j)/q_j & \text{if } q_j < 0 \\ (d_j^* - x_j)/q_j & \text{if } q_j > 0 \\ \infty & \text{if } q_j = 0, j \in J_{sup}. \end{cases}$$

$$\theta_f = \min\{\theta_0, \theta_{l_0}\} , \theta_0 = \frac{f(x)}{B(x, K_{sup})} , \theta_{l_0} = \min\{\theta_l^+, \theta_l^-\}, l \in L_n ,$$

$$\theta_l^+ = \begin{cases} (-c_l^t - d_l + x_0)/(c_l^t q + B(x, K_{sup})) & \text{if } c_l^t q + B(x, K_{sup}) > 0 , \\ \infty & \text{else} \quad , \quad l \in L_n , \end{cases}$$

$$\theta_l^- = \begin{cases} (-c_l^t - d_l - x_0)/(c_l^t q - B(x, K_{sup})) & \text{if } c_l^t q - B(x, K_{sup}) < 0 , \\ \infty & \text{else} \quad , \quad l \in L_n , \end{cases}$$

The suboptimality estimate of the support feasible solution $\{\bar{x}, K_{sup}\}$ is equal to :

$$\mathrm{B}(x, Ksup) = \sum_{\Delta_{j>0, j \in J_n}} \Delta_j \, (x_j - d_{*j}) + \sum_{\Delta_{j<0, j \in J_n}} \Delta_j \, (x_j - d_j^*)$$

$$-\sum_{l \in L_{sup}} u_l \, w_l = \mathrm{B}(x, Ksup) + \theta \sum_{j \in J_n} \Delta_j \, q_j = (1-\theta) B(x, K_{sup})$$

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If the support feasible solution $\{x, K_{sup}\}$ is primal nondegenerate , then $\vartheta > 0$ consequently:

$$B(\overline{x}, K_{sup}) < B(x, K_{sup}).$$

We compute ϑ , we have the various following cases:

.First case : $\vartheta = \vartheta_0$, then $\overline{x} = x + \vartheta q$ is optimal and $f(\overline{x}) = 0$ or $\overline{x} = x + \vartheta q$ is ϵ-optimal with $f(\overline{x}) \leq \epsilon$.

.Second case : $\vartheta = 1$, then $\overline{x} = \boldsymbol{\kappa}$ is optimal .

.Third case : $(1 - \vartheta)\, B(x, K_{sup}) > \epsilon$, then we past to the second procedure change of the support K_{sup} to $\overline{K_{sup}}$ decreasing the mesure of the support nonoptimality .

7.2 Changing K_{sup} ($K_{sup} \to \overline{K_{sup}}$)

The change of support $K_{sup} \to \overline{K_{sup}}$ involves the change of the dual feasible solution $\lambda \to \overline{\lambda}$, i.e:

$$\underline{\overline{u}}(L) = u(L) + \sigma_0 t(L)\ ;$$
$$\Delta(J) = \Delta(J) + \sigma_0 t(J)\ ,$$

where the vector $t(J, L)$ is the direction of the dual cost function increase and σ_0 is the greatest step along this direction .

This change of the support is achieved according to the value of ϑ. Hence there are two distinct cases:

a/ $\vartheta = \vartheta_{j_0}$, $j_0 \in J_{sup}$:

Let:

$$\mu_* = \begin{cases} d_{*j_0} - \kappa_{j_0} & \text{if} \quad \kappa_{j_0} < d_{*j_0}\ , \\ d^*_{j_0} - \kappa_{j_0} & \text{if} \quad \kappa_{j_0} > d^*_{j_0}\ . \end{cases}$$

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and assume that:

$$\begin{cases} t_{j_0} = sign(\mu_*) & , \\ t(J_{sup} \backslash j_0) = 0 & , \\ t(L_n) = 0 & , \end{cases}$$

from the admissibility of $\overline{u}$, $\bar{\Delta}$ we obtain:

$$(t(L_{sup}), t(I)) = (0, t(J_{sup}))^t \ B_{sup}^{-1} \quad ,$$
$$t^t(J_n) = t^t(L_{sup}) \ C(L_{sup}, J_n) + t^t(I) \ A(I, J_n) \ .$$

We calculate the steps σ_j , $j \in J_n$; σ_k , $k \in L_{sup}$:

$$\sigma_j = \begin{cases} -\Delta_j / t_j & for \quad \Delta_j \ t_j < 0 \\ 0 \quad for \quad \Delta_j = 0, t_j > 0, \kappa_j \neq d_{*j} \ or \ \Delta_j = 0, t_j > 0, \kappa_j \neq d_j^* \\ \infty \quad in \ other \ cases & , k \in L_{sup} \end{cases}$$

$$\sigma_k = \begin{cases} -u_k / t_k \quad for \quad t_k < 0 \ , k \in L_{sup}^+ & , \\ -u_k / t_k \quad for \quad t_k > 0 \ , k \in L_{sup}^- & , \\ \infty \quad in \ other \ cases \quad , \quad k \in L_{sup} & . \end{cases}$$

Let the finite values σ_j , $j \in J_n$; σ_k , $k \in L_{sup}$ in to nondecreasing order:

$$\sigma_{j_1} \leq \sigma_{j_2} \leq \ldots \leq \sigma_{j_p} \ , \qquad j_k \in \{J_n \cup L_{sup}\} \ .$$

For each j_k , we compute the jump of the dual objective function:

$$\triangle \mu_k = -|t_{j_k}|(d^*_{j_k} - d_{*j_k}) \ .$$

The dual cost function behaves along the direction t as a concave continuous piece-wise-linear function . Its slope in the interval $[\sigma_k, \sigma_{k+1}]$ is equal to:

$$\mu_k = \mu_{k-1} - |t_{j_k}|(d^*_{j_k} - d_{*j_k}) \ , \quad k = 1, \ldots, \bar{p} \ ;$$
$$\mu_0 = |\mu_*| \ ,$$

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where $\bar{p} \leq p$ is a number such that:

$$\begin{aligned} &\bar{p} = p \qquad \text{if } j \in J \quad , \quad \forall i = \overline{1, p}\,; \\ &or \\ &\bar{p} < p \qquad \text{if } j_{\bar{p}+1} \in L_{sup}\,. \end{aligned}$$

Remark: μ_0 is the initial speed of change of the dual cost function .

We move along the direction t until the dual cost function is increasing. Following the principle of full relaxation of the dual cost function we find the step σ_0 and the index s_0 . Two cases are possible ((1) $s_0 \in J_n$, (2) $s_0 \in L_{sup}$.)

1- If $\mu_{\bar{p}} > 0$, then it is necessary that $\bar{p} < p$. Denote $\sigma_0 = \sigma_{j_{\bar{p}+1}}$, $s_0 = j_{\bar{p}+1}$, $s = \bar{p} + 1$.

2- If $\mu_{\bar{p}} \leq 0$, we find an index v , $0 < v \leq \bar{p}$ such that $\mu_{v-1} > 0$ and $\mu_v \leq 0$. Assume $\sigma_0 = \sigma_{j_v}$, $s_0 = j_v$, $s = v$.

The iteration ends by building the new multipliers and the reduced costs as follows:

$$\begin{aligned} \overline{\Delta}(J) &= \Delta(J) + \sigma_0\, t(J) \quad ; \\ \bar{u}(L) &= u(L) + \sigma_0\, t(L) \quad . \end{aligned} \tag{7.2}$$

and the new support $\overline{K}_{sup} = \{\bar{\bar{J}}_{sup}, \overline{L}_{sup}\}$: $\bar{\bar{J}}_{sup} = \{0\} \cup \bar{J}_{sup}$,

$\bar{J}_{sup} = J_{sup} \backslash j_0$, $\qquad \overline{J_{sup}} = (J_{sup} \backslash j_0) \cup s_0$,

$L_{sup} = L_{sup} \backslash s_0$, if $s_0 \in L_{sup}$ $\qquad \overline{L}_{sup} = L_{sup}$, if $s_0 \in J_n$.

,

Let us now study the second possible situation that can appear after the first procedure,

b/ $\vartheta = \vartheta_{l_0}$ **,** $l_0 \in L_n$ **:**

Let us compute:

$$\mu_* = \begin{cases} -\kappa_0 + c_{l_0}^t \kappa + d_{l_0} & \text{if } \theta_{l_0} = \theta_{l_0}^+ , \\ \kappa_0 + c_{l_0}^t \kappa + d_{l_0} & \text{if } \theta_{l_0} = \theta_{l_0}^- . \end{cases}$$

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and assume :

$$t = sign\mu \;;$$
$$t(L_n \backslash l_0) \equiv 0\;;$$
$$t(J_{sup}) \equiv 0 \;,$$

and from the admissibility of $\overline{u}$, $\overline{\Delta}$ we obtain:

$$(t(L_{sup}), t(I)) = (1, -t_{l_0}\, c^t_{l_0}\, (J_{sup}))\, B^{-1}_{sup} \;,$$
$$t^t(J_n) = t^t(L_{sup})\; C(L_{sup}, J_n) + \; t_{l_0}\, c^t_{l_0}\; (J_n) + t^t(I)A(I,J_n)$$

We apply the same method used of (a) in order to find the index s_0 and the step σ_0 .

The iteration ends in constructing the reduced costs and the multipliers $\overline{\Delta}(J)$, $\overline{u}(L)$ given by (7.2) and the new support:

$$\overline{K}_{sup} = \{\overline{\overline{J}}_{sup} \;, \overline{L}_{sup}\} \quad , \quad \overline{\overline{J}}_{sup} = \overline{J}_{sup} \cup \{0\}\,,$$

$$\begin{cases} \overline{J}_{sup} = J_{sup} \;, \\ \overline{L}_{sup} = (L_{sup} \backslash s_0) \cup l_0 \;, \;\; if\; s_0 \in L_{sup}\;, \end{cases} , \begin{cases} \overline{J}_{sup} = J_{sup} \cup s_0 \;, \\ \overline{L}_{sup} = L_{sup} \cup l_0\;, \qquad if\; s_0 \in J_n\,. \end{cases}$$

Such that:

$$\begin{cases} l_0 \in \overline{L}^{+}_{sup} \qquad if \qquad \theta_{l_0} = \theta^{+}_{l_0}\;, \\ l_0 \in \overline{L}^{-}_{sup} \qquad if \qquad \theta_{l_0} = \theta^{-}_{l_0}\;. \end{cases}$$

By construction, the new support is coordinated with the feasible solution $\overline{x}$.

The suboptimality estimate of the new support feasible solution $\{\overline{x}\,, \overline{K}_{sup}\}$ is equal to:

$$\mathrm{B}(\overline{x}, \overline{\mathrm{K}}_{\mathrm{sup}}) = (1-\theta)B(x, K_{sup}) - \sum_{k=0}^{s-1} \mu_k\,(\sigma_{j_{k+1}} - \sigma_{j_{k+1}}), \sigma_{j_0} = 0$$

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Example: Let us consider the following Min-Max problem:

$$\max\left(|2x_1 - 2x_2 + x_3|,\ |x_1 + 4x_2 - x_3 - 3|,\ |x_1 - x_2 - x_3 + 2| \right) \rightarrow \min,$$

$$\text{s.t} \qquad \begin{pmatrix} 2 & -1 & 3 \\ -1 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (1)$$

$$-1 \le x_1 \le 3,\quad -1 \le x_2 \le 1,\quad -1 \le x_3 \le 1.$$

First iteration :

$x = (\frac{6}{7}, \frac{5}{7}, 0)$ is a feasible solution of problem of (1) and

$K_{sup} = \{ \overline{J}_{sup}, I, L_{sup} \}$ is the support coordinated with x,

Such that: $\overline{J}_{sup} = J_{sup} \cup \{0\}$, $J_{sup} = \{1,2\}$, $I = \{1,2\}$,

$L_{sup} = \{3\} = L^{+}_{sup}$. $\{x, K_{sup}\}$ is a nondegenerate support

feasible solution. $B(x, K_{sup}) = \frac{31}{14} > 0$, then the support

feasible solution $\{x, K_{sup}\}$ is not optimal.

1. Changing of feasible solution : $x \rightarrow \bar{x} = x + \vartheta q$

The pseudo-plan $\kappa = (\kappa_1, \kappa_2, \kappa_3) = (\frac{-1}{14}, \frac{-13}{14}, \frac{1}{7})$ is not optimal.

$q = (\frac{-25}{14}, \frac{-4}{7}, 1), \theta = \frac{20}{51} = \theta_2^-, 2 \in L_n, then\ \bar{x} = (\frac{56}{357}, \frac{175}{357} \frac{20}{51})$

and $B(\bar{x}, K_{sup}) = \frac{961}{714} < B(x, K_{sup}) = \frac{31}{14}$.

$B(\bar{x}, K_{sup}) > 0$, then $\{\bar{x}, K_{sup}\}$ is not optimal.

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2. Changing of support : $K_{sup} \rightarrow \overline{K_{sup}}$

$\sigma_0 = \frac{31}{102} = \sigma_3$, $3 \in J_n$. *Then the new support is:*

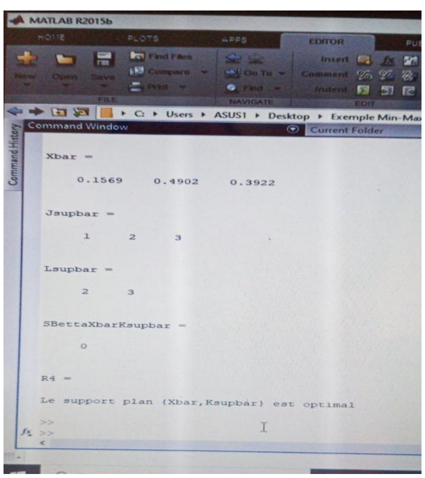
$\overline{K_{sup}} = \{\overline{\overline{J_{sup}}}, \overline{L_{sup}}\}$ such that $\overline{\overline{J_{sup}}} = \overline{J_{sup}} \cup \{0\}$, $\overline{J_{sup}} = \{1, 2, 3\}$;

$I = \{1, 2\}$; $\overline{L}_{sup} = \{2, 3\}$, $2 \in \overline{L}_{sup}^{-}$ and $3 \in \overline{L}_{sup}^{+}$.

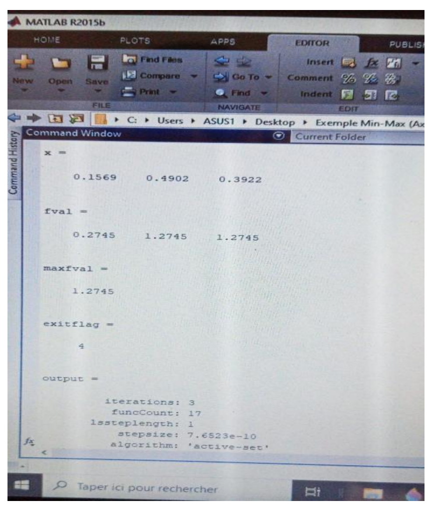
$B(\overline{x}, K_{sup}) = 0$, then the support feasible solution $\{\overline{x}, \overline{K}_{sup}\}$

is optimal, with: $x^0 = \overline{x} = (\frac{56}{357}, \frac{175}{357}, \frac{20}{51})$ and $x_0^0 = f(x^0) = \frac{455}{357}$.

Remark : When using the function "fminimax" command, we obtain the same solution after three iterations, but with our program after one iteration we obtain the support optimal solution as shown in figure 1 .



(a) Adaptive method



(b) fminimax command

Figure 1

8 Conclusion :

In the paper we have used the principles of the adaptive method, at the same time specific features of Min-Max problems have been taken into consideration. After giving some definitions, we have

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constructed the support, then calculated the increment of the objective functional. Optimality and suboptimality criteria have been formulated to describe the iteration of the adaptive method. This method is based on two procedures, one is the change of the feasible solution, the other is the coordinator support. We have implemented it in the programming language Matlab (R 2015 b) and the results are illustrated with a numerical example.

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