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THE CHROMATIC NUMBER OF HOMOGENEOUS FUZZY CATERPILLAR AND ITS RELATED GRAPHS

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Abstract

This paper aims to derive the chromatic number of the homogeneous fuzzy caterpillar $C(n;m)$ and its some related graphs such as middle graph, splitting graph, shadow graph, line graph, total graph and the subdivision graph. Some significant properties of the fuzzy coloring of the fuzzy graphs are derived.

Keywords: Chromatic number, Homogeneous fuzzy caterpillar, Middle graph, Splitting graph, Shadow graph, Line graph, Total graph, Subdivision graph.

1. Introduction

Fuzzy coloring serves as a significant extension of classical graph coloring into the domain of fuzzy graph theory, enabling more flexible modeling and effective problem-solving in systems characterized by uncertainty or imprecise relationships. As numerous practical problems can be formulated as coloring problems, this area has become a central topic of research within graph theory. The concept of coloring of fuzzy graph was initially introduced by Susana Munoz et al.[1], who proposed a method for coloring the vertices of fuzzy graphs with a crisp vertex set and a fuzzy edge set and also defined the chromatic number of a fuzzy graph as a fuzzy subset of its vertex. Later, Eslahchi and Onagh [2] proposed a coloring technique that considers strong adjacency between vertices to determine the chromatic number of a fuzzy graph. Following this, Sovan Samanta et al. [3] introduced fuzzy colors and they developed an alternative approach in which the fuzzy colors are assigned based on the strength of an edges incident to each vertex. This new method offers a more refined and application-oriented way of coloring fuzzy graphs.

In our earlier work, we determined the chromatic number of certain families of fuzzy graphs using fuzzy colors based on the strength of an edge incident to a vertex, and derived various properties of fuzzy coloring [4]. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy coloring is an assignment of basic or fuzzy colors to the vertices of G, and it is a proper coloring,

(i) if two vertices are connected by a *strong* edge, then they either have different basic or fuzzy colors (if necessary), or one vertex can have a basic color and the other can have a

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fuzzy color corresponding to different basic color.

(ii) if two vertices are connected by a *weak* edge, then they either have same or different fuzzy colors, or one vertex can have a basic color and other can have a fuzzy color corresponding to the same basic color. The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the chromatic number of G, is denoted by $\chi_f(G)$. We also computed the chromatic number of some related graphs of fuzzy paths and fuzzy cycles [5],[6]. In the present study, we extend our research by determining the chromatic number of the homogeneous fuzzy caterpillar graph and its some related graphs such as middle graph, splitting graph, shadow graph, line graph, total graph, and subdivision graph. The chromatic number is obtained by using fuzzy colors based on the strength of an edge incident to a vertex. The organization of this paper is as follows. Section 1 presents an overview of fuzzy coloring within the context of fuzzy graph theory. Section 2 outlines the fundamental definitions and preliminary concepts relevant in this study. In Section 3, we focus on determining the chromatic numbers of some related graphs of $C(n;m)$ such as, middle graph, splitting graph, shadow graph, line graph, total graph, and subdivision graph. Section 4 presents the final conclusion of this study.

2. Preliminaries

This section begins with a review of some definitions from the fuzzy graph theory and the fuzzy coloring, which helps to find the chromatic number of homogeneous fuzzy caterpillar and its some related graphs.

Definition 2.1. [7] Let $G = (V, E)$ be a graph. Chromatic number of G is the minimum number of colors needed to color the vertices of G, such that no two adjacent vertices can have the same color (proper coloring) and is denoted by $\chi(G)$.

Definition 2.2. [8] A connected graph $G = (V, E)$ with no cycle is called a tree.

Definition 2.3. [9] A caterpillar graph is a tree having a central path P with n vertices $v_1, v_2, \ldots, v_n$ and m leaf vertices that are attached to every vertex $v_i, 1 \le i \le n$ of the central path P.

Definition 2.4. [10] The homogeneous caterpillar is a caterpillar in which an equal number of leaf vertices are attached to each v_i, for $1 \le i \le n$, and is denoted by $C(n;m)$. The vertex set and edge set of homogeneous caterpillar $C(n;m)$ is given as $V\big(C(n;m)\big) = V \cup U$, where $V = \{v_i : 1 \le i \le n\}$ and $U = \{u_{ij} : 1 \le i \le n, 1 \le j \le m\}$, and $E\big(C(n;m)\big) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_i u_{ij} : 1 \le i \le n, 1 \le j \le m\}$.

Theorem 2.1. [7] *Let G be a tree graph, then $\chi(G) = 2$.*

Theorem 2.2. *Let $C(n;m)$ be a homogeneous caterpillar graph. Since it is a tree [7], by Theorem 2.1, $\chi(C(n;m)) = 2$.*

Definition 2.5. [11] Let X be universe of discourse, then a fuzzy set A in X is a set of ordered pairs : $A = \{\big(x, \mu_A(x)\big) \mid x \in X\}$, where $\mu_A(x): X \to [0,1]$ is called the membership function (generalized characteristic function). i.e., membership function assigns a fuzzy index $\mu_A(x)$ to every member x of a fuzzy set A in the interval of $[0,1]$. Which is often called membership value of x in A.

Definition 2.6. [12] A fuzzy graph $G = (V, \sigma, \mu)$ is a pair of functions (σ, μ), where $\sigma : V \to [0,1]$ is a fuzzy subset of a non-empty set V, and $\mu : V \to [0,1]$ is a symmetric fuzzy relation on σ, such that the relation $\mu(v_i, v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$ is satisfied for all $v_i, v_j \in V$ and $(v_i, v_j) \in E \subset V \times V$.

Here, $\sigma(v_i)$ denote the degree of membership of the vertex v_i, and $\mu(v_i, v_j)$ denotes the degree of membership of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.

Note : In this paper, we denote $\sigma(v_i) \wedge \sigma(v_j) = min\{\sigma(v_i), \sigma(v_j)\}$, and

$\sigma(v_i) \vee \sigma(v_j) = max\{\sigma(v_i), \sigma(v_j)\}$.

Definition 2.7. [1] Let $G = (V, \sigma, \mu)$ be a fuzzy graph and an edge $e = (v_i, v_j) \in G$ is called **strong** if $\frac{1}{2}\{\sigma(v_i) \wedge \sigma(v_j)\} \leq \mu(v_i, v_j)$ and it is called **weak** otherwise.

Definition 2.8. [1] Let $G = (V, \sigma, \mu)$ be a fuzzy graph and the **strength of an edge** $(v_i, v_j) \in G$ is denoted by,

$$I(v_i, v_j) = \frac{\mu(v_i, v_j)}{\sigma(v_i) \wedge \sigma(v_j)}.$$

Definition 2.9. [13] A fuzzy graph $G = (V, \sigma, \mu)$ is called a strong fuzzy graph if each edge in G is a strong edge.

Definition 2.10. [14] Let $G = (V, \sigma, \mu)$ be a fuzzy graph with underlying crisp graph G^*. A fuzzy path P_n in G is a sequence of distinct vertices $v_0, v_1, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0, 1 \leq i \leq n$. Here $n \geq 1$ is called the length of the path P_n.

Definition 2.11. [11] A fuzzy graph $G = (V, \sigma, \mu)$ is a fuzzy star S_n, if there exists only one vertex $v_0 \in V$ such that $\mu(v_0, v_i) > 0$ and $\mu(v_i, v_j) = 0 \ \forall \ v_i, v_j \in V$.

Definition 2.12. [14] A connected fuzzy graph $G = (V, \sigma, \mu)$ with no fuzzy cycle is called a fuzzy tree.

Definition 2 .13. [5] The middle graph $M_f(G)(V_M, \sigma_M, \mu_M)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $M(G)(V_M, E_M)$, with the vertex set $V_M = V \cup V_{ij}$ where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij} \ \forall \ (v_i, v_j) \in E\}$ and $\nu(M_f(G)) = n + 1 + n = 2n + 1$ and the edge set

$$E_M = \begin{cases} (v_{ij}, v_i), (v_{ij}, v_j) & \forall \ i \ and \ j, \\ (v_{ij}, v_{rs}) & if \ the \ edges \ (v_i, v_j) \ and \ (v_r, v_s) \\ & are \ adjacent \ in \ G. \end{cases}$$

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Then, $\sigma_M(v_i) = \sigma(v_i)\ if\ v_i \in V, 0 \leq\ i \leq\ n,$

$\sigma_M(v_{ij}) = \mu(v_i, v_j)\ if\ (v_i, v_j) \in E\ \forall\ \ i\ and\ j,$

$\mu_M(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,

and $\mu_M(v_i, v_{ij}) = \mu_M(v_j, v_{ij}) = \mu(v_i, v_j)\ \forall\ i\ and\ \ j.$

Definition 2.14. [5] The splitting graph $S_f(G)(V_S, \sigma_S, \mu_S)$ of a fuzzy graph $G(V, \sigma, \mu\ )$ is a fuzzy graph with underlying crisp graph $S(G)(V_S, E_S)$, with the vertex set $V_S =\ V \cup V'$ where $V =\ \{v_i \mid v_i \in V\}$ and $V' = \{v_i'\ \forall\ v_i \in V\}$ and the edge set

$$E_S = \begin{cases} (v_i, v_j) & if\ v_i\ and\ v_j\ are\ adjacent\ in\ G, \\ (v'_i, v_j) & if\ v_i' \in V' and\ v_j \in V\ that\ are\ adjacent\ to\ v_i \in V. \end{cases}$$

Then, $\sigma_S(v_i) = \sigma_S(v_i') = \sigma(v_i)\ for\ v_i \in V\ and\ v_i' \in V',$

$\mu_S(v_i, v_j) = \mu(v_i, v_j)\ if\ v_i\ and\ v_j\ are\ adjacent\ in\ V,$

and $\mu_S(v_i', v_j) = \sigma_S(v_i') \wedge \sigma_S(v_j)\ if\ v_i' \in V' and\ v_j \in V\ that\ are\ adjacent\ to\ v_i \in V.$

Definition 2.15. [5] The shadow graph $D_{2_f}(G)(V_{D_2}, \sigma_{D_2}, \mu_{D_2})$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $D_2(G)(V_{D_2}, E_{D_2})$ is obtained by taking two copies of G namely G' and G'' with the vertex set $V_{D_2} = V' \cup V''$ where $V' = \{v_i'\ \forall\ v_i \in V\}$ and $V'' = \{v_i''\ \forall\ v_i \in V\}$ and the edge set

$$E_{D_2} = \begin{cases} (v_i', v_j'), (v_i'', v_j'') & if\ v_i\ and\ v_j\ are\ adjacent\ in\ G, \\ (v_{ij}, v_{rs}) & if\ v_i' \in V' and\ v_j'' \in V'' that\ are\ adjacent\ \ to\ v_i'' \in V''. \end{cases}$$

Then, $\sigma_{D_2}(v') = \sigma_{D_2}(v'') = \sigma(v) for\ v \in V, v' \in V', v'' \in V'',$

$\mu_{D_2}(v_i'v_j') = \mu_{D_2}(v_i''v_j'') = \mu(v_iv_j)$, for $v_i', v_j' \in V', v_i'', v_j'' \in V'', v_i, v_j \in V,$

$\mu_{D_2}(v_i'v_j'') = \sigma(v_i') \wedge \sigma(v_j'')\ if\ v_i' \in V' and\ v_j'' \in V''$ that are adjacent to $v_i'' \in V''$.

Definition 2 .16. [5] The line graph $L_f(G)(V_L, \sigma_L, \mu_L)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $L(G)(V_L, E_L)$, where the vertex set $V_L = \{v_{ij} \forall\ (v_i, v_j) \in E\}$ and edge set $E_L = \{(v_{ij}, v_{rs}) the\ edges\ (v_i, v_j)\ and\ (v_r, v_s)\ are\ adjacent\ in\ G\}$. Then, $\sigma_L(v_{ij}) = \mu(v_i, v_j)$ if $v_{ij} \in V_L$ and $\mu_L(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G.

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Definition 2.17. [5] The total graph $T_f(G)(V_T, \sigma_T, \mu_T)$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $T(G)(V_T, E_T)$, with the vertex set $V_T = V \cup V_{ij}$, where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij} \ \forall \ (v_i, v_j) \in E\}$ and the edge set

$$E_T = \begin{cases} (v_i, v_j) & \textit{if } v_i \textit{ and } v_j \textit{ are adjacent in } G, \\ (v_{ij}, v_i), (v_{ij}, v_j) & \forall \ i \textit{ and } j, \\ (v_{ij}, v_{rs}) & \textit{if the edges } (v_i, v_j) \textit{and } (v_r, v_s) \textit{are adjacent in } G. \end{cases}$$

Then, $\sigma_T(v_i) = \sigma(v_i) \ if \ v_i \in V$,

$\sigma_T(v_{ij}) = \mu(v_i, v_j) if \ (v_i, v_j) \in E \ \forall \ i \ and \ j$,

$\mu_T(v_i, v_j) = \mu(v_i, v_j) \ if \ v_i \ and \ v_j \ are \ adjacent \ in \ G$,

$\mu_T(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,

and $\mu_T(v_i, v_{ij}) = \mu_T(v_j, v_{ij}) = \mu(v_i, v_j) \forall \ i \ and \ j$.

Definition 2.18. [5] The subdivision graph $sd_f(G)(V_{sd}, \sigma_{sd}, \mu_{sd})$ of a fuzzy graph $G(V, \sigma, \mu)$ is a fuzzy graph with underlying crisp graph $sd(G)(V_{sd}, E_{sd})$, with the vertex set $V_{sd} = V \cup V_{ij}$ where $V = \{v_i \mid v_i \in V\}$ and $V_{ij} = \{v_{ij} \ \forall \ (v_i, v_j) \in E\}$ and the edge set $E_{sd} = (v_{ij}, v_i), (v_{ij}, v_j) \forall \ i \ \& \ j\}$. Then, $\sigma_{sd}(v_{ij}) = \mu(v_i, v_j)$ if $(v_i, v_j) \in E \ \forall \ i \ \& \ j$ and $\mu_{sd}(v_i, v_{ij}) = \mu_{sd}(v_j, v_{ij}) = (v_i, v_j) \ \forall \ i \ \& \ j$.

Definition 2.19. [4] Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy coloring is an assignment of basic or fuzzy colors to the vertices of a fuzzy graph G and it is proper,

(i) if two vertices are connected by a strong edge, then they either have different basic or fuzzy colors(if necessary), or one vertex can have a basic color and the other can have a fuzzy color corresponding to different basic color.

(ii) if two vertices are connected by a weak edge, then they either have same or different fuzzy colors, or one vertex can have a basic color and other can have a fuzzy color corresponding to the same basic color.

Definition 2.20. [4] Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Perfect fuzzy coloring (optimal fuzzy coloring) is an assignment of minimum number of colors (basic or fuzzy) for a proper fuzzy coloring of G.

Definition 2.21. [4] Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the chromatic number of G and is denoted by $\chi_f(G)$.

Lemma 2.1. [4] *Let P_n be a fuzzy path of length n. If all the edges are strong in P_n, then $\chi_f(P_n)$= 2.*

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Lemma 2.2. [4] *Let K_n be a complete fuzzy graph. If all the edges are strong in K_n, then $\chi_f(K_n) = n$.*

Lemma 2.3. [4] *Let S_n be a fuzzy star. If all the edges are strong in S_n, then $\chi_f(S_n) = 2$.*

Theorem 2.3. [5] $\chi_f(G) \geq max\,\{\chi_f(G_i) : 1 \leq i \leq k\}$, *where* $G = G_1 \cup G_2 \cup \ldots \cup G_k$ *and* $G_i, 1 \leq i \leq k$ *are fuzzy graphs.*

Corollary 2.3.1. [5] $\chi_f(G) \geq max\,\{\chi_f(G_i) : 1 \leq i \leq k\}$, *where* $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ *and* $G_i, 1 \leq i \leq k$ *are edge disjoint fuzzy graphs.*

Theorem 2.4. [15] *The complete graph K_n has Hamiltonian decomposition for all n.*

Note [4] : i.e., $K_{2n+1} = \oplus\, nC_{2n+1}$ and $K_{2n} = C_{2n} \oplus nP_1$, where $\oplus$ denotes edge disjoint union.

3. The Chromatic Number of Homogeneous Fuzzy Caterpillar and Its Some Related Graphs

In this section, we determine the chromatic number of the homogeneous fuzzy caterpillar and some of its related graphs, such as the middle graph $M_f(C(n;m))$, splitting graph $S_f(C(n;m))$, shadow graph $D_{2_f}(C(n;m))$, line graph $L_f(C(n;m))$, total graph $T_f(C(n;m))$, and the subdivision graph $sd_f\,(C(n;m))$, using fuzzy coloring based on the strength of edges incident to vertices.

3.1. The Chromatic Number of a Fuzzy Tree

Lemma 3.1.1. *Let G be a fuzzy tree. If all edges are weak in G, then $\chi_f(G) = 1$.*

(by the fuzzy coloring procedure [4]).

Lemma 3.1.2. *Let G be a fuzzy tree. If all the edges are strong in G, then $\chi_f(G) = 2$.*

Proof. Let G be a fuzzy tree. Since all the edges are strong in G, then the fuzzy coloring of G resembles the coloring of the crisp tree (by the fuzzy coloring procedure [4]). Therefore by Theorem 2.1, $\chi_f(G) = 2$.

Theorem 3.1.1. *Let G be a fuzzy tree. If atleast one edge is strong in G, then $\chi_f(G) = 2$.*

Proof. Let G be a fuzzy tree. Since atleast one edge is strong in G, then only two colors are required to color the fuzzy tree (by the fuzzy coloring procedure [4]).

Therefore $\chi_f(G) = 2$.

3.2. The Chromatic Number of a Homogeneous Fuzzy Caterpillar

Definition 3.2.1. The fuzzy caterpillar graph is a fuzzy tree having a central fuzzy path P with n vertices $v_1, v_2, \ldots, v_n$ and any number of leaf vertices can be attached to each vertex $v_i, 1 \leq i \leq n$, of the central fuzzy path P.

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Example 1.

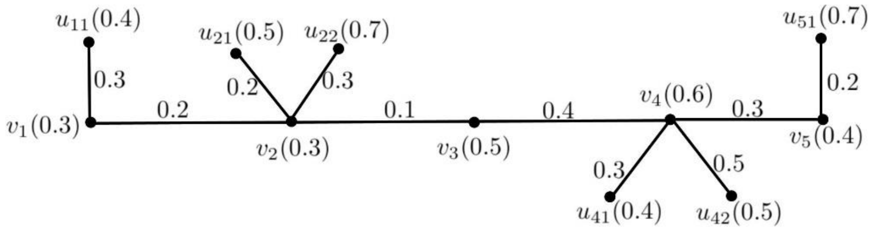

Figure 1. The Fuzzy Caterpillar

Definition 3.2.2. The homogeneous fuzzy caterpillar is a fuzzy caterpillar in which m number of leaf vertices are attached to each vertex $v_i, 1 \leq i \leq n$, of the central fuzzy path P and is denoted by $C(n;m)$. The vertex set and edge set of homogeneous fuzzy caterpillar $C(n;m)$ is given as $V\big(C(n;m)\big) = V \cup U$, where $V = \{v_i : 1 \leq i \leq n\}$ and $U = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$, and $E\big(C(n;m)\big) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$.

Example 2.

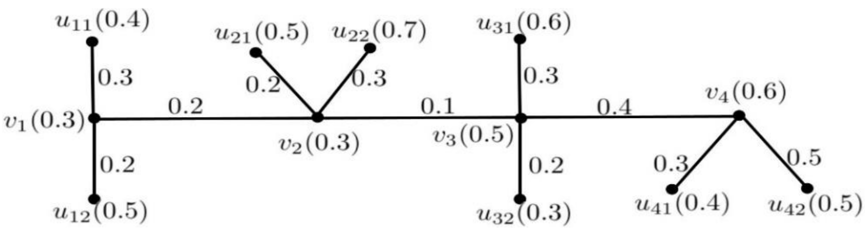

Figure 2. The Homogeneous Fuzzy Caterpillar

Theorem 3.2.1. *Every fuzzy caterpillar is a fuzzy tree.*

Proof. Every fuzzy caterpillar is a connected fuzzy graph with no fuzzy cycle (by definition 3.2.1). Therefore, every fuzzy caterpillar is a fuzzy tree (by definition 2.12).

Lemma 3.2.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all edges are weak in $C(n;m)$, then $\chi_f(C(n;m)) = 1$.* (by the fuzzy coloring procedure [4]).

Lemma 3.2.2. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are strong in $C(n;m)$, then $\chi_f(C(n;m)) = 2$.*

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Since all the edges are strong in $C(n;m)$, then the fuzzy coloring of $C(n;m)$ resembles the coloring of the crisp homogeneous caterpillar (by the fuzzy coloring procedure [4]). Therefore by Theorem 2.2, $\chi_f(C(n;m)) = 2$.

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Theorem 3.2.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If atleast one edge is strong in $C(n;m)$, then $\chi_f(C(n;m)) = 2$.*

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Since $C(n;m)$ is a fuzzy tree (by Theorem 3.2.1), then by Theorem 3.1.1, $\chi_f(C(n;m)) = 2$.

3.3. The Chromatic Number of $M_f(C(n;m))$

Remark 3.3.1. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $M_f\big(C(n;m)\big) = \oplus\, 2K_{m+2} \oplus (n-2)K_{m+3} \oplus mnP_1$ (by Theorem 2.4), where K_{m+2} and K_{m+3} are complete fuzzy graphs and P_1 be a fuzzy path of length 1.

Lemma 3.3.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $M_f(C(n;m))$ is a strong fuzzy graph.*

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. By the definition of middle graph of a fuzzy graph, we have $\sigma_M(v_i) = \sigma(v_i) if\ v_i \in V, 1 \le i \le n,$

$\sigma_M(v_{ij}) = \mu(v_i, v_j) if\ (v_i, v_j) \in E\ \forall\ i\ \&\ j,$

$\mu_M(v_{ij}, v_{rs}) = \mu(v_i, v_j) \wedge \mu(v_r, v_s)$ if the edges (v_i, v_j) and (v_r, v_s) are adjacent in G,

and $\mu_M(v_i, v_{ij}) = \mu_M(v_j, v_{ij}) = \mu(v_i, v_j)\ \forall\ i\ \&\ j$.

Then each edges of $M_f(C(n;m))$ satisfies the condition of a strong edge (by definition 2.7). Therefore, $M_f(C(n;m))$ is a strong fuzzy graph.

Theorem 3.3.1. *If $M_f(C(n;m))$ is a strong fuzzy graph,*

$$\chi_f\big(M_f(C(n;m))\big) = \begin{cases} m+2 & if\ n = 2, \\ m+3 & if\ n \ge 3. \end{cases}$$

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar and by Lemma 3.3.1, $M_f(C(n;m))$ is a strong fuzzy graph.

Case 1 : In $C(n;m)$, if $n = 2$.

Then $M_f\big(C(2;m)\big) = \oplus\, 2K_{m+2} \oplus 2mP_1$ (by Remark 3.3.1). Then by Lemma 2.2 we have, $\chi_f(K_{m+2}) = m+2$ and by Lemma 2.1 we have, $\chi_f(P_1) = 2$.

Therefore by Corollary 2.3.1,

$$\begin{aligned} \chi_f(M_f(C(2;m))) &= max\{\chi_f(K_{m+2}), \chi_f(P_1)\} \\ &= max\{m+2, 2\} \\ &= m+2. \end{aligned}$$

Case 2 : In $C(n;m)$, if $n \ge 3$.

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Then $M_f(C(n;m)) = \oplus\ 2K_{m+2} \oplus (n-2)K_{m+3} \oplus mnP_1$ (by Remark 3.3.1). Then by Lemma 2.2 we have, $\chi_f(K_{m+2}) = m+2$ &

$\chi_f(K_{m+3}) = m+3$ and by Lemma 2.1 we have, $\chi_f(P_1) = 2$.

Therefore by Corollary 2.3.1,

$$\begin{aligned}\chi_f\left(M_f(C(n;m))\right) &= max\{\chi_f(K_{m+2}), \chi_f(K_{m+3}), \chi_f(P_1)\}\\ &= max\{m+2, m+3, 2\}\\ &= m+3.\end{aligned}$$

3.4. The Chromatic Number of $S_f(C(n;m))$

Remark 3.4.1. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $S_f(C(n;m)) = C(n;m) \oplus 2S_{m+1} \oplus (n-2)S_{m+2} \oplus mnP_1$ (by Theorem 2.4), where $C(n;m)$ be the homogeneous fuzzy caterpillar, S_{m+1} and S_{m+2} are fuzzy stars and P_1 be the fuzzy path of length 1.

Lemma 3.4.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are weak in $C(n;m)$, then the edges of $C(n;m) \in S_f(C(n;m))$ are weak, while the edges of all fuzzy stars $S_{m+1} \in S_f(C(n;m))$ and $S_{m+2} \in S_f(C(n;m))$, as well as the edges of all fuzzy paths $P_1 \in S_f(C(n;m))$, are strong.*

Proof. Proof follows from the definition of splitting graph of a fuzzy graph and the definition of weak and strong edges.

Theorem 3.4.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are weak in $C(n;m)$, then $\chi_f(S_f(C(n;m))) = 2$.*

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $S_f(C(n;m)) = C(n;m) \oplus 2S_{m+1} \oplus (n-2)S_{m+2} \oplus mnP_1$ (by Remark 3.4.1) and by Lemma 3.4.1, the edges of homogeneous fuzzy caterpillar $C(n;m) \in S_f(C(n;m))$ are weak while the edges of all fuzzy stars $S_{m+1} \in S_f(C(n;m))$ and $S_{m+2} \in S_f(C(n;m))$, as well as the edges of all fuzzy paths $P_1 \in S_f(C(n;m))$, are strong. Then by Lemma 3.3 we have, $\chi_f(C(n;m)) = 1$, by Lemma 2.3 we have, $\chi_f(S_{m+1}) = \chi_f(S_{m+2}) = 2$ and by Lemma 2.1 we have, $\chi_f(P_1) = 2$.

Therefore by Corollary 2.3.1,

$$\begin{aligned}\chi_f(S_f(C(n;m))) &= max\{\chi_f(S_{m+1}), \chi_f(S_{m+2}), \chi_f(P_1)\}\\ &= max\{2,2,2\}\\ &= 2.\end{aligned}$$

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Lemma 3.4.2. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are strong in $C(n;m)$, then $S_f(C(n;m))$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of splitting graph of a fuzzy graph and the definition of strong edge.

Theorem 3.4.2. *If $S_f(C(n;m))$ is a strong fuzzy graph, then $\chi_f(S_f(C(n;m))) = 2$.*

(The proof will be similar as above theorem).

Lemma 3.4.3. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If weak and strong edges are distributed in any sequence in $C(n;m)$, then the edges of $C(n;m) \in S_f(C(n;m))$ are weak and strong, which are distributed in any sequence in $S_f(C(n;m))$. However, the edges of all fuzzy stars $S_{m+1} \in S_f(C(n;m))$ and $S_{m+2} \in S_f(C(n;m))$, as well as the edges of all fuzzy paths $P_1 \in S_f(C(n;m))$, are strong.* (The proof will be similar as above lemma).

Theorem 3.4.3. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If weak and strong edges are distributed in any sequence in $C(n;m)$, then $\chi_f(S_f(C(n;m))) = 2$.*

(The proof will be similar as above theorem).

3.5. The Chromatic Number of $D_{2_f}(C(n;m))$

Remark 3.5.1. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $D_{2_f}\big(C(n;m)\big) = \oplus\, 2C(n;m) \oplus 2S_{m+1} \oplus (n-2)S_{m+2} \oplus mnP_1$ (by Theorem 2.4), where S_{m+1} and S_{m+2} are fuzzy stars and P_1 be a fuzzy path of length 1.

Lemma 3.5.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are weak in $C(n;m)$, then the edges of both $C(n;m) \in D_{2_f}(C(n;m))$ are weak, while the edges of all fuzzy stars $S_{m+1} \in D_{2_f}(C(n;m))$ and $S_{m+2} \in D_{2_f}(C(n;m))$, as well as the edges of all fuzzy paths $P_1 \in D_{2_f}(C(n;m))$, are strong.*

Proof. Proof follows from the definition of shadow graph of a fuzzy graph and the definition of weak and strong edges.

Theorem 3.5.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are weak in $C(n;m)$, then $\chi_f(D_{2_f}(C(n;m))) = 2$.*

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $D_{2_f}(C(n;m) = \oplus\, 2C(n;m) \oplus 2S_{m+1} \oplus (n-2)S_{m+2} \oplus mnP_1$ (by Remark 3.5.1) and by Lemma 3.5.1, the edges of both $C(n;m) \in D_{2_f}(C(n;m))$ are weak, while the edges of all fuzzy stars $S_{m+1} \in D_{2_f}(C(n;m))$ and $S_{m+2} \in D_{2_f}(C(n;m))$, as well as the edges of all fuzzy paths $P_1 \in D_{2_f}(C(n;m))$, are strong. Then by Lemma 3.2.1 we have, $\chi_f(C(n;m)) = 1$, by Lemma 2.3 we have, $\chi_f(S_{m+1}) = \chi_f(S_{m+2}) = 2$ and by Lemma 2.1 we have, $\chi_f(P_1) = 2$. Therefore by Corollary 2.3.1, $\chi_f(D_{2_f}(C(n;m)) = 2$.

International Journal of Applied Mathematics
Volume 38 No. 3s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Lemma 3.5.2. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are strong in $C(n;m)$, then $D_{2_f}(C(n;m))$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of shadow graph of a fuzzy graph and the definition of strong edge.

Theorem 3.5.2. *If $D_{2_f}(C(n;m))$ is a strong fuzzy graph, then $\chi_f(D_{2_f}(C(n;m))) = 2$.*

(The proof will be similar as above theorem).

Lemma 3.5.3. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If weak and strong edges are distributed in any sequence in $C(n;m)$, then the edges of both $C(n;m) \in D_{2_f}(C(n;m))$ are weak and strong, which are distributed in any sequence in $D_{2_f}(C(n;m))$. However, the edges of all fuzzy stars $S_{m+1} \in D_{2_f}(C(n;m))$ and $S_{m+2} \in D_{2_f}(C(n;m))$, as well as the edges of all fuzzy paths $P_1 \in D_{2_f}(C(n;m))$, are strong.* (The proof will be similar as above lemma).

Theorem 3.5.3. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If weak and strong edges are distributed in any sequence in $D_{2_f}(C(n;m))$, then $\chi_f(D_{2_f}(C(n;m))) = 2$.*

(The proof will be similar as above theorem).

3.6. The Chromatic Number of $L_f(C(n;m))$

Remark 3.6.1. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $L_f\big(C(n;m)\big) = \oplus (n-2)K_{m+2} \oplus 2K_{m+1}$ (by Theorem 2.4), where K_{m+1} and K_{m+2} are complete fuzzy graphs.

Lemma 3.6.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $L_f(C(n;m))$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of line graph of a fuzzy graph and the definition of strong edge.

Theorem 3.6.1. *If $L_f(C(n;m))$ is a strong fuzzy graph, then*

$$\chi_f\big(L_f(C(n;m))\big) = \begin{cases} m+1 & if\ n = 2, \\ m+2 & if\ n \geq 3. \end{cases}$$

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then by Lemma 3.6.1, $L_f(C(n;m))$ is a strong fuzzy graph.

Case 1 : In $C(n;m)$, if $n = 2$.

Then $L_f\big(C(2;m)\big) = \oplus\, 2K_{m+1}$ (by Remark 3.6.1). Then by Lemma 2.2 we have, $\chi_f(K_{m+1}) = m+1$. Therefore, by Corollary 2.3.1, $\chi_f(L_f(C(n;m))) = \chi_f(K_{m+1}) = m+1$.

International Journal of Applied Mathematics
Volume 38 No. 3s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Case 2 : In $C(n;m)$, if $n \geq 3$.

Then $L_f\big(C(n;m)\big) = \oplus\ (n-2)K_{m+2} \oplus 2K_{m+1}$ (by Remark 3.6.1). Then by Lemma 2.2 we have, $\chi_f(K_{m+1}) = m+1$ and $\chi_f(K_{m+2}) = m+2$.

Therefore by Corollary 2.3.1,

$$\begin{aligned}\chi_f(L_f(C(n;m))) &= max\{\chi_f(K_{m+1}), \chi_f(K_{m+2})\}\\ &= max\{m+1, m+2\}\\ &= m+2.\end{aligned}$$

3.7. The Chromatic Number of $T_f(C(n;m))$

Remark 3.7.1. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $T_f\big(C(n;m)\big) = C(n;m) \oplus L_f\big(C(n;m)\big) \oplus [mn+(n-1)]P_2$ (by Theorem 2.4), where $C(n;m)$ be the homogeneous fuzzy caterpillar, $L_f(C(n;m))$ be the line graph of $C(n;m)$ and P_2 be a fuzzy path of length 2.

Lemma 3.7.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are weak in $C(n;m)$, then the edges of $C(n;m) \in T_f(C(n;m))$ are weak while the edges of $L_f\big(C(n;m)\big) \in T_f(C(n;m))$ and the edges of all paths $P_2 \in T_f(C(n;m))$ are strong.*

Proof. Proof follows from the definition of total graph of a fuzzy graph and the definition of weak and strong edges.

Theorem 3.7.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are weak in $C(n;m)$, then*

$$\chi_f\big(T_f(C(n;m))\big) = \begin{cases} m+2 & if\ n=2,\\ m+3 & if\ n \geq 3.\end{cases}$$

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $T_f\big(C(n;m)\big) = C(n;m) \oplus L_f\big(C(n;m)\big) \oplus [mn+(n-1)]P_2$ (by Remark 3.7.1) and by Lemma 3.7.1, the edges of homogeneous fuzzy caterpillar $C(n;m) \in T_f(C(n;m))$ are weak while the edges of $L_f\big(C(n;m)\big) \in T_f(C(n;m))$ and the edges of all paths $P_2 \in T_f(C(n;m))$ are strong. Then by Lemma 3.2.1 we have, $\chi_f(C(n;m)) = 1$, by Lemma 2.1 we have, $\chi_f(P_2) = 2$ and by Theorem 3.6.1, we have

$$\chi_f\big(L_f(C(n;m))\big) = \begin{cases} m+1 & if\ n=2,\\ m+2 & if\ n \geq 3.\end{cases}$$

Case 1 : In $C(n;m)$, if $n = 2$.

Then by Corollary 2.3.1,

$$\chi_f(T_f\big(C(2;m)\big) = max\{\chi_f\big(C(2;m)\big), \chi_f\big(L_f\big(C(2;m)\big), \chi_f(P_2)\} + 1$$

International Journal of Applied Mathematics
Volume 38 No. 3s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$= max\{1, m+1, 2\} + 1$$
$$= m + 2.$$

Case 2 : In $C(n;m)$, $n \geq 3$.

Then by Corollary 2.3.1,

$$\chi_f(T_f(C(n;m)) = max\{\chi_f(C(n;m)), \chi_f(L_f(C(n;m)), \chi_f(P_2)\} + 1$$
$$= max\{1, m+2, 2\} + 1$$
$$= m + 3.$$

Lemma 3.7.2. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If all the edges are strong in $C(n;m)$, then $T_f(C(n;m))$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of total graph of a fuzzy graph and the definition of strong edge.

Theorem 3.7.2. *If $T_f(C(n;m))$ is a strong fuzzy graph, then*

$$\chi_f(T_f(C(n;m))) = \begin{cases} m+2 & if\ n = 2, \\ m+3 & if\ n \geq 3. \end{cases}$$

(The proof will be similar as above Theorem).

Lemma 3.7.3. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If weak and strong edges are distributed in any sequence in $C(n;m)$, then the edges of $C(n;m) \in T_f(C(n;m))$ are weak and strong, which are distributed in any sequence in $T_f(C(n;m))$, while the edges of $L_f(C(n;m)) \in T_f(C(n;m))$ and the edges of all paths $P_2 \in T_f(C(n;m))$ are strong.* (The proof will be similar as above lemma).

Theorem 3.7.3. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. If weak and strong edges are distributed in any sequence in $C(n;m)$, then*

$$\chi_f(T_f(C(n;m))) = \begin{cases} m+2 & if\ n = 2, \\ m+3 & if\ n \geq 3. \end{cases}$$

(The proof will be similar as above Theorem).

3.8. The Chromatic Number of $sd_f(C(n;m))$

Remark 3.7.1. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $sd_f(C(n;m)) = P_{2n-2} \oplus mnP_2$ (by Theorem 2.4), where P_{2n-2} and P_2 are fuzzy paths of lengths $2n-2$ and 2, respectively.

Lemma 3.7.1. *Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $sd_f(C(n;m))$ is a strong fuzzy graph.*

Proof. Proof follows from the definition of subdivision graph of a fuzzy graph and the definition of strong edge.

Theorem 3.7.1. *If $sd_f(C(n;m))$ is a strong fuzzy graph, then $\chi_f(sd_f(C(n;m))) = 2$.*

Proof. Let $C(n;m)$ be a homogeneous fuzzy caterpillar. Then $sd_f\big(C(n;m)\big) = P_{2n-1} \oplus mnP_3$ (by Remark 3.8.1) and by Lemma 3.8.1, all edges are strong in $sd_f(C(n;m))$. Then by Lemma 2.1 we have, $\chi_f(P_{2n-2}) = \chi_f(P_2) = 2$. Then by Corollary 2.3.1, $\chi_f(sd_f(C(n;m))) = 2$.

4. Conclusion

In this paper, we determined the chromatic number of the homogeneous fuzzy caterpillar graph and some related graphs of $C(n;m)$ such as middle graph, splitting graph, shadow graph, line graph, total graph and the subdivision graph. The results are determined by using fuzzy coloring based on the strength of the edges incident on each vertex.

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