International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

HOMOGENIZATION OF A COUPLED STOKES VISCOELASTIC SYSTEM IN A PERIODIC COMPOSITE WITH MEMORY EFFECTS

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Abstract

This paper studies the homogenization of a coupled Stokes–viscoelastic system posed in a three-dimensional periodically perforated domain. At the microscopic level, an incompressible Stokes fluid occupies a periodic network of channels, while a viscoelastic solid with an internal damping term fills the complement. The two phases interact through kinematic and dynamic transmission conditions on the interface. By combining the method of multiple scales with a Laplace-transform approach in time, a homogenized integro–differential model with memory is derived and rigorously justified by the energy method. The effective macroscopic equation features a tensor-valued convolution kernel that encodes both the spatial microstructure and the viscoelastic time dependence of the solid phase. A first-order corrector is constructed, and strong convergence in $L^2(0,T;H^1(\Omega)^3)$ is obtained. In addition, novel structural properties of the homogenized memory kernel are proved: symmetry and positive definiteness, a quantitative exponential decay rate, and optimal time-regularity in terms of the data. These results yield uniqueness, stability, and continuous dependence on the microscopic geometry for the macroscopic model, and thus provide a robust mathematical framework for the effective description of such composite media.

Keywords: homogenization; viscoelastic composites; Stokes flow; periodic structures; memory effects; integro–differential equations; corrector estimate.

MSC 2020: 35B27; 35Q35; 74Q15; 76M50; 76D07.

1. Introduction

Composite materials consisting of a viscous fluid interacting with a viscoelastic solid phase arise in a wide spectrum of applications, including biological tissues, polymer suspensions and engineered porous media. At the microscopic level, the dynamics is governed by a fluid–structure interaction problem coupling the Stokes equations with a second-order-in-time viscoelastic equation. However, the fine geometric details of the microstructure are often not accessible in practice and, even when they are known, direct numerical simulation of the full microscopic model is prohibitively expensive. This motivates the derivation of effective macroscopic models via homogenization methods.

The mathematical theory of homogenization for elliptic and parabolic problems in periodically perforated domains is by now classical and is well documented in monographs and survey papers devoted to composite materials, porous media and fluid–structure

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
interaction. In particular, homogenization of Stokes or Navier–Stokes flows in connected or perforated porous media has been extensively investigated, and several approaches have been developed, ranging from the energy method to two-scale convergence. The extension of these techniques to time-dependent viscoelastic behaviour, where memory effects appear in the homogenized model, has received significant attention more recently, especially in the context of linear viscoelastic composites and more complex fluid–structure interactions.

In this work, a linear microscopic model is considered in which an incompressible Stokes fluid occupies a periodically perforated subdomain Ω_f^ε of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, while a linearly viscoelastic solid with internal damping fills the complementary region $\Omega_s^\varepsilon = \Omega \backslash \overline{\Omega_f^\varepsilon}$. The interface between the two phases is assumed to be smooth and periodically distributed, with characteristic scale proportional to a small parameter $\varepsilon > 0$. The microscale problem couples the Stokes system in Ω_f^ε with a second-order equation in time in Ω_s^ε through standard continuity of normal stress and continuity of velocity/displacement conditions on the interface. The external boundary of the domain is clamped.

The first aim of the paper is to derive, using a multiple-scales expansion in space combined with a Laplace transform in time, a homogenized effective equation satisfied by the macroscopic displacement field. The resulting limit model is an integro– differential equation with a memory kernel that depends on both the microscopic geometry and the viscoelastic parameters of the solid phase. This equation can be viewed as a poro-viscoelastic analogue of classical homogenized Stokes or elasticity systems. The formal asymptotic derivation is then rigorously justified by the energy method in the Laplace domain, which yields convergence of the microscopic solutions towards the solution of the homogenized model.

The second aim is to go beyond the mere convergence result and to obtain a quantitative description of the effective memory kernel. In particular, we prove that the kernel is symmetric and positive definite, that it decays exponentially in time under natural spectral assumptions on the microscopic cell operators, and that its time regularity is optimal with respect to the regularity of the data. These properties play a key role in ensuring well-posedness and stability of the homogenized equation, and they provide a clear structural link between the microscale viscoelastic dynamics and the macroscale response.

Finally, a first-order corrector is constructed and used to obtain a strong convergence result in $L^2(0,T;H^1(\Omega)^3)$ for the difference between the microscopic displacement and its homogenized approximation. This result not only justifies the asymptotic expansion but also provides a rigorous error estimate for numerical and modelling purposes.

Outline of the paper

Section 3 introduces the microscopic coupled Stokes–viscoelastic problem, its variational formulation on the fixed domain Ω, and establishes existence, uniqueness and a priori energy estimates that are uniform in the microstructural parameter ε. Section 4 describes the periodic geometric setting and presents the formal multiple-scales expansion, leading to the cell problems and the definition of the effective coefficients. The energy method in the Laplace domain and the rigorous homogenization result are carried out in Section 5, where the

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
homogenized integro–differential equation is identified and the convergence of the microscopic solutions is proved. Section 6 introduces a first-order corrector and establishes a strong convergence result in $L^2(0,T;H^1(\Omega)^3)$.

The novel contributions of the paper are presented in Section 7. There, structural properties of the homogenized memory kernel are derived, including symmetry, positive definiteness, exponential decay and optimal time-regularity. As a consequence, uniqueness, stability and continuous dependence on the microstructure are obtained for the homogenized model.

2. Functional-analytic preliminaries

In this section we collect a few functional-analytic tools that will be used throughout the paper. We only state and prove those results that are directly needed in the well-posedness and homogenization analysis. For general background, we refer to [10, 2].

2.1. Bochner spaces and basic properties

Let X be a Banach space and $T > 0$. For $1 \le p \le \infty$ we denote by $L^p(0,T;X)$ the space of (equivalence classes of) strongly measurable functions $f: (0,T) \to X$ such that

$$\| f \|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \| f(t) \|_X^p \, dt \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t\in(0,T)} \| f(t) \|_X, & p = \infty, \end{cases}$$

is finite. Endowed with this norm, $L^p(0,T;X)$ is a Banach space.

If $\Omega \subset \mathbb{R}^n$ is an open set and $X = L^q(\Omega)$, then Fubini's theorem yields

$$L^p\big(0,T;L^q(\Omega)\big) = L^p\big(\Omega \times (0,T)\big),$$

with equality of norms.

2.2. Distributional time derivatives and a Lions–Magenes lemma

We will frequently differentiate X–valued functions with respect to time in the distributional sense. Denote by $\mathcal{D}(0,T)$ the space of scalar C^∞ functions with compact support in $(0,T)$, and by $\mathcal{D}'(0,T;X)$ the space of continuous linear maps $\mathcal{D}(0,T) \to X$.

If $v \in L^1(0,T;X)$, we associate to v the vector-valued distribution $T_v \in \mathcal{D}'(0,T;X)$ defined by

$$T_v(\varphi) = \int_0^T v(t)\, \varphi(t)\, dt, \qquad \forall\, \varphi \in \mathcal{D}(0,T),$$

which provides a continuous linear embedding $L^1(0,T;X) \hookrightarrow \mathcal{D}'(0,T;X)$.

Definition 1. Let $f \in \mathcal{D}'(0,T;X)$. Its time derivative $f' \in \mathcal{D}'(0,T;X)$ is defined by

$$f'(\varphi) = -f(\varphi'), \qquad \forall\, \varphi \in \mathcal{D}(0,T).$$

The following regularity result, due to Lions and Magenes, will be used to justify energy identities and time continuity of solutions (see [10, Chap. III]).

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
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Theorem 2 (Lions–Magenes). *Let H be a separable Hilbert space and V a reflexive Banach space such that V is continuously and densely embedded in H, and H in turn is continuously and densely embedded in V' (the dual of V), that is,*

$$V \hookrightarrow H \hookrightarrow V'.$$

Assume that

$$g \in L^2(0,T;V), \qquad g' \in L^2(0,T;V'),$$

where g' is the distributional time derivative of g. Then:

1. *g admits a representative (still denoted by g) which belongs to $C([0,T];H)$;*
2. *for almost every $t \in [0,T]$ one has*

$$\langle g'(t), g(t)\rangle_{V' \times V} = \frac{1}{2}\frac{d}{dt}\| g(t) \|_H^2.$$

2.2. Lax–Milgram and Poincaré inequality

We will also use the Lax–Milgram theorem to solve elliptic and parabolic variational problems at both the microscopic and homogenized levels, and the Poincaré inequality to derive a priori estimates. For completeness we recall the statements and refer to [10, Chap. I], [2, Chap. 1].

Theorem 3 (Lax–Milgram). *Let V be a Hilbert space and $a: V \times V \to \mathbb{R}$ a bilinear form which is continuous and coercive, i.e., there exist constants $C, \alpha > 0$ such that*

$$|a(u,v)| \le C \| u \|_V \| v \|_V, \qquad a(u,u) \ge \alpha \| u \|_V^2, \quad \forall\, u,v \in V.$$

For every $L \in V'$ there exists a unique $u \in V$ such that

$$a(u,v) = L(v), \qquad \forall\, v \in V,$$

and $\| u \|_V \le \alpha^{-1} \| L \|_{V'}$.

Lemma 4 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists a constant $C_P > 0$, depending only on Ω, such that $\| u \|_{L^2(\Omega)} \le C_P \| \nabla u \|_{L^2(\Omega)}, \quad \forall\, u \in H_0^1(\Omega)$.*

3. The coupled Stokes–viscoelastic problem

3.1. Geometric setting and unknowns

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $T > 0$ a fixed final time. For each $\varepsilon > 0$, the domain Ω is decomposed into a fluid part Ω_f^ε and a solid (viscoelastic) part Ω_s^ε, with a smooth interface $\partial\Omega_s^\varepsilon$, such that

$$\Omega = \Omega_f^\varepsilon \cup \Omega_s^\varepsilon, \qquad \Omega_f^\varepsilon \cap \Omega_s^\varepsilon = \emptyset.$$

The unknowns are:

- $u^\varepsilon: \Omega_f^\varepsilon \times (0,T) \to \mathbb{R}^3$, the fluid velocity;
- $p^\varepsilon: \Omega_f^\varepsilon \times (0,T) \to \mathbb{R}$, the fluid pressure;

Received: Aug 14, 2025

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- $w^\varepsilon : \Omega_s^\varepsilon \times (0,T) \to \mathbb{R}^3$, the solid displacement.

We denote by n the unit normal to $\partial\Omega_s^\varepsilon$ pointing into the fluid region.

3.2. Microscopic model

The coupled system $(\mathcal{P}^\varepsilon)$ reads:

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon + \nabla p^\varepsilon &= f && \text{in } \Omega_f^\varepsilon \times (0,T), && (3.1)\\ \nabla \cdot u^\varepsilon &= 0 && \text{in } \Omega_f^\varepsilon \times (0,T), && (3.2)\\ \frac{\partial^2 w^\varepsilon}{\partial t^2} - \Delta w^\varepsilon + w^\varepsilon &= f && \text{in } \Omega_s^\varepsilon \times (0,T), && (3.3) \end{aligned}$$

with coupling conditions on the interface $\partial\Omega_s^\varepsilon$:

$$\begin{aligned} u^\varepsilon &= \frac{\partial w^\varepsilon}{\partial t} && \text{on } \partial\Omega_s^\varepsilon \times (0,T), && (3.4)\\ \frac{\partial u_i^\varepsilon}{\partial n} - p^\varepsilon n_i &= \frac{\partial w_i^\varepsilon}{\partial n} && \text{on } \partial\Omega_s^\varepsilon \times (0,T), && (3.5) \end{aligned}$$

and homogeneous Dirichlet condition on the external boundary:

$$u^\varepsilon = 0 \qquad \text{on } \partial\Omega \times (0,T). \qquad (3.6)$$

Initial conditions are

$$u^\varepsilon(x,0) = 0 \text{ in } \Omega_f^\varepsilon, \qquad w^\varepsilon(x,0) = 0, \quad \frac{\partial w^\varepsilon}{\partial t}(x,0) = 0 \text{ in } \Omega_s^\varepsilon. \quad (3.7)$$

We assume the body force f satisfies

$$f \in L^2(0,T; L^2(\Omega)^3).$$

3.3. Variational formulation

To obtain a formulation on the whole domain Ω, we extend w^ε to Ω by imposing in Ω_f^ε

$$\frac{\partial w^\varepsilon}{\partial t} = u^\varepsilon,$$

and by choosing $w^\varepsilon(x,0) = 0$ in Ω_f^ε, consistently with (3.7). In this way, w^ε is defined on all of Ω, and we can express the dynamics in terms of w^ε only.

Introduce the space

$$V_{\mathrm{div},\Omega_f^\varepsilon} := \{ v \in H_0^1(\Omega)^3 \,;\, \nabla \cdot v = 0 \text{ in } \Omega_f^\varepsilon \}.$$

Definition 5 (Variational problem $(\mathcal{P}^\varepsilon)$). Find

$$w^\varepsilon \in L^\infty\left(0,T; V_{\mathrm{div},\Omega_f^\varepsilon}\right), \quad \partial_t w^\varepsilon \in L^\infty(0,T; L^2(\Omega)^3),$$

such that for almost every $t \in (0,T)$ and all $\varphi \in V_{\mathrm{div},\Omega_f^\varepsilon}$,

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$$\begin{aligned}\frac{d^2}{dt^2}(w^\varepsilon(t),\varphi)_{0,\Omega} \quad &+\frac{d}{dt}(\nabla w^\varepsilon(t),\nabla\varphi)_{0,\Omega_f^\varepsilon} \\ &+(\nabla w^\varepsilon(t),\nabla\varphi)_{0,\Omega_s^\varepsilon}+(w^\varepsilon(t),\varphi)_{0,\Omega_s^\varepsilon}=(f(t),\varphi)_{0,\Omega},\end{aligned} \tag{3.8}$$

with initial conditions

$$w^\varepsilon(0)=0, \qquad \partial_t w^\varepsilon(0)=0 \text{ in } \Omega. \tag{3.9}$$

Here $(\cdot,\cdot)_{0,D}$ denotes the $L^2(D)$ inner product.

3.4. Existence, uniqueness and a priori estimates

Theorem 6 (Well-posedness of $(\mathcal{P}^\varepsilon)$). *Let* $f\in L^2(0,T;L^2(\Omega)^3)$. *Then, for each* $\varepsilon>0$, *there exists a unique solution*

$$w^\varepsilon\in L^\infty\left(0,T;V_{\mathrm{div},\Omega_f^\varepsilon}\right),\quad \partial_t w^\varepsilon\in L^\infty(0,T;L^2(\Omega)^3),$$

of (3.8)–(3.9), *and there exists a constant* $C>0$, *independent of* ε, *such that*

$$\| w^\varepsilon \|_{L^\infty\left(0,T;V_{\mathrm{div},\Omega_f^\varepsilon}\right)}+\| \partial_t w^\varepsilon \|_{L^\infty(0,T;L^2(\Omega)^3)}\le C\ \| f \|_{L^2(0,T;L^2(\Omega)^3)}.$$

Proof. The proof follows a standard Galerkin scheme.

Step 1: Galerkin approximation. Since $V_{\mathrm{div},\Omega_f^\varepsilon}$ is a separable Hilbert space, there exists an orthonormal basis $\{\varphi_k\}_{k\in\mathbb{N}}$ in $V_{\mathrm{div},\Omega_f^\varepsilon}$ which is also orthogonal in $L^2(\Omega)^3$. For $m\in\mathbb{N}$, consider the finite-dimensional subspace $V_m=\mathrm{span}\{\varphi_1,\dots,\varphi_m\}$ and seek an approximate solution of the form

$$w_m^\varepsilon(t)=\sum_{k=1}^m d_k^{(m)}(t)\,\varphi_k,$$

with scalar coefficients $d_k^{(m)}:[0,T]\to\mathbb{R}$.

Imposing (3.8) for test functions $\varphi\in V_m$ (i.e. taking $\varphi=\varphi_j$ for $j=1,\dots,m$), we obtain a system of second-order ordinary differential equations for $d_k^{(m)}(t)$ with homogeneous initial conditions. The bilinear form

$$B_\varepsilon(w,\varphi):=(\nabla w,\nabla\varphi)_{0,\Omega_f^\varepsilon}+(\nabla w,\nabla\varphi)_{0,\Omega_s^\varepsilon}+(w,\varphi)_{0,\Omega_s^\varepsilon}$$

is symmetric, continuous and coercive on $V_{\mathrm{div},\Omega_f^\varepsilon}$ (by the Poincaré inequality restricted to Ω_f^ε and Ω_s^ε). Thus the ODE system is well-posed and admits a unique solution $w_m^\varepsilon\in C^2([0,T];V_m)$.

Step 2: Energy estimate, uniform in m. Take $\varphi=\partial_t w_m^\varepsilon(t)$ in the Galerkin equation and sum over the basis. This yields

$$\frac{d^2}{dt^2}(w_t^\varepsilon,\partial_t w_m^\varepsilon)_{0,\Omega}+\frac{d}{dt}\left(\| \nabla w_m^\varepsilon \|^2_{L^2\left(\Omega_f^\varepsilon\right)}+\| \nabla w_m^\varepsilon \|^2_{L^2(\Omega_s^\varepsilon)}+\| w_m^\varepsilon \|^2_{L^2(\Omega_s^\varepsilon)}\right)=2(f,\partial_t w_m^\varepsilon)_{0,\Omega}.$$

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Using

$$\frac{d}{dt} \| \partial_t w_m^\varepsilon \|_{L^2(\Omega)}^2 = 2\,(\partial_t^2 w_m^\varepsilon, \partial_t w_m^\varepsilon)_{0,\Omega},$$

we obtain the energy identity

$$\frac{d}{dt} \mathcal{E}_m^\varepsilon(t) = 2\left(f(t), \partial_t w_m^\varepsilon(t)\right)_{0,\Omega},$$

where

$$\mathcal{E}_m^\varepsilon(t) = \frac{1}{2} \| \partial_t w_m^\varepsilon(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla w_m^\varepsilon(t) \|_{L^2\left(\Omega_f^\varepsilon\right)}^2 + \frac{1}{2} \| \nabla w_m^\varepsilon(t) \|_{L^2(\Omega_S^\varepsilon)}^2 + \frac{1}{2} \| w_m^\varepsilon(t) \|_{L^2(\Omega_S^\varepsilon)}^2.$$

Using Cauchy–Schwarz and Young inequalities, we get

$$\frac{d}{dt} \mathcal{E}_m^\varepsilon(t) \leq \| f(t) \|_{L^2(\Omega)^3}^2 + \| \partial_t w_m^\varepsilon(t) \|_{L^2(\Omega)^3}^2 \leq \| f(t) \|_{L^2(\Omega)^3}^2 + 2\,\mathcal{E}_m^\varepsilon(t).$$

By Gronwall's lemma and the initial conditions $w_m^\varepsilon(0) = \partial_t w_m^\varepsilon(0) = 0$, we deduce a bound

$$\mathcal{E}_m^\varepsilon(t) \leq C \int_0^t \| f(s) \|_{L^2(\Omega)^3}^2 \ ds, \qquad 0 \leq t \leq T,$$

for some $C > 0$ independent of m (and, as one checks, independent of ε under uniform geometry assumptions). This yields the uniform estimates

$$\| \partial_t w_m^\varepsilon \|_{L^\infty(0,T;L^2(\Omega)^3)} + \| w_m^\varepsilon \|_{L^\infty(0,T;H^1(\Omega)^3)} \leq C \ \| f \|_{L^2(0,T;L^2(\Omega)^3)}.$$

Step 3: Passage to the limit and uniqueness. By the uniform bounds and standard compactness arguments, there exists a subsequence (still denoted w_m^ε) and a function w^ε such that

$$w_m^\varepsilon \rightharpoonup w^\varepsilon \text{ weak-* in } L^\infty(0,T;H^1(\Omega)^3),$$
$$\partial_t w_m^\varepsilon \rightharpoonup \partial_t w^\varepsilon \text{ weak-* in } L^\infty(0,T;L^2(\Omega)^3).$$

Passing to the limit in the Galerkin formulation shows that w^ε satisfies (3.8)–(3.9). Uniqueness follows by a standard energy estimate on the difference of two solutions. The uniform bound is inherited from the energy estimate. □

4. Periodic structure and multiple scales

4.1. Periodic geometry

Let $Y = (0,1)^3$ be the unit cell and suppose Y is decomposed into a fluid part Y_f and a solid part Y_s, with smooth interface ∂Y_s, such that

$$Y = Y_f \cup Y_s, \qquad Y_f \cap Y_s = \emptyset.$$

Assume that the microstructure of Ω is obtained by periodically repeating the pattern Y. For $\varepsilon > 0$ small, we define

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\Omega_f^\varepsilon := \Omega \cap \bigcup_{k\in\mathbb{Z}^3} \varepsilon\big(Y_f + k\big), \qquad \Omega_s^\varepsilon := \Omega \cap \bigcup_{k\in\mathbb{Z}^3} \varepsilon(Y_s + k).$$

Thus the fluid and solid domains oscillate with period ε.

4.2. Formal two-scale expansions

We seek an asymptotic expansion of the solution in powers of ε:

$$\begin{cases} u^\varepsilon(x,t) = u^0(x,y,t) + \varepsilon u^1(x,y,t) + \varepsilon^2 u^2(x,y,t) + \cdots, \\ w^\varepsilon(x,t) = w^0(x,y,t) + \varepsilon w^1(x,y,t) + \varepsilon^2 w^2(x,y,t) + \cdots, \\ p^\varepsilon(x,t) = p^0(x,y,t) + \varepsilon p^1(x,y,t) + \varepsilon^2 p^2(x,y,t) + \cdots, \end{cases}$$

where $y = x/\varepsilon \in Y$ is the fast variable. The functions u^j, w^j, p^j are assumed Y–periodic with respect to y and sufficiently regular.

When differentiating, we use the chain rule

$$\nabla_x \mapsto \nabla_x + \frac{1}{\varepsilon}\nabla_y,$$

and insert the expansions into the microscopic system (3.1)–(3.3). Collecting terms of order $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^0, \ldots$ yields a hierarchy of problems on Y_f and Y_s.

4.3. The leading-order problem and independence of y

At order ε^{-2}, one obtains (in Y_f and Y_s):

$$-\Delta_y u^0 = 0, \quad \nabla_y \cdot u^0 = 0 \text{ in } Y_f \times (0,T),$$

$$-\Delta_y w^0 = 0 \text{ in } Y_s \times (0,T),$$

with appropriate interface and periodic conditions on ∂Y_s and ∂Y.

Lemma 7. *At the leading order, u^0 and w^0 do not depend on the fast variable y, i.e. $u^0 = u^0(x,t), \qquad w^0 = w^0(x,t)$.*

Proof. We sketch the argument for u^0; the proof for w^0 is analogous.

Fix (x,t) and consider $u^0(\cdot): Y_f \to \mathbb{R}^3$ as a function of y. It satisfies

$$-\Delta_y u^0 = 0, \quad \nabla_y \cdot u^0 = 0$$

in Y_f, with Y–periodic boundary conditions and appropriate conditions on ∂Y_s. Multiply the equation $-\Delta_y u^0 = 0$ by u^0 and integrate over Y_f. Integrating by parts (and using periodicity plus the interface conditions) yields

$$\int_{Y_f} \left|\nabla_y u^0\right|^2 \, dy = 0.$$

Hence $\nabla_y u^0 = 0$ in Y_f, so u^0 is independent of y in Y_f. Periodicity and transmission conditions then imply that u^0 is independent of y in the whole cell. The argument for w^0 is identical. □

Thus the leading-order macroscopic fields u^0 and w^0 depend only on (x,t), as expected.

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
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4.4. Cell problem and homogenized coefficients

At the next order, one obtains linear cell problems for the correctors u^1, w^1, p^0. After Laplace transform in time (to handle the time derivatives under the transmission conditions), one can write w^1, p^0 in the form

$$\widehat{w}_i^1(x,y,p) = -\sum_{k,h=1}^{3} \chi_i^{kh}(y,p)\frac{\partial \widehat{w}_k^0}{\partial x_h}(x,p),$$

$$\hat{p}^0(x,y,p) = \sum_{k,h=1}^{3} \gamma^{kh}(y,p)\frac{\partial \widehat{w}_k^0}{\partial x_h}(x,p),$$

where (χ^{kh}, γ^{kh}) solve a cell problem on Y (in y), with parameter p.

We do not repeat the full system here; the important point for the homogenization is that this cell problem is well-posed and determines the effective coefficients.

Theorem 8 (Cell problem). *Let $Y = (0,1)^3$ be the reference cell, decomposed into a fluid part Y_f and a solid part Y_s with smooth interface ∂Y_s, and let $p \in \mathbb{C}$ with $\Re p > 0$. For each $1 \le i,k,h \le 3$, there exists a unique pair*

$$\chi^{kh}(\cdot,p) \in H^1_{\mathrm{per}}(Y)^3, \quad \gamma^{kh}(\cdot,p) \in L^2_{\mathrm{per}}(Y),$$

with χ^{kh} Y–periodic and of zero mean over Y, solving the cell system associated with the order-ε^{-1} equations. Moreover, the maps

$$p \mapsto \chi^{kh}(\cdot,p), \qquad p \mapsto \gamma^{kh}(\cdot,p),$$

are continuous on $\{p \in \mathbb{C}: \Re p > 0\}$ with values in $H^1_{\mathrm{per}}(Y)^3$ and $L^2_{\mathrm{per}}(Y)$, respectively, and satisfy uniform energy estimates of the form

$$\| \chi^{kh}(\cdot,p) \|_{H^1(Y)^3} + \| \gamma^{kh}(\cdot,p) \|_{L^2(Y)} \le C(1+|p|),$$

for some constant $C > 0$ independent of p with $\Re p > 0$.

Proof. **1. Explicit form of the cell problem.** For clarity, we write the cell system (in the Laplace variable p) associated with a fixed triple of indices (k,h), and we omit the superscript kh on χ and γ when no confusion arises.

In the fluid region Y_f, the microscopic Stokes-type part gives, after Laplace transform and extraction of the order-ε^{-1} terms,

$$-p\,\Delta_y \chi_i(y,p) + \frac{\partial \gamma(y,p)}{\partial y_i} = p\,\Delta_y \pi_i^{kh}(y), \qquad y \in Y_f, \tag{4.1}$$

together with the incompressibility constraint

$$\nabla_y \cdot \chi(y,p) := \sum_{j=1}^{3} \frac{\partial \chi_j}{\partial y_j}(y,p) = -\sum_{j=1}^{3} \frac{\partial \pi_j^{kh}}{\partial y_j}(y), \qquad y \in Y_f. \tag{4.2}$$

In the solid region Y_s, the viscoelastic part leads to

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
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$$-\Delta_y \chi_i(y,p) = \Delta_y \pi_i^{kh}(y), \qquad y \in Y_s. \qquad (4.3)$$

On the interface ∂Y_s, we have the transmission conditions

$$\chi \text{ and } \pi^{kh} \quad \text{are continuous across } \partial Y_s, \qquad (4.4)$$

$$p\sum_{j=1}^{3} \frac{\partial}{\partial y_j}\left(\chi_i - \pi_i^{kh}\right) n_j - \gamma\, n_i \quad = \sum_{j=1}^{3} \frac{\partial}{\partial y_j}\left(\chi_i - \pi_i^{kh}\right) n_j, \quad y \in \partial Y_s, \qquad (4.5)$$

where n is the unit normal to ∂Y_s (for instance, pointing out of Y_s into Y_f). Finally, χ and γ are Y–periodic and we impose the normalization

$$\int_Y \chi\,(y,p)\, dy = 0. \qquad (4.6)$$

The functions π^{kh} are the affine functions

$$\pi_i^{kh}(y) := y_h\, \delta_{ik}, \qquad i,k,h \in \{1,2,3\}.$$

They are fixed, smooth and Y–periodic modulo affine terms; in the variational formulation only their gradients $\nabla_y \pi^{kh}$ will enter, which are constant.

2 Functional setting and variational formulation. Define

$$V = \left\{\varphi \in H^1_{per}(Y)^3 \; ; \int_Y \varphi\,(y)\, dy = 0\right\}.$$

This is a closed subspace of $H^1_{\text{per}}(Y)^3$ and thus a Hilbert space with the H^1-norm.

For the fluid part, the incompressibility condition (4,2) is treated in the usual Stokes way: the pressure γ appears as a Lagrange multiplier enforcing $\nabla_y \cdot (\chi - \pi^{kh}) = 0$ in Y_f. However, for the purpose of obtaining χ, it is enough to work with the weak formulation obtained by testing (4.1)–(4.3) and the interface condition (4.5) against smooth Y–periodic vector fields and integrating by parts.

Let $\varphi \in V$ be an arbitrary test function. Multiplying (4.1) by $\left(\chi_i - \pi_i^{kh}\right)$, integrating over Y_f, and then integrating by parts, using the periodic boundary conditions on ∂Y and the interface conditions on ∂Y_s, and doing the same with (4.3) over Y_s, one arrives at the identity

$$a_p(\chi,\varphi) = \ell_p^{kh}(\varphi), \qquad \forall\, \varphi \in V, \qquad (4.7)$$

where

$$a_p(\chi,\varphi) := \int_{Y_f} p\ \nabla_y \chi : \nabla_y \varphi\, dy + \int_{Y_s} \nabla_y\, \chi : \nabla_y \varphi\, dy,$$

and

$$\ell_p^{kh}(\varphi) := \int_{Y_f} p\ \nabla_y \pi^{kh} : \nabla_y \varphi\, dy + \int_{Y_s} \nabla_y\, \pi^{kh} : \nabla_y \varphi\, dy.$$

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
Here $A:B = \sum_{i,j} A_{ij} B_{ij}$ denotes the Frobenius product of matrices. Note that the pressure γ does not appear explicitly in (4.7) because its contribution cancels after integration by parts and using the divergence constraint of the test functions in Y_f; we will reconstruct γ later.

The bilinear form $a_p: V \times V \to \mathbb{C}$ and the linear functional $\ell_p^{kh}: V \to \mathbb{C}$ are well defined.

3. Continuity of a_p and ℓ_p^{kh}. Let $\chi, \varphi \in V$. By Cauchy–Schwarz,

$$|a_p(\chi, \varphi)| \leq |p| \int_{Y_f} |\nabla_y \chi| |\nabla_y \varphi| \, dy + \int_{Y_s} |\nabla_y \chi| |\nabla_y \varphi| \, dy \leq (|p| + 1) \parallel \nabla \chi \parallel_{L^2(Y)} \parallel \nabla \varphi \parallel_{L^2(Y)}.$$

Thus a_p is continuous on $V \times V$, with

$$|a_p(\chi, \varphi)| \leq C_1(1 + |p|) \parallel \chi \parallel_{H^1(Y)^3} \parallel \varphi \parallel_{H^1(Y)^3}.$$

Similarly, since $\nabla_y \pi^{kh}$ is constant,

$$|\ell_p^{kh}(\varphi)| \leq (|p| + 1) \parallel \nabla \pi^{kh} \parallel_{L^2(Y)} \parallel \nabla \varphi \parallel_{L^2(Y)} \leq C_2(1 + |p|) \parallel \varphi \parallel_{H^1(Y)^3}.$$

Hence ℓ_p^{kh} is a bounded linear functional on V.

4. Coercivity of a_p on V. We now show that a_p is coercive on V when $\Re p > 0$. For $\chi \in V$,

$$a_p(\chi, \chi) = \int_{Y_f} p \, |\nabla_y \chi|^2 \, dy + \int_{Y_s} |\nabla_y \chi|^2 \, dy.$$

Taking real parts,

$$\Re a_p(\chi, \chi) = (\Re p) \int_{Y_f} |\nabla_y \chi|^2 \, dy + \int_{Y_s} |\nabla_y \chi|^2 \, dy \geq \min\{\Re p, 1\} \parallel \nabla \chi \parallel_{L^2(Y)}^2.$$

Since χ has zero mean over Y and is Y–periodic, the Poincaré inequality on the torus yields

$$\parallel \chi \parallel_{L^2(Y)} \leq C_P \parallel \nabla \chi \parallel_{L^2(Y)}.$$

Therefore

$$\parallel \chi \parallel_{H^1(Y)^3}^2 \leq (1 + C_P^2) \parallel \nabla \chi \parallel_{L^2(Y)}^2.$$

Combining with the previous estimate, we obtain

$$\Re a_p(\chi, \chi) \geq \alpha_p \parallel \chi \parallel_{H^1(Y)^3}^2,$$

for some $\alpha_p > 0$ depending only on $\Re p$ and Y. In particular, since $\Re p > 0$ is fixed, we have coercivity of a_p on V:

$$|a_p(\chi, \chi)| \geq \Re a_p(\chi, \chi) \geq \alpha_p \parallel \chi \parallel_{H^1(Y)^3}^2, \qquad \forall \chi \in V.$$

5. Existence and uniqueness of χ^{kh}. We are now in the setting of the Lax–Milgram theorem on the Hilbert space V:

- $a_p: V \times V \to \mathbb{C}$ is continuous and coercive; - $\ell_p^{kh}: V \to \mathbb{C}$ is continuous and linear.

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
Therefore, there exists a unique $\chi^{kh}(\cdot,p) \in V$ such that

$$a_p(\chi^{kh},\varphi) = \ell_p^{kh}(\varphi), \qquad \forall\, \varphi \in V.$$

Moreover, the Lax–Milgram estimate gives

$$\| \chi^{kh}(\cdot,p) \|_{H^1(Y)^3} \leq \frac{1}{\alpha_p} \| \ell_p^{kh} \|_{V'} \leq C(1+|p|),$$

for some constant $C > 0$ independent of p with $\Re p > 0$.

6. Reconstruction and uniqueness of γ^{kh}. Having obtained χ^{kh}, we reconstruct γ^{kh} from the fluid equation and the divergence constraint. In Y_f, define γ^{kh} (up to an additive constant) by

$$\frac{\partial \gamma^{kh}}{\partial y_i} := p\, \Delta_y\big(\chi_i^{kh} - \pi_i^{kh}\big), \qquad y \in Y_f.$$

This definition is consistent because the right-hand side has zero curl: the Laplacian of a gradient has zero curl, and the functions involved are sufficiently smooth (in the weak sense).

To extend γ^{kh} to Y_s, we use the flux transmission condition (4.5). The additive constant is fixed by imposing the zero-mean condition over Y:

$$\int_Y \gamma^{kh}\,(y,p)\, dy = 0.$$

This gives a unique $\gamma^{kh} \in L^2_{\mathrm{per}}(Y)$.

Uniqueness follows from the linearity of the cell system: if (χ_1,γ_1) and (χ_2,γ_2) are two solutions with zero mean for χ, then their difference $(\chi,\gamma) = (\chi_1-\chi_2,\gamma_1-\gamma_2)$ solves the homogeneous cell system. The variational identity (4.7). with $\ell_p^{kh} = 0$ implies $a_p(\chi,\chi) = 0$, and by coercivity we obtain $\chi = 0$. The equations then give $\nabla_y \gamma = 0$ in Y_f, and periodicity plus the zero-mean condition imply $\gamma = 0$. Hence the solution is unique.

7. Continuous dependence on p. Let $p_n \to p$ in $\{p\colon \Re p > 0\}$ and let $\chi_n^{kh}(\cdot) := \chi^{kh}(\cdot,p_n)$ be the corresponding solutions in V. From the uniform Lax–Milgram estimate, the sequence $\{\chi_n^{kh}\}$ is bounded in V. By weak compactness, there exists $\chi_\infty \in V$ and a subsequence (denoted again χ_n^{kh}) such that

$$\chi_n^{kh} \rightharpoonup \chi_\infty \quad \text{weakly in } H^1(Y)^3.$$

Passing to the limit in the variational formulation,

$$a_{p_n}(\chi_n^{kh},\varphi) = \ell_{p_n}^{kh}(\varphi), \qquad \forall\, \varphi \in V,$$

using the continuity of a_p and ℓ_p^{kh} with respect to p, we obtain

$$a_p(\chi_\infty,\varphi) = \ell_p^{kh}(\varphi), \qquad \forall\, \varphi \in V.$$

By uniqueness, $\chi_\infty = \chi^{kh}(\cdot,p)$. Since the limit is unique, the whole sequence converges weakly. The coercivity and the uniform bounds then allow us to upgrade the weak

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
convergence to strong convergence in $H^1(Y)^3$ by a standard argument (for instance, by testing the difference of the equations for p_n and p with $\chi_n^{kh} - \chi^{kh}(\cdot,p)$ and using the coercivity). This shows that the map $p \mapsto \chi^{kh}(\cdot,p)$ is continuous from $\{\Re p > 0\}$ into $H^1_{\text{per}}(Y)^3$.

The continuity of $\gamma^{kh}(\cdot,p)$ in $L^2(Y)$ follows from its definition in terms of χ^{kh} and π^{kh} and the continuity of the trace and extension operators on Y_f and Y_s.

This completes the proof of the theorem. □

From the cell solutions, one defines the (Laplace-transformed) homogenized tensor

$$\alpha_{ijkh}(p) := \frac{1}{|Y|}\int_Y \left(p\chi_{Y_f} + \chi_{Y_s}\right)\nabla_y(-\chi^{kh} + \pi^{kh}) : \nabla_y\left(-\chi^{ij} + \pi^{ij}\right) dy,$$

and the scalar

$$\theta := \frac{|Y_s|}{|Y|}.$$

5. Energy method and convergence

To justify rigorously the formal asymptotics, we use the energy method combined with Laplace transform in time.

5.1. Laplace-transformed microscopic problem

Let $\widehat{w}^\varepsilon(x,p)$ and $\hat{p}^\varepsilon(x,p)$ denote the Laplace transforms in time of w^ε and p^ε:

$$\widehat{w}^\varepsilon(x,p) = \int_0^{+\infty} e^{-pt}\, w^\varepsilon(x,t)\, dt, \qquad \Re p > 0.$$

Transforming (3.8) with respect to t and using the initial conditions yields the stationary variational problem $\left(\widehat{\mathcal{P}^\varepsilon}\right)$: find

$$\widehat{w}^\varepsilon(p) \in V_{\text{div},\Omega_f^\varepsilon}, \quad \hat{p}^\varepsilon(p) \in L^2\left(\Omega_f^\varepsilon\right),$$

such that for all $\varphi \in H^1_0(\Omega)^3$,

$$\begin{aligned} &(p^2\widehat{w}^\varepsilon, \varphi)_{0,\Omega} + (p\nabla\widehat{w}^\varepsilon, \nabla\varphi)_{0,\Omega_f^\varepsilon} \\ &\quad + (\nabla\widehat{w}^\varepsilon, \nabla\varphi)_{0,\Omega_s^\varepsilon} + (\widehat{w}^\varepsilon, \varphi)_{0,\Omega_s^\varepsilon} = (\hat{f}, \varphi)_{0,\Omega} + (\hat{p}^\varepsilon, \nabla\cdot\varphi)_{0,\Omega_f^\varepsilon}. \end{aligned} \tag{5.1}$$

The existence and uniqueness of $(\widehat{w}^\varepsilon, \hat{p}^\varepsilon)$ follow from Lax–Milgram and a standard Stokes-type argument (using the divergence constraint and a pressure reconstruction lemma).

Uniform a priori estimates in ε are inherited from Theorem 6 together with the boundedness of the Laplace transform for $\Re p > 0$.

5.2. Weak convergence and identification of the limit problem

Denote by $(\widehat{w}^\varepsilon, \hat{p}^\varepsilon)$ the solutions of (5.1) for each $\varepsilon > 0$ and fixed p with $\Re p > 0$. From the uniform bounds and standard compactness results in periodic homogenization, we deduce:

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Lemma 9. *There exist functions $\widehat{w}^\star(x,p) \in H_0^1(\Omega)^3$, $\hat{p}^\star(x,p) \in L^2(\Omega)$, and a subsequence (not relabelled) such that, as $\varepsilon \to 0$,* $\widehat{w}^\varepsilon \rightharpoonup \widehat{w}^\star$ weakly in $H_0^1(\Omega)^3$, $\hat{p}^\varepsilon \rightharpoonup \hat{p}^\star$ weakly in $L^2(\Omega)$.

To identify the limit problem satisfied by $\widehat{w}^\star$, we use oscillating test functions in (5.1), involving the cell correctors (χ^{kh}, γ^{kh}), and apply the convergence results for periodic structures (Tartar's lemma and related tools). The details follow standard homogenization arguments and we state the main result.

Theorem 10 (Homogenized equation in the Laplace domain). *For each p with $\Re p > 0$, the limit $\widehat{w}^\star(x,p)$ satisfies the homogenized equation*

$$(p^2+\theta)\,\widehat{w}_i^\star(x,p) - \sum_{j,k,h=1}^{3} \alpha_{ijkh}(p)\frac{\partial^2 \widehat{w}_k^\star}{\partial x_h\,\partial x_j}(x,p) = \hat{f}_i(x,p),$$

in the sense of distributions on Ω, together with homogeneous Dirichlet boundary condition $\widehat{w}^\star = 0$ on $\partial\Omega$. Moreover, $\widehat{w}^\star$ is uniquely determined by this equation.

Idea of the proof. The proof relies on testing (5.1) with functions of the form

$$\varphi^\varepsilon(x) = \varphi(x) + \varepsilon \sum_{k,h} \chi^{kh}\left(\frac{x}{\varepsilon}, p\right)\frac{\partial \varphi_k}{\partial x_h}(x),$$

where $\varphi \in C_0^\infty(\Omega)^3$. Passing to the limit $\varepsilon \to 0$ using the convergence of $\widehat{w}^\varepsilon$, the periodicity of the cell functions and the definition of $\alpha_{ijkh}(p)$, one obtains the homogenized variational formulation for $\widehat{w}^\star$. Lax–Milgram in the homogenized setting ensures uniqueness, hence the whole sequence converges. □

5.3. Inverse Laplace transform and time-domain homogenized equation

Let $c_{ijkh}(t)$ denote the inverse Laplace transform of $\alpha_{ijkh}(p)$ in t, and consider the inverse Laplace transform of Theorem 10. This yields:

Proof. The proof proceeds in four main steps: uniform a priori estimates for w^ε, passage to the limit in the Laplace domain, identification of the time-domain homogenized equation via inverse Laplace transform, and uniqueness in the time domain.

1. Uniform a priori estimates and compactness. By Theorem 6, for each $\varepsilon > 0$ and $f \in L^2(0,T;L^2(\Omega)^3)$ there exists a unique solution

$$w^\varepsilon \in L^\infty(0,T;H_0^1(\Omega)^3), \quad \partial_t w^\varepsilon \in L^\infty(0,T;L^2(\Omega)^3)$$

of $(\mathcal{P}^\varepsilon)$, and the energy estimate

$$\| w^\varepsilon \|_{L^\infty(0,T;H_0^1(\Omega)^3)} + \| \partial_t w^\varepsilon \|_{L^\infty(0,T;L^2(\Omega)^3)} \leq C \, \| f \|_{L^2(0,T;L^2(\Omega)^3)}$$

holds with a constant $C > 0$ independent of ε.

From these uniform bounds and the Banach–Alaoglu theorem, there exist $w^\star \in L^\infty(0,T;H_0^1(\Omega)^3)$ and a subsequence (still denoted by w^ε) such that

$$w^\varepsilon \overset{*}{\rightharpoonup} w^\star \quad \text{in } L^\infty(0,T;H_0^1(\Omega)^3), \qquad (5.2)$$

and

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\partial_t w^\varepsilon \overset{*}{\rightharpoonup} \partial_t w^\star \quad \text{in } L^\infty(0,T;L^2(\Omega)^3),$$

up to extraction. The initial conditions $w^\varepsilon(0) = \partial_t w^\varepsilon(0) = 0$ imply that $w^\star$ inherits the same initial conditions in the sense of $L^2(\Omega)^3$.

2. Homogenized equation in the Laplace domain. Fix a complex parameter p with $\Re p > 0$ and consider the Laplace transform in time of w^ε:

$$\widehat{w}^\varepsilon(x,p) := \int_0^{+\infty} e^{-pt}\, w^\varepsilon(x,t)\, dt.$$

The uniform estimate and the fact that $w^\varepsilon(t) = 0$ for $t < 0$ ensure that this integral converges in $H_0^1(\Omega)^3$ and that $\widehat{w}^\varepsilon$ is analytic in p for $\Re p > 0$.

Applying the Laplace transform to the variational formulation of $(\mathcal{P}^\varepsilon)$ and using the zero initial data yields the stationary problem $\left(\widehat{\mathcal{P}^\varepsilon}\right)$ in the Laplace domain (equation (5.1): for each fixed p with $\Re p > 0$, find

$$\widehat{w}^\varepsilon(\cdot,p) \in V_{\mathrm{div},\Omega_f^\varepsilon}, \quad \hat{p}^\varepsilon(\cdot,p) \in L^2\left(\Omega_f^\varepsilon\right),$$

such that for all $\varphi \in H_0^1(\Omega)^3$,

$$\begin{aligned}&(p^2\widehat{w}^\varepsilon,\varphi)_{0,\Omega} + (p\nabla\widehat{w}^\varepsilon,\nabla\varphi)_{0,\Omega_f^\varepsilon}\\ &\quad+(\nabla\widehat{w}^\varepsilon,\nabla\varphi)_{0,\Omega_s^\varepsilon} + (\widehat{w}^\varepsilon,\varphi)_{0,\Omega_s^\varepsilon} = (\hat{f},\varphi)_{0,\Omega} + (\hat{p}^\varepsilon,\nabla\cdot\varphi)_{0,\Omega_f^\varepsilon},\end{aligned} \tag{5.3}$$

where $\hat{f}$ is the Laplace transform of f.

From the estimates in the time domain, one deduces uniform bounds on $\widehat{w}^\varepsilon(\cdot,p)$ in $H_0^1(\Omega)^3$, uniformly in ε and locally uniformly in p in the half-plane $\Re p > \delta > 0$. Using standard compactness results in periodic homogenization (and the cell problem of Theorem 8), one shows that, up to extracting a further subsequence, for each fixed p,

$$\widehat{w}^\varepsilon(\cdot,p) \rightharpoonup \widehat{w}^\star(\cdot,p) \quad \text{weakly in } H_0^1(\Omega)^3,$$

and the limit $\widehat{w}^\star$ satisfies the homogenized equation in the Laplace domain (Theorem 10): for all $\varphi \in H_0^1(\Omega)^3$,

$$\begin{aligned}&\int_\Omega (p^2+\theta)\ \widehat{w}_i^\star(x,p)\,\varphi_i(x)\,dx\\ &\quad+\sum_{j,k,h=1}^{3}\int_\Omega \alpha_{ijkh}(p)\,\frac{\partial\widehat{w}_k^\star}{\partial x_h}(x,p)\,\frac{\partial\varphi_i}{\partial x_j}(x)\,dx = \int_\Omega \hat{f}_i(x,p)\,\varphi_i(x)\,dx.\end{aligned} \tag{5.4}$$

By Theorem 10, this problem has a unique solution $\widehat{w}^\star(\cdot,p)$. Hence, in fact, the whole family $\{\widehat{w}^\varepsilon(\cdot,p)\}_\varepsilon$ converges weakly to $\widehat{w}^\star(\cdot,p)$ for each fixed p with $\Re p > 0$.

3. Inverse Laplace transform and identification in time. We now show that the family $\{w^\star(\cdot,t)\}_{t\geq 0}$ obtained by inverse Laplace transform of $\widehat{w}^\star$ solves the homogenized integro–differential equation in the time domain and that $w^\varepsilon \to w^\star$ in the sense stated.

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
3.1. Inverse Laplace transform. For each fixed $x \in \Omega$, the function $p \mapsto \widehat{w}^\star(x,p)$ is analytic in $\Re p > 0$ and has at most polynomial growth in $|p|$ (because of the a priori estimates on $\widehat{w}^\star$ and the structure of (5.4)). Therefore, the inverse Laplace transform is well defined and we can write

$$w^\star(x,t) = \frac{1}{2\pi i}\int_{\Re p=\sigma} e^{pt}\, \widehat{w}^\star(x,p)\, dp, \qquad t > 0,$$

for any $\sigma > 0$. By standard properties of the Laplace transform, $w^\star$ belongs to $L^\infty(0,T;H_0^1(\Omega)^3) \cap W^{1,\infty}(0,T;L^2(\Omega)^3)$ and its Laplace transform is exactly $\widehat{w}^\star$.

Similarly, since $\alpha_{ijkh}(p)$ is the Laplace transform of the kernel $c_{ijkh}(t)$,

$$\alpha_{ijkh}(p) = \int_0^{+\infty} e^{-pt}\, c_{ijkh}(t)\, dt,$$

and

$$\left(c_{ijkh} * g\right)(t) = \int_0^t c_{ijkh}\,(t-s) g(s)\, ds$$

has Laplace transform

$$\mathcal{L}\left[c_{ijkh} * g\right](p) = \alpha_{ijkh}(p)\, \hat{g}(p).$$

3.2. Transforming back (5.4). Equation (5.4)can be rewritten as

$$\mathcal{L}\left[\partial_t^2 w_i^\star + \theta w_i^\star - \sum_{j,k,h=1}^{3}\left(c_{ijkh} * \partial_{x_h x_j}^2 w_k^\star\right)\right](x,p) = (x,p).$$

for every p with $\Re p > 0$, in the sense of $H^{-1}(\Omega)$–valued functions. Since Laplace transform is injective on the space of functions considered, it follows that

$$\partial_t^2 w_i^\star(x,t) + \theta\, w_i^\star(x,t) - \sum_{j,k,h=1}^{3}\left(c_{ijkh} * \partial_{x_h x_j}^2 w_k^\star\right)(x,t) = f_i(x,t),$$

for almost every $(x,t) \in \Omega \times (0,T)$, with $w^\star = 0$ on $\partial\Omega \times (0,T)$ and $w^\star(x,0) = \partial_t w^\star(x,0) = 0$ in Ω.

4. Convergence and uniqueness in the time domain. We now show that the limit $w^\star$ is unique and that the entire family w^ε converges (not just a subsequence).

Uniqueness of $w^\star$ follows from standard energy estimates for the homogenized integro–differential equation: if $w^{\star,1}$ and $w^{\star,2}$ are two solutions corresponding to the same f, their difference $z := w^{\star,1} - w^{\star,2}$ satisfies the homogeneous equation

$$\partial_t^2 z_i + \theta z_i - \sum_{j,k,h}\left(c_{ijkh} * \partial_{x_h x_j}^2 z_k\right) = 0,$$

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
with zero initial data. Testing this equation with $\partial_t z$ in the appropriate variational setting and using the symmetry and positivity properties of the effective tensor (as in Theorem 12) and the properties of the kernel c_{ijkh}, one obtains an energy identity implying $z \equiv 0$. Thus the solution $w^\star$ is unique.

Finally, since for each p the solution $\widehat{w}^\star(\cdot,p)$ of the homogenized elliptic problem in the Laplace domain is unique (Theorem 10), the whole sequence $\widehat{w}^\varepsilon(\cdot,p)$ converges weakly to $\widehat{w}^\star(\cdot,p)$ for each fixed p, not just a subsequence. By the uniqueness of the inverse Laplace transform, this implies that the entire family w^ε converges to $w^\star$ in the sense of (5.2). This completes the proof. □

6. Corrector and strong convergence

To improve the convergence result and obtain an estimate for the error $w^\varepsilon - w^\star$, we introduce a first-order corrector.

6.1. Definition of the corrector

Let $w^\star$ be the solution of the homogenized problem . Define the corrector $w^\varepsilon_{\mathrm{corr}}$ by

$$w^\varepsilon_{\mathrm{corr}}(x,t) := w^\star(x,t) + \varepsilon \sum_{k,h=1}^{3} \chi^{kh}\left(\frac{x}{\varepsilon},\partial_t\right)\frac{\partial w_k^\star}{\partial x_h}(x,t),$$

where the notation $\chi^{kh}(\cdot,\partial_t)$ is schematic and reflects the time-convolution structure coming from the inverse Laplace transform of $\chi^{kh}(y,p)$. In practice, one works in the Laplace domain, defines the corrector there, and then transforms back.

6.2. Strong convergence in H^1 with corrector

Proof. The proof follows the classical corrector strategy in periodic homogenization, adapted to the present time-dependent viscoelastic setting via the Laplace transform. It proceeds in three main steps: (i) construction of an approximate solution in the Laplace domain, (ii) an energy estimate for the difference in the Laplace domain, (iii) transfer of the strong convergence to the time domain and removal of subsequences.

1. Corrector in the Laplace domain and error equation. Fix p with $\Re p > 0$. Denote by $\widehat{w}^\varepsilon(\cdot,p)$ and $\widehat{w}^\star(\cdot,p)$ the Laplace transforms of w^ε and $w^\star$ with respect to time. From Section 4 and the cell analysis (Theorem 8), we know that the effective tensor $\alpha_{ijkh}(p)$ and the correctors $\chi^{kh}(y,p)$ describe the first-order oscillations in space.

We define the *corrected* approximation in the Laplace domain by

$$\widehat{\mathrm{w}}^\varepsilon_{\mathrm{corr}}(x,p) = \widehat{\mathrm{w}}^\star(x,p) + \varepsilon \sum_{k,h=1}^{3} \chi^{kh}\left(\frac{x}{\varepsilon},\mathrm{p}\right)\frac{\partial \widehat{\mathrm{w}}_{\mathrm{k}}^\star}{\partial x_h}(x,p), \quad \mathrm{x} \in \Omega$$

The functions $\chi^{kh}(\cdot,p)$ are Y–periodic in their argument, have zero mean, and belong to $H^1_{\mathrm{per}}(Y)^3$, while $\widehat{w}^\star(\cdot,p) \in H^1_0(\Omega)^3$ solves the homogenized variational problem (5.4).

Define the error

$$e^{\varepsilon}(x,p) := \widehat{w}^{\varepsilon}(x,p) - \widehat{w}^{\varepsilon}_{\mathrm{corr}}(x,p).$$

Our goal is to show that

$$\| e^{\varepsilon}(\cdot,p) \|_{H_0^1(\Omega)^3} \to 0 \quad \text{as } \varepsilon \to 0,$$

uniformly for p in appropriate vertical lines $\Re p = \sigma > 0$.

2. Variational identity for the error and energy estimate.

2.1. Microscopic and homogenized variational problems. Recall the microscopic Laplace-variational formulation (5.3), which we rewrite here for convenience: for all $\varphi \in H_0^1(\Omega)^3$,

$$\begin{aligned} a_p^{\varepsilon}(\widehat{w}^{\varepsilon},\varphi) &= (p^2\widehat{w}^{\varepsilon},\varphi)_{0,\Omega} + (p\nabla\widehat{w}^{\varepsilon},\nabla\varphi)_{0,\Omega_f^{\varepsilon}} \\ &\quad +(\nabla\widehat{w}^{\varepsilon},\nabla\varphi)_{0,\Omega_s^{\varepsilon}} + (\widehat{w}^{\varepsilon},\varphi)_{0,\Omega_s^{\varepsilon}} \qquad (6.1) \\ &= \left(\hat{f},\varphi\right)_{0,\Omega} + (\hat{p}^{\varepsilon},\nabla\cdot\varphi)_{0,\Omega_f^{\varepsilon}}. \end{aligned}$$

Here $\hat{p}^{\varepsilon}$ is the (Laplace-transformed) pressure.

On the other hand, the homogenized variational problem (5.4) states that, for all $\psi \in H_0^1(\Omega)^3$,

$$\begin{aligned} a_p^{\mathrm{hom}}(\widehat{w}^{\star},\psi) &= (p^2\widehat{w}^{\star},\psi)_{0,\Omega} + \theta(\widehat{w}^{\star},\psi)_{0,\Omega} \\ &\quad + \sum_{j,k,h=1}^{3} \int_{\Omega} \alpha_{ijkh}(p) \frac{\partial \widehat{w}_k^{\star}}{\partial x_h} \frac{\partial \psi_i}{\partial x_j}\, dx \qquad (6.2) \\ &= \left(\hat{f},\psi\right)_{0,\Omega}. \end{aligned}$$

2.2. Testing with the corrector-based function. We shall insert $\widehat{w}^{\varepsilon}_{\mathrm{corr}}$ into the microscopic variational formulation (6.1), and compare with (6.2). The two-scale expansion and the cell problem guarantee that, modulo terms of order ε, the bilinear form a_p^{ε} acting on $\widehat{w}^{\varepsilon}_{\mathrm{corr}}$ reproduces the homogenized bilinear form a_p^{hom} acting on $\widehat{w}^{\star}$.

More precisely, for any test function $\varphi \in H_0^1(\Omega)^3$, we can write

$$a_p^{\varepsilon}(\widehat{w}^{\varepsilon}_{\mathrm{corr}},\varphi) = a_p^{\mathrm{hom}}(\widehat{w}^{\star},\varphi) + R_p^{\varepsilon}(\varphi),$$

where the remainder $R_p^{\varepsilon}(\varphi)$ comes from:

- the gradients of $\chi^{kh}(x/\varepsilon,p)$ and the mismatch between the true microstructure and the cell average,
- the cut-off near the boundary (if one uses a standard technique to localize the corrector near the interior of Ω),
- the pressure term and divergence constraint in Ω_f^{ε}.

Classical periodic homogenization results (see, e.g., the proofs of corrector theorems for elliptic or parabolic problems) show that

$$\left|R_p^{\varepsilon}(\varphi)\right| \le C\varepsilon \, \| \widehat{w}^{\star}(\cdot,p) \|_{H^2(\Omega)^3} \| \varphi \|_{H^1(\Omega)^3},$$

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
for some constant $C > 0$ independent of ε and p in vertical lines $\Re p = \sigma > 0$, provided $\widehat{w}^{\star}$ has enough regularity in x (which holds for smooth Ω and smooth coefficients, or by density and approximation).

Now subtract (6.1) with test function φ and (6.2) with test function $\psi = \varphi$ to obtain the error equation in the Laplace domain:

$$\begin{aligned} a_p^{\varepsilon}(e^{\varepsilon}, \varphi) &= a_p^{\varepsilon}(\widehat{w}^{\varepsilon}, \varphi) - a_p^{\varepsilon}(\widehat{w}_{\text{corr}}^{\varepsilon}, \varphi) \\ &= \left(\hat{f}, \varphi\right)_{0,\Omega} + (\hat{p}^{\varepsilon}, \nabla \cdot \varphi)_{0,\Omega_f^{\varepsilon}} - a_p^{\text{hom}}(\widehat{w}^{\star}, \varphi) - R_p^{\varepsilon}(\varphi). \end{aligned}$$

Using (6.2) and the fact that the homogenized pressure does not appear explicitly (because of the divergence-free constraint in the effective equation), we get

$$a_p^{\varepsilon}(e^{\varepsilon}, \varphi) = (\hat{p}^{\varepsilon}, \nabla \cdot \varphi)_{0,\Omega_f^{\varepsilon}} - R_p^{\varepsilon}(\varphi).$$

At this point, we choose $\varphi = e^{\varepsilon}$ in the variational equation above. The incompressibility constraint on $\widehat{w}^{\varepsilon}$ in Ω_f^{ε} implies that $\nabla \cdot e^{\varepsilon}$ is controlled by the divergence of $\widehat{w}_{\text{corr}}^{\varepsilon}$, which is of order ε thanks to the cell problem; one shows

$$|(\hat{p}^{\varepsilon}, \nabla \cdot e^{\varepsilon})_{0,\Omega_f^{\varepsilon}}| \leq C\varepsilon \ \| \hat{p}^{\varepsilon} \|_{L^2\left(\Omega_f^{\varepsilon}\right)} \| \nabla e^{\varepsilon} \|_{L^2(\Omega)^3}.$$

On the other hand, the coercivity of a_p^{ε} gives

$$\Re a_p^{\varepsilon}(e^{\varepsilon}, e^{\varepsilon}) \geq \alpha\left(\| \nabla e^{\varepsilon} \|_{L^2(\Omega)^3}^2 + \| e^{\varepsilon} \|_{L^2(\Omega)^3}^2\right),$$

for some $\alpha > 0$ depending on $\Re p$ but independent of ε. Combining these estimates, we arrive at

$$\alpha\left(\| \nabla e^{\varepsilon} \|_{L^2}^2 + \| e^{\varepsilon} \|_{L^2}^2\right) \leq C\varepsilon(\| \hat{p}^{\varepsilon} \|_{L^2} \| \nabla e^{\varepsilon} \|_{L^2} + \| \widehat{w}^{\star} \|_{H^2} \| \nabla e^{\varepsilon} \|_{L^2}),$$

for suitable norms, all depending on x and p. Using Young's inequality and the uniform bounds on $\hat{p}^{\varepsilon}$ and $\widehat{w}^{\star}$, we obtain

$$\| \nabla e^{\varepsilon}(\cdot, p) \|_{L^2(\Omega)^3} + \| e^{\varepsilon}(\cdot, p) \|_{L^2(\Omega)^3} \leq C\varepsilon,$$

with a constant C independent of ε and locally uniform in p (for $\Re p \geq \sigma > 0$). Hence

$$\| e^{\varepsilon}(\cdot, p) \|_{H_0^1(\Omega)^3} \leq C\varepsilon \quad \text{for all } p \text{ with } \Re p \geq \sigma > 0 \qquad (6.3)$$

3. Transfer to the time domain and conclusion.

3.1. Inverse Laplace transform of the error. Define the time-domain corrector

$$w_{\text{corr}}^{\varepsilon}(x, t) := w^{\star}(x, t) + \varepsilon \sum_{k,h=1}^{3} \chi^{kh}\left(\frac{x}{\varepsilon}, \partial_t\right) \frac{\partial w_k^{\star}}{\partial x_h}(x, t),$$

understood via convolution in time, i.e., the operator $\chi^{kh}(\cdot, \partial_t)$ is defined as the inverse Laplace transform of the multiplication by $\chi^{kh}(\cdot, p)$ in the Laplace domain. The error in time domain is

$$e^{\varepsilon}(x, t) := w^{\varepsilon}(x, t) - w_{\text{corr}}^{\varepsilon}(x, t),$$

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
whose Laplace transform in t is exactly the $e^\varepsilon(x,p)$ above.

Estimate (6.3) shows that the Laplace transform of e^ε is uniformly bounded by $C\varepsilon$ in $H_0^1(\Omega)^3$ for $\Re p \geq \sigma$. By standard properties of the Laplace transform (Plancherel-type identities for vertical lines), one obtains

$$\int_0^T \| \nabla e^\varepsilon(\cdot,t) \|_{L^2(\Omega)^3}^2 \; dt \leq C'\varepsilon^2,$$

for some $C' > 0$ independent of ε. Similarly for $\| e^\varepsilon(\cdot,t) \|_{L^2}$. Therefore

$$\| e^\varepsilon \|_{L^2\left(0,T;H_0^1(\Omega)^3\right)} \leq C''\varepsilon \underset{\varepsilon\to 0}{\to} 0.$$

This proves the strong convergence

$$w^\varepsilon - w_{\mathrm{corr}}^\varepsilon \to 0 \quad \text{in } L^2(0,T;H_0^1(\Omega)^3),$$

as $\varepsilon \to 0$.

3.2. Removal of subsequences. The argument above yields an estimate valid for the full family $\{e^\varepsilon\}_\varepsilon$, not merely for a subsequence, since the corrector construction and the a priori estimates are uniform in ε. Hence no diagonal extraction is needed, and the entire sequence converges strongly in $L^2(0,T;H_0^1(\Omega)^3)$. □

Remark 11. The corrector result not only justifies the first-order two-scale expansion but also provides an error estimate: one can typically show that

$$\| w^\varepsilon - w_{\mathrm{corr}}^\varepsilon \|_{L^2\left(0,T;H^1(\Omega)\right)} \leq C\,\varepsilon\, \| f \|_{L^2\left(0,T;L^2(\Omega)\right)},$$

under additional regularity assumptions on f and the geometry. We do not detail these refinements here.

7. Structural properties of the homogenized memory kernel

In this section we investigate qualitative properties of the effective memory kernel appearing in the homogenized model . These results are crucial for the well-posedness and stability of the macroscopic problem, and they also clarify how the microscopic viscoelastic behaviour and geometry are encoded at the macroscale.

7.1. Representation and symmetry

Recall from Section 4 that the homogenized model in the Laplace domain has the form

$$(p^2+\theta)\,\widehat{w}_i^\star(x,p) - \sum_{j,k,h=1}^{3} \alpha_{ijkh}(p) \frac{\partial^2 \widehat{w}_k^\star}{\partial x_h\,\partial x_j}(x,p) = \hat{f}_i(x,p),$$

where $\theta = |Y_s|/|Y|$ and $\alpha_{ijkh}(p)$ are defined by the cell problems. In the time domain, the corresponding model reads

$$\frac{\partial^2 w_i^\star}{\partial t^2}(x,t) + \theta\, w_i^\star(x,t) - \sum_{j,k,h=1}^{3} \left(c_{ijkh} * \partial_{x_h x_j}^2 w_k^\star\right)(x,t) = f_i(x,t),$$

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
where $c_{ijkh}(t)$ is the inverse Laplace transform of $\alpha_{ijkh}(p)$.

We first show that the tensor $\alpha(p) = \{\alpha_{ijkh}(p)\}$ is symmetric and positive definite for each p with $\Re p > 0$.

Theorem 12 (Symmetry and positivity of the effective tensor). *For each fixed* p *with* $\Re p > 0$*, the fourth-order tensor* $\alpha(p) = \{\alpha_{ijkh}(p)\}$ *satisfies* $\alpha_{ijkh}(p) = \alpha_{khij}(p)$ *and there exists a constant* $c_0 > 0$*, independent of* p*, such that for all symmetric matrices* $\xi = (\xi_{kh}) \in \mathbb{R}^{3\times 3}$*,* $\sum_{i,j,k,h=1}^{3} \alpha_{ijkh}(p)\, \xi_{kh}\, \xi_{ij} \geq c_0\, |\xi|^2$.

Proof. Let $\xi \in \mathbb{R}^{3\times 3}$ be symmetric and consider the auxiliary cell function

$$\Phi_i^{\xi}(y,p) := \sum_{k,h=1}^{3} \left(-\chi_i^{kh}(y,p) + \pi_i^{kh}(y)\right) \xi_{kh},$$

which is a linear combination of the correctors. By the definition of $\alpha_{ijkh}(p)$, one can rewrite

$$\sum_{i,j,k,h} \alpha_{ijkh}(p)\, \xi_{kh}\, \xi_{ij} = \frac{1}{|Y|} \int_Y \left(p\chi_{Y_f} + \chi_{Y_s}\right) \left|\nabla_y \Phi^{\xi}(y,p)\right|^2 dy.$$

The integrand is nonnegative and vanishes only if $\nabla_y \Phi^{\xi} = 0$ in Y, that is, Φ^{ξ} is constant in y. However, the periodicity and zero-mean condition on the correctors imply that Φ^{ξ} has zero mean over Y, so the only constant solution is $\Phi^{\xi} \equiv 0$. This forces $\xi = 0$, which shows that the quadratic form is strictly positive for $\xi \neq 0$. A standard compactness argument on the unit sphere of symmetric matrices then yields the existence of $c_0 > 0$ such that the stated inequality holds.

Symmetry $\alpha_{ijkh} = \alpha_{khij}$ follows from the fact that the underlying cell bilinear form is symmetric (it involves only the L^2 inner product of gradients and the same weight $p\chi_{Y_f} + \chi_{Y_s}$). Writing the tensor explicitly in terms of the bilinear form and swapping the roles of indices (i,j) and (k,h) yields the desired symmetry. □

7.2. Exponential decay of the memory kernel

We next show that the memory kernel $c_{ijkh}(t)$ decays exponentially in time, under a mild spectral assumption on the microscopic operators.

Assumption 1 (Spectral gap). The microscopic cell operators associated with the viscoelastic solid phase and the Stokes fluid phase have discrete spectra with strictly positive real parts, and there exists $\lambda_0 > 0$ such that all eigenvalues satisfy $\Re\lambda \geq \lambda_0$.

Theorem 13 (Exponential decay). *Under Assumption 1, there exist constants* $M \geq 1$ *and* $\beta > 0$ *such that the memory kernel* $c_{ijkh}(t)$ *satisfies*

$$\left|c_{ijkh}(t)\right| \leq M\, e^{-\beta t}, \qquad t \geq 0,$$

for all $1 \leq i,j,k,h \leq 3$.

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
Proof. The proof relies on the spectral representation of the resolvent of the cell operators and on standard results on Laplace transforms of exponentially decaying functions.

1. Operator formulation of the cell problem. Fix (i, j, k, h) and p with $\Re p > 0$. For simplicity of notation we suppress the indices (i, j, k, h) and write $\chi = \chi^{kh}$, $\gamma = \gamma^{kh}$, $\alpha = \alpha_{ijkh}$, $c = c_{ijkh}$.

The cell problem (Theorem 8) can be written as

$$\mathcal{A}(p)u = f,$$

where $u = (\chi, \gamma)$ belongs to a suitable Hilbert space $\mathcal{H}$ (essentially $H^1_{\mathrm{per}}(Y)^3 \times L^2_{\mathrm{per}}(Y)$, with appropriate constraints), f is a fixed element encoding the affine function π^{kh}, and $\mathcal{A}(p): \mathcal{D}(\mathcal{A}(p)) \subset \mathcal{H} \to \mathcal{H}$ is the (parameter-dependent) cell operator coming from the PDE system. More concretely, for each fixed p with $\Re p > 0$, $\mathcal{A}(p)$ is elliptic in y, with transmission conditions on ∂Y_s and periodic boundary conditions on ∂Y.

Assumption 1 states that the spectrum of $\mathcal{A}(p)$ (as a function of p) lies in a half-plane with a strictly positive real part, uniformly with respect to p in vertical strips. Equivalently, there exists $\lambda_0 > 0$ such that all eigenvalues λ of the underlying (time-independent) spatial operators satisfy

$$\Re\lambda \geq \lambda_0.$$

In particular, the resolvent

$$R(z) := (\mathcal{L} - zI)^{-1}$$

of the (time-independent) spatial operator $\mathcal{L}$ is well defined and bounded for $\Re z > -\lambda_0$.

2. Representation of $\alpha(p)$ via the resolvent. By solving the cell problem in the Laplace domain, one can express $\chi(y, p)$ and $\gamma(y, p)$ in terms of the resolvent of an operator of the form

$$\mathcal{L}_p := p\chi_{Y_f}(-\Delta_y) + \chi_{Y_s}(-\Delta_y)$$

acting on periodic vector fields, together with lower-order (mass-type) terms. For the purpose of this proof, it is enough to note that $\alpha(p)$ can be written as

$$\alpha(p) = \langle \mathcal{R}(p)\phi, \phi \rangle_{\mathcal{H}}, \qquad (7.1)$$

where ϕ is a fixed element of $\mathcal{H}$ (depending on (i, j, k, h) but not on p), and

$$\mathcal{R}(p) := (\mathcal{L} + p\mathcal{M})^{-1},$$

with $\mathcal{L}$ a sectorial (elliptic) operator on $\mathcal{H}$ and $\mathcal{M}$ a bounded, coercive operator (encoding the weight $p\chi_{Y_f} + \chi_{Y_s}$). The precise form is not essential; what matters is that $\mathcal{R}(p)$ is the resolvent of a sectorial operator with spectrum contained in $\{z: \Re z \geq \lambda_0\}$ and that ϕ is fixed.

From standard resolvent estimates for sectorial operators, it follows that there exist constants $C > 0$ and $\beta \in (0, \lambda_0)$ such that

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\| \mathcal{R}(p) \|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{1+|p|}, \qquad \Re p > -\beta. \qquad (7.2)$$

Indeed, the resolvent of a sectorial operator decays like $1/|p|$ along lines parallel to the imaginary axis, as long as one stays to the right of the spectral bound.

Combining (7.1) and (7.2) , we obtain

$$|\alpha(p)| \leq \| \mathcal{R}(p) \|_{\mathcal{L}(\mathcal{H})} \| \phi \|_{\mathcal{H}}^2 \leq \frac{C'}{1+|p|}, \qquad \Re p > -\beta, \qquad (7.3)$$

for some constant $C' > 0$ independent of p.

3. α as Laplace transform of c and analytic extension. By definition of the memory kernel $c(t)$, the function $\alpha(p)$ is the Laplace transform of $c(t)$ for $\Re p > 0$:

$$\alpha(p) = \int_0^{+\infty} e^{-pt}\, c(t)\, dt, \qquad \Re p > 0.$$

The estimate (7.3) shows that $\alpha(p)$ extends holomorphically to the larger half-plane $\Re p > -\beta$ and satisfies a polynomial decay estimate in $|p|$ there. Thus the Laplace transform of $c(t)$ admits an analytic continuation to the half-plane $\Re p > -\beta$, with the bound (7.3).

4. Exponential decay of $c(t)$ via Laplace inversion. We now use a standard result from Laplace-transform theory: if a function α is the Laplace transform of a finite measure (or a tempered distribution) c supported on $[0, \infty)$, and extends holomorphically to $\Re p > -\beta$ with at most polynomial growth in $|p|$ in every vertical strip, then c is exponentially decaying with rate at least β.

More precisely, consider the Bromwich inversion formula with a vertical line $\Re p = \sigma \in (0, \beta)$:

$$c(t) = \frac{1}{2\pi i} \int_{\Re p = \sigma} e^{pt}\, \alpha(p)\, dp, \qquad t \geq 0.$$

Using the analytic continuation of $\alpha(p)$ to $\Re p > -\beta$, we can shift the line of integration to $\Re p = -\beta + \varepsilon$, for any fixed $\varepsilon \in (0, \beta)$, without crossing any singularities (there are none in the half-plane $\Re p > -\beta$). Thus

$$c(t) = \frac{1}{2\pi i} \int_{\Re p = -\beta + \varepsilon} e^{pt}\, \alpha(p)\, dp.$$

Hence

$$|c(t)| \leq \frac{1}{2\pi} \int_{\Re p = -\beta + \varepsilon} |e^{pt}|\, |\alpha(p)|\, |dp|.$$

On the line $\Re p = -\beta + \varepsilon$, we have $|e^{pt}| = e^{(-\beta + \varepsilon)t}$, and the bound (7.3) gives

$$|\alpha(p)| \leq \frac{C'}{1+|p|}, \qquad \Re p = -\beta + \varepsilon.$$

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
Therefore

$$|c(t)| \leq C'' e^{(-\beta+\varepsilon)t} \int_{\mathbb{R}} \frac{1}{1+|\xi|}\, d\xi \leq M e^{-(\beta-\varepsilon)t}, \qquad t \geq 0,$$

for some constants $C'', M > 0$ independent of t. Since $\varepsilon > 0$ is arbitrary in $(0, \beta)$, we may replace $\beta - \varepsilon$ by a (smaller) positive constant $\tilde{\beta}$, and obtain

$$|c(t)| \leq M e^{-\tilde{\beta} t}, \qquad t \geq 0.$$

Renaming $\tilde{\beta}$ as β yields the claimed exponential decay.

5. Dependence on the indices. The constants M and β obtained above depend on the spectral gap λ_0 of the spatial operators and on the norm of the fixed vector ϕ in (7.1). Since there are only finitely many index combinations (i, j, k, h) and the underlying operators are the same for all these combinations, we can choose M and β uniformly for all $1 \leq i, j, k, h \leq 3$.

This completes the proof of Theorem 13. □

7.3. Well-posedness and stability of the homogenized model

Thanks to Theorems 12 and 13, the homogenized integro–differential equation of Theorem 6 can be treated as a damped hyperbolic equation with memory, for which well-posedness and stability are classical.

Theorem 14 (Well-posedness and stability). *Let $f \in L^2(0,T; L^2(\Omega)^3)$ and assume that the conditions of Theorems 12 and 13 hold. Then the homogenized problem of Theorem 6 admits a unique solution*

$$w^\star \in L^\infty(0,T; H_0^1(\Omega)^3) \cap W^{1,\infty}(0,T; L^2(\Omega)^3),$$

and there exists a constant $C > 0$, independent of f, such that

$$\| w^\star \|_{L^\infty(0,T;H_0^1(\Omega)^3)} + \| \partial_t w^\star \|_{L^\infty(0,T;L^2(\Omega)^3)} \leq C \ \| f \|_{L^2(0,T;L^2(\Omega)^3)}.$$

Moreover, the solution depends continuously on the data and on the microscopic geometry (through the tensor α and the scalar θ).

Sketch of the proof. Define the energy functional

$$E(t) := \frac{1}{2} \| \partial_t w^\star(t) \|_{L^2(\Omega)^3}^2 + \frac{\theta}{2}$$
$$\| w^\star(t) \|_{L^2(\Omega)^3}^2 + \frac{1}{2} \sum_{i,j,k,h} \int_0^t \int_\Omega c_{ijkh}\,(t-s)\ \partial_{x_h x_j}^2 w_k^\star(x,s)\ \partial_{x_h x_j}^2 w_k^\star(x,t)\, dx\, ds.$$

Using the symmetry and positive definiteness of α and the exponential decay of c, one shows that $E(t)$ is equivalent to the natural norm in $H_0^1(\Omega)^3 \times L^2(\Omega)^3$ and that $E(t)$ satisfies an energy inequality of the form

$$\frac{d}{dt} E(t) \leq C\, E(t) + \| f(t) \|_{L^2}^2.$$

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
Gronwall's lemma then yields the stated estimate. A Galerkin approximation argument, combined with this energy inequality, gives existence and uniqueness in the stated functional spaces. Continuous dependence on the microscopic geometry follows from the continuity of α and θ with respect to the cell coefficients. □

7.4. Optimal time-regularity

Finally, we record a simple time-regularity result which shows that the memory term does not degrade the time regularity of the solution beyond what is dictated by the forcing term.

Theorem 15 (Time-regularity). *If, in addition, $f \in H^1(0,T;L^2(\Omega)^3)$, then the solution $w^\star$ of Theorem 14 satisfies*

$$w^\star \in H^2(0,T;H^{-1}(\Omega)^3) \cap H^1(0,T;H_0^1(\Omega)^3),$$

and the following estimate holds:

$$\| w^\star \|_{H^1(0,T;H_0^1(\Omega)^3)} + \| w^\star \|_{H^2(0,T;H^{-1}(\Omega)^3)} \leq C \, \| f \|_{H^1(0,T;L^2(\Omega)^3)},$$

for some constant $C > 0$ depending only on the coefficients of the homogenized model and on Ω.

Proof. We recall that $w^\star$ is the unique solution of the homogenized integro–differential problem

$$\partial_t^2 w_i^\star(x,t) + \theta\, w_i^\star(x,t) - \sum_{j,k,h=1}^{3} \left(c_{ijkh} * \partial_{x_h x_j}^2 w_k^\star \right)(x,t) = f_i(x,t), \tag{7.4}$$

in $\Omega \times (0,T)$, with $w^\star = 0$ on $\partial\Omega \times (0,T)$ and zero initial data

$$w^\star(x,0) = 0, \qquad \partial_t w^\star(x,0) = 0 \quad \text{in } \Omega.$$

We work in the weak (variational) setting. Set

$$V := H_0^1(\Omega)^3, \qquad H := L^2(\Omega)^3, \qquad V' := H^{-1}(\Omega)^3.$$

Then $V \hookrightarrow H \hookrightarrow V'$ with continuous and dense embeddings. Theorem 14 ensures that

$$w^\star \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H),$$

and that the variational identity corresponding to (7.4) holds in V'.

1. Variational formulation and notation. For each $\varphi \in V$ and almost every $t \in (0,T)$, the variational form of (7.4) reads

$$\begin{aligned} \langle \partial_t^2 w^\star(t), \varphi \rangle_{V' \times V} \quad &+ \theta\, (w^\star(t), \varphi)_H \\ &- \sum_{i,j,k,h=1}^{3} \int_\Omega \left(c_{ijkh} * \partial_{x_h x_j}^2 w_k^\star \right)(x,t)\, \varphi_i(x)\, dx \\ &= (f(t), \varphi)_H. \end{aligned} \tag{7.5}$$

We may rewrite the memory term as

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$(c * Aw^\star)(t) := \sum_{i,j,k,h} \int_0^t c_{ijkh}\,(t-s)\,\frac{\partial^2 w_k^\star}{\partial x_h\,\partial x_j}(s)\,ds,$$

where A is the (vector-valued) second-order elliptic operator

$$(A\psi)_i := -\sum_{j,k,h=1}^{3} a_{ijkh}\,\partial^2_{x_h x_j}\psi_k,$$

with coefficients a_{ijkh} corresponding to the homogenized tensor (carefully defined in Section 4). Under the symmetry and coercivity assumptions of Theorem 12, A is a self-adjoint, strongly elliptic operator from V to V'.

Thus the homogenized equation can be viewed abstractly as

$$\partial_t^2 w^\star(t) + \theta w^\star(t) + \int_0^t K\,(t-s) A w^\star(s)\,ds = f(t), \qquad (7.6)$$

in V', where $K(t)$ is a scalar (actually tensor-valued) kernel built from $c_{ijkh}(t)$.

By Theorem 13, for each fixed i, j, k, h,

$$\left|c_{ijkh}(t)\right| \le M\,e^{-\beta t}, \qquad t \ge 0,$$

for some $M, \beta > 0$. Therefore the associated operator-valued kernel $K(t)$ satisfies

$$\| K(t) \|_{\mathcal{L}(V,V')} \le C_0 e^{-\beta t}, \qquad t \ge 0, \qquad (7.7)$$

for some constant $C_0 > 0$.

2. Differentiation of the equation and regularity for $\partial_t w^\star$. Assume now $f \in H^1(0,T;H)$. Formally differentiating (7.6) with respect to t, we obtain

$$\partial_t^3 w^\star(t) + \theta\,\partial_t w^\star(t) + \int_0^t K\,(t-s) A\,\partial_t w^\star(s)\,ds = \partial_t f(t) + K(0) A w^\star(t), \qquad (7.8)$$

where we have used the identity

$$\frac{d}{dt}\int_0^t K\,(t-s) A w^\star(s)\,ds = \int_0^t K\,(t-s) A\,\partial_t w^\star(s)\,ds + K(0) A w^\star(t),$$

valid in V' for kernels $K(t)$ satisfying (7.7) and $w^\star \in W^{1,\infty}(0,T;H)$.

Define $v := \partial_t w^\star$. Then (7.8) can be rewritten as

$$\partial_t^2 v(t) + \theta v(t) + \int_0^t K\,(t-s) A v(s)\,ds = g(t), \qquad (7.9)$$

where

$$g(t) := \partial_t f(t) + K(0) A w^\star(t).$$

We claim that $g \in L^2(0,T;V')$, and then apply Theorem 14 to (7.9).

First, since $f \in H^1(0,T;H)$, we have $\partial_t f \in L^2(0,T;H) \subset L^2(0,T;V')$.

Received: Aug 14, 2025

International Journal of Applied Mathematics
Volume 39 No. 1s, 2026
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
Second, $K(0)$ is a bounded operator from V to V' (it involves only the values $c_{ijkh}(0)$ and the elliptic operator A), and $w^\star \in L^\infty(0,T;V)$ by Theorem 14. Hence

$$t \mapsto K(0)Aw^\star(t) \in L^\infty(0,T;V'),$$

and in particular in $L^2(0,T;V')$. Thus

$$g \in L^2(0,T;V').$$

Now, (7.9) has the same structure as the original homogenized equation (7.6), with unknown v, forcing term g and zero initial data:

$$v(0) = \partial_t w^\star(0) = 0, \qquad \partial_t v(0) = \partial_t^2 w^\star(0) = f(0),$$

where $f(0)$ is well defined in H since $f \in H^1(0,T;H)$ (continuous in time with values in H). Applying Theorem 14 to (7.9), we obtain

$$v \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H),$$

with the estimate

$$\| v \|_{L^\infty(0,T;V)} + \| \partial_t v \|_{L^\infty(0,T;H)} \leq C_1 \| g \|_{L^2(0,T;V')}. \tag{7.10}$$

Recalling that $v = \partial_t w^\star$, this implies

$$\partial_t w^\star \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H),$$

which is equivalent to

$$w^\star \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H). \tag{7.11}$$

In particular,

$$w^\star \in H^1(0,T;V), \qquad w^\star \in H^2(0,T;H).$$

3. From H to H^{-1} regularity. Since $H \hookrightarrow V'$ continuously, $H^2(0,T;H)$ embeds into $H^2(0,T;V')$. Therefore

$$w^\star \in H^2(0,T;V').$$

Combining this with $w^\star \in H^1(0,T;V)$ from (7.11), we obtain

$$w^\star \in H^1(0,T;V) \cap H^2(0,T;V').$$

Recalling that $V = H_0^1(\Omega)^3$ and $V' = H^{-1}(\Omega)^3$, this is precisely

$$w^\star \in H^1(0,T;H_0^1(\Omega)^3) \cap H^2(0,T;H^{-1}(\Omega)^3).$$

4. A priori estimate. We now derive the estimate in the statement. From (7.10) and the definition of g, we have

$$\| \partial_t w^\star \|_{L^\infty(0,T;V)} + \| \partial_t^2 w^\star \|_{L^\infty(0,T;H)} \leq C_1 \left(\| \partial_t f \|_{L^2(0,T;H)} + \| K(0)Aw^\star \|_{L^2(0,T;V')} \right).$$

The second term is bounded by

$$\| K(0) \|_{\mathcal{L}(V,V')} \| A \|_{\mathcal{L}(V,V')} \| w^\star \|_{L^2(0,T;V)} \leq C_2 \| f \|_{L^2(0,T;H)},$$

Received: Aug 14, 2025

using the basic a priori estimate of Theorem 14. Hence

$$\| \partial_t w^\star \|_{L^\infty(0,T;V)} + \| \partial_t^2 w^\star \|_{L^\infty(0,T;H)} \leq C_3 \big(\| \partial_t f \|_{L^2(0,T;H)} + \| f \|_{L^2(0,T;H)} \big).$$

Together with the basic estimate for $w^\star$ itself, this yields

$$\| w^\star \|_{H^1(0,T;V)} + \| w^\star \|_{H^2(0,T;V')} \leq C \ \| f \|_{H^1(0,T;H)},$$

for some constant $C > 0$ depending only on the coefficients of the homogenized model and on Ω. This is exactly the estimate in the statement. □

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