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APPLICATION OF SEMIGROUP THEORY IN MODELING AND MITIGATING CORRUPTION IN GOVERNMENT PROCUREMENT

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Abstract

Government procurement is among the most corruption – prone functions of governance, with major risks including favoritism, bid-rigging, specification manipulation, collusion among bidders, and monitoring failures. Traditional approaches such as game theory, network analysis, dynamic models have offered insights, but often treat corruption acts or agents in isolation or via probalistic transitions, rather than formally modeling how corrupt acts enable or reinforce one another structurally. This paper presents a novel algebraic framework based on semigroup theory to model government procurement corruption. In this work, acts or states of corruption constitute a semigroup, with the binary operation modeling the enabling of corrupt acts or conditions and key algebraic features are mapped to procurement corruption phenomena. This interdisciplinary study offers a novel algebraic perspective to compliment existing political and economic models of corruption.

Keywords: Semigroup theory, government procurement, corruption modeling, absorbing states, anti-corruption policy.

AMS Classification: 20M10

1. Introduction

Government procurement is responsible for large shares of public spending. Because of the volume of funds, the necessity of discretion, interactions between public and private actions, and frequent information asymmetries, procurement is highly susceptible to corruption. Corruption in procurement wastes resources, leads to poor quality public works, undermines public trust, distorts markets, and can entrench inefficiencies.

Many studies have characterized corruption risks factors (lack of transparency, weak oversight, specifications manipulation, collusion), for example, Gnaldi and Del [3] analyzed how corruption rise increase during emergency procurement situation when oversight is weaker and finding level rise. The authors develop indicators to measure corruption vulnerability across procurement stages and emphasis the importance of monitoring system during crises. Waxenecker and Prell [10] applied stochastic network modeling to examine corruption patterns in public procurement contracts. This study showed that collusion, political influence, and

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spending concentration play key roles in sustaining corruption networks in procurement systems. Common forms of procurement corruption such as bid-rigging, market sharing, and favoritism were studied by Zhu [11] while the most corruption-prone stages in public procurement systems using the analytic hierarchy process (AHP) were studied by Adjorlolo et al in [1]. In 2025, Santos et al [8] analyzed modern data- driven methods used to detect fraud and corruption in public procurement. It highlights the growing use of machine learning, data analytics, and predictive models for identifying suspicious procurement activities.

There is less work that models how corrupt acts compose or interact in chains or enables one another, which is essential to understanding why some procurement systems become deeply corruption (system capture), while others remain relatively resilient. Valverde et al [9] developed an agent-based mathematical model to analyze corruption dynamics between government officials and contractors. Their results showed that institutional controls and salary structures can significantly influence corruption levels in public contract systems. Medina et al [6] used machine learning and network analysis to identify fraudulent suppliers in government procurement systems. Their model significantly improved the detection of corruption contracts compared with traditional risk indicators.

However, mathematical abstractions especially from algebraic structures offer under utilized yet powerful tools to model complex, interacting systems such as corruption networks.

Semigroup theory studies set with associative binary operations. It offers tools for modeling structure – how elements, in particular, corrupt acts or states compose, how some combinations become self-reinforcing, how subsystems emerge.

Such tools can help identify persistent and dangerous corruption patterns (e.g, acts which once they occur, lead inexorably towards system capture), model the enabling relationships among acts (which acts tend to follow/reinforce others), understand how interventions (reforms, transparency, oversight) might block particular composition paths, or “neutralize” certain corrupt enabling acts, provide complementary qualitative structural analysis to probalistic/dynamic/graph/network models. There has been recent mathematical of corruption and semigroup application in modeling, for example, see [7], [1] and [2].

This paper presents the application of semigroup theory to model and analyze corruption dynamics.

2. Preliminaries

Definition 2.1. A semigroup is an algebraic structure $(S, \circ)$ where S is a non-empty set and $\circ$ is an associative binary operation : $\forall\, a, b, c \in S,\ (a \circ b) \circ c = a \circ (b \circ c)$.

Definition 2.2. An element $a \in S$ is called an idempotent if $a \circ a = a^2 = a$. Viewing elements as corrupt acts, self-sustaining corruption (e.g. institutional bribery) can be seen as idempotents.

Definition 2.3. An element $a \in S$ is called an absorbing or zero element if if $a \circ b = a, \forall\, b \in S$. In our study, absorbing or zero elements are irreversible corruption nodes (e.g. godfatherism).

Definition 2.4. An element $1 \epsilon S$ is called an identity element if $a \circ 1 = 1 \circ e = a$. In our study, identity element can be seen as anti-corruption acts that do not change the system state.

Definition 2.5. Let $(S_1, \circ)$ and $(S_2, *)$ be two semigroups. A mapping $\theta : S_1 \rightarrow S_2$ is a homomorphism if $\forall\, a, b \,\epsilon\, S$,

$$\theta(a \circ b) = \theta(a) * \theta(b)\,.$$

Suppose corruption in different agencies (say health and education) can be modeled as semigroups S_1 and S_2, then a homomorphism $\theta : S_1 \rightarrow S_2$ can model how corruption patterns transfer between departments.

Definition 2.6. Let S be a finite or countable set of corruption states or acts on procurement. Consider a simplified procurement system with ten elements namely;

b − bribe

c − collusion among bidders

s − specification manipulation/unfair technical requirements

f − fraudulent misrepresentation in contract execution

m – monitoring/over sight failure

v – vendor favoritism

n – nepotism

w – whistle blower/auditing/transparency/reform act

z − system capture (absorbing state)

1 − reform identity element.

Define a binary operation $* : S \times S \rightarrow S$ by the rule; $a * b =$ state/ act corresponding to "after act/state a, act/state b is either enabled, facilitated, or its effect is amplified or continues in presence of a".

Definition 2.7 [4]. A non-empty subset A of S is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$ and two sided or an ideal if it is both a left and a right ideal.

Definition 2.8 [4]. Let S be a semigroup. A relation ρ on the set S is said to be left compatible with respect to the operation on S if

$$(\forall\, s, t, a \,\epsilon\, S), \quad (s, t) \,\epsilon\, \rho \text{ implies } (as, at) \,\epsilon\, \rho,$$

and right compatible if

$$(\forall\, s, t, a \,\epsilon\, S), \quad (s, t) \,\epsilon\, \rho \text{ implies } (as, at) \,\epsilon\, \rho\,.$$

It is said to be compatible if it is both left and right compatible.

An equivalence relation (a relation that is reflexive, symmetric and transitive) ρ is said to be a congruence if for all $a, b, c, d \,\epsilon\, S$, $a \,\rho\, b$ and $c \,\rho\, d$ implies $ac \,\rho\, bd$.

Definition 2.9 [4]**.** Let S be a semigroup and let $a \epsilon S$. The principal right ideal general generated by $'a'$ is the set $aS' = aS \cup \{a\}$. Also, the left principle ideal generated by $'a'$ is $S'a = Sa \cup \{a\}$. Now suppose S is a semigroup and $a, b \epsilon S$. Then a and b are $\mathcal{R}$-related written as $a\,\mathcal{R}\,b$ if a and b generate the same principal right ideal. Similarly, a and b are $\mathcal{L}$-related if they generate the same principal left ideal, in which case we write $a\,\mathcal{L}\,b$. More so, a and b are said to be $\mathcal{J}$-related if they generate the same principle two-sided ideal $(S'aS' = aS \cup Sa \cup SaS \cup \{a\})$. By the above definition,

$a\,\mathcal{L}\,b$ if and only if for some $x, y \epsilon S', a = xb$ and $b = ya$.

$a\,\mathcal{R}\,b$ if and only if for some $x, y \epsilon S', a = bx$ and $b = ay$.

$a\,\mathcal{J}\,b$ if and only if for some $x, y, u, v \epsilon S', xay = b$ and $ubv = a$.

An important property of $\mathcal{L}$ and $\mathcal{R}$ is that $\mathcal{L}$ is a right congruence while $\mathcal{R}$ is a left congruence. More so, $a\,\mathcal{H}\,b$ if and only if $a\,\mathcal{L}\,b$ and $a\,\mathcal{R}\,b$. In fact, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Definition 2.10 [4]**.** Let S be a semigroup. An element $a \epsilon S$ is regular if there exists $x \epsilon S$ such that $a = axa$ and we say S is regular if every element of S is regular.

3. Semigroup Construction

Now let $S = \{b, c, s, f, m, v, n, w, z, 1\}$ be a set representing the procurement system element. We define a binary operation $* : S \times S \rightarrow S$ that models the intersection between these elements such that $\forall\, a, b, c \epsilon S, (a * b) * c = a * (b * c)$.

To define the Cayley table, we make the following assumption:

i. Corrupt acts reinforce each other: combination of corrupt elements lead to inverse corruption.

ii. Whistle-blowers (w) can reduce or nullify certain corrupt acts.

iii. Reform identity (1) acts as a right and left identity;

$$a * 1 = 1 * a = a \;\; \forall\, a \epsilon S.$$

iv. System capture (z) absorbs other corrupt acts;

$$a * z = z * a = z, \forall\, a \epsilon S,$$

modeling how deep corruption overides others.

v. $w * a = 1$ for $a \epsilon S$, meaning whistle-blowing can neutralize individual corruption types.

$*$	b	c	s	f	m	v	n	w	z	1
b	z	z	z	z	z	z	z	z	z	b
c	z	z	z	z	z	z	z	z	z	c
s	z	z	z	z	z	z	z	z	z	s
f	z	z	z	z	z	z	z	z	z	f
m	z	z	z	z	z	z	z	z	z	m

v	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*v*
n	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*n*
w	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*w*	*1*	*w*
z	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*z*	*z*
1	*b*	*c*	*s*	*f*	*m*	*v*	*n*	*z*	*z*	*1*

The absorbing nature of the system capture element (z) indicates that once corruption has fully infiltrated the system, no further reforms or interventions can alter the state, highlighting the importance of early intervention. The whistle-blower resets corrupt elements to reform (1) while reform elements act neutrally.

We now have the following results;

Lemma 3.1. $(S, \ *)$ *is a semigroup.*

Proof. The proof follows from the Cayley table. Moreover, the idempotents in S are w, z an 1.

In fact, it can be easily seen that every corrupt element $x \epsilon b, c, s, f, m, v, n$ is nilpotent with $x^2 = z$.

Lemma 3.2. z *is a two- sided ideal.*

Proof. For any $a \epsilon S, \ a * z = z, \ z * a = z,$ so that $S * z \subseteq z,$ and $z * S \subseteq z.$

Hence, *z* is the minimal two-sided ideal.

It is known in [4] that every finite semigroup has at least one minimal ideal. In this case, *z* is minimal.

Theorem 3.3. z is $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}$ *– minimal.*

Proof. Since for all $x \epsilon S, x \ * z = z$ and $z * x = x,$ it follows that, left and right ideals containing z are all mapped to $z,$ so that $S'z = z$ and z$S' = z.$

Hence, $z \, \mathcal{L} \, z, \ z \, \mathcal{R} \, z, \ z \, \mathcal{J} \, z,$ and it is in its own minimal class.

Lemma 3.4. *The whistle-blower* w *forms its own* $\mathcal{H}$*-class.*

Proof. The proof follows from the fact that no corruption element can map back to w through left or right multiplication.

Lemma 3.5. 1 *is the center of the semigroup* S *and lies in a singleton* $\mathcal{H}$*-class.*

Proof. From definition, we have that, $x * 1 = 1 * x = x \ \forall \, x \epsilon S,$ so that $1 \epsilon Z(S),$ the center of the semigroup. Thus, no other element has this identity behavior.

So, 1 is in its own $\mathcal{H}$-class.

Lemma 3.6. *Whistle-blower w is not* $\mathcal{L}$*-related to any element.*

Proof. Since $S' * w = 1 \neq z = S'b,$ it follows that $\{w, b\} \ \notin \ \mathcal{L}.$

It can be easily checked that all other corrupt elements namely; b, c, s, f, m, v, n, z are $\mathcal{L}$-related.

Let us now consider the congruence partition of S. We define ρ as the equivalence relation that partition S into the following equivalent classes;

i) Corruption classes $(C): C = b, c, s, f, m, v, n, z$

ii) Whistle -blower class $(W): W = u$

iii) Identity class $(I) = 1$

Theorem 3.7. ρ *is a congruence on* $(S, *)$.

Proof. We are to show that for any $x_1 \, \rho \, x_2$ and $y \in S$, we have that $x_1 * y \, \rho \, x_2 * y$ and $y * x_1 \, \rho \, y * x_2$. We consider all combination of classes:

Case 1. $x_1, x_2 \in C$. *Then* $x_1 * y, x_2 * y \in C$ *because all interactions within* C *lead to* $z \in C$.

So, we have that $x_1 * y \, \rho \, x_2 * y$.

Case 2. $x_1, x_2 \in W$ (*both are w*). Then $x_1 = x_2 = w$, *so trivially congruent.*

Case 3. $x_1, x_2 \in I$ (*both are I*). *Again, trivially congruent under all operation.*

Consequently, for $x \in W, \; y \in C, \; w * y \in 1, m \subseteq C \cup I$.

All such products say within the union of classes, and their equivalence is preserved. Also, for $x \in C, \; y \in W, \; x * w = 1 \in I$ – same for all $x \in C$.

Thus, ρ satisfies the compatibility condition for a congruence.

It is known in Definition 2.5 that homomorphisms preserve structure and allow attraction of complex systems.

Now suppose we have a simplier target semigroup $P = C, R, I$ where $C =$ corruption, $R =$ Reform (whistle-blower) and $I =$ identity. Define a map $\theta : S \to P$ by the rule

$$\theta(x) = \begin{cases} C & \text{if } x = \{b, c, s, f, m, v, n, z\} \\ R & \text{if } x = w \\ I & \text{if } x = 1 \end{cases}$$

for $x \in S$.

We define operation $\circledast$ on P based on θ:

$\circledast$	C	R	I
C	C	I	C
R	C	R	R
I	C	R	I

Theorem 3.8. $\theta : S \to P$ *is a semigroup homomorphism.*

Proof. We are to show that

$$\theta(x * y) = \theta(x) \circledast \theta(y), \ \forall\, x, y \in S.$$

We check the key cases:

i) $x, y \in b, c, s, f, m, v, n, z :\ \ x * y = z,$ and $\theta(z) = C,\ \theta(x) = \theta(y) = C$ and $C \circledast C = C.$

ii) $x = w,\ y \in b, c, s, f, v, n\ : w * y = 1,$ so that $\theta(w * y) = I$ and $\theta(w) = R,\ \theta(y) = C,$ so that we have $R \circledast C = I.$

iii) $x = w,\ y = m$: $w * m = m,\ so\ \ \theta(w * m) = C,$ and $\theta(w) = R,\ \theta(m) = C,$ so $R \circledast C = C.$

Obviously, the kernel of the map above $\ker(\theta)$ respects the semigroup operation, it is a congruence. It partitions S into three congruence classes $[b] = \{b, c, s, f, m, v, n, z\},\ [w] = \{w\},\ [1] = \{1\}.$

Theorem 3,9. *S is not a regular semigroup.*

Proof. The proof is a routine check.

Theorem 3.10. *The whistle-blower w is a left-inverse for the primary corrupt elements.*

Proof. The proof follows from the fact that $\forall\, x \in \{b, c, s, f, m, v, n, z\},\ w * x = 1.$

Theorem 3.10 above shows that the presence of a whistle-blower mechanism neutralizes individual corruption events, returning the system to the reformed state. This does not apply to *z*, as system capture is irreversible.

Theorem 3.11. *The semigroup* $(S,\ *)$ *admits a faithful representation.*

Proof. Define $X = S$ and $\varphi : S \to \mathcal{T}(X)$ by the rule;

$\varphi(a) : X \to X,\ \ \varphi(a)(x) = a * x.$ For each $a \in S,\ \varphi(a)$ is the left multiplication map by *a*. The map φ is a homomorphism since

$$\varphi(a * b)(x) = \varphi(a * b) * x = a * (b * x)$$
$$= \varphi(a)\big(\varphi(a)(x)\big).$$

To prove faithfulness, assume $\varphi(a) = \varphi(b).$ Then for all $x \in X,$ we have that;

$$a * x = \varphi(a)(x) = \varphi(b)(x) = b * x.$$

In particular, for $x = 1,$ the identity, $a * 1 = a = b * 1 = b.$

Thus, $a = b,$ proving φ is injective.

Remark 3.12. From Theorem 3.11 above, we have the following remark,

(i) The faithful representation models each corruption element as a transformation on the procurement state.

(ii) Left multiplication by $'a'$ models the impact of corruption action $'a'$ on any procurement state.

(iii) This representation enables studying corruption evolution as transformation corruption.

(iv) The identity reform 1 corresponds to no change (reform stabilizers).

(v) Whistle-blower w transformation acts as corrective or revitalizing operations, reducing corruption states.

(vi) The presence of a faithful representation ensures no loss of information in algebraic modeling, enabling algebraic and computational tools to analyze corruption.

We conclude this section by presenting some results that will enable us to mitigate corruption.

Lemma 3.13. *Repeated application of* 1 and w *restores a mitigated corruption state.*

Proof. It can be easily seen from the Cayley table that

$$w * 1 = w = 1 * w, \qquad w * w = w.$$

Hence, sequences of reform and whistle-blower activities keep the system in a non-corruptive or reform state.

The Lemma below is an immediate consequence of Lemma 3.13.

Lemma 3.14. *Let* $x_1, x_2, \ldots, x_n \in S$, *and for some* $i, x_i = w$. If $x_{i+1} \in b, c, f, m, v, n, then$

$$x_1 * x_2 * \ldots * w * x_{i+1} * \ldots * \ x_n = x_1 * \ldots * 1 * \ldots * x_n.$$

4. Analysis and Discussion

From the results obtained in section 3, we have the following analysis;

4.1. Systematic corruption; The role of *z*

(a) The Cayley table shows that any interaction between two corruption related actions (e.g. bribe + collusion, or favoritism + nepotism) escalates into system capture (z).

(b) This reflects the real-world observation where isolated incidents of corruption can snowball into institutionalize corruption, difficult to reverse.

4.2. Reform mechanisms: The role of *w* and 1.

(a) The whistle-blower (w) acts as a reset mechanism. Interaction with any corrupt element returns the system to reform (1).

(b) This reflects, the power of transparency and exposure to disrupt corruption cycles.

(c) identity reform element (1) allows continuity of reform. It sustains the current system state without escalation.

4.3. Fragility of oversight (*m*)

m combines with any other corruption to form z, indicating that oversight failure is a key accelerant of system corruption.

4.4. Visual insights

The Cayley table has three distinct regions namely;

(a) Corruption interaction zone: leads to z.

(b) Reform zone (w, 1): neutral or restorative

(c) Whistle-blower interactions: always result in 1.

4.5. Mitigation insights

(a) Early intervention is key: if w is applied early (before system capture), corruption can be neutralized.

(b) Once system capture (z) occurs, recovery is nearly impossible within the model highlighting the importance of prevention.

(c) Reform (1) is not enough alone, without mechanisms like w, corruption interactions persist.

From our algebraic modeling, we can now derive several actionable insights, and policy recommendations.

Insight	**Policy Recommendation**
Systematic corruption compounds.	Target root causes before they combine.
Whistle-blowing is powerful.	Perfect whistle-blowers.
Oversight failure is critical.	then independent monitoring bodies.
Reform needs active defense.	Combine policy reform (1) and enforcement (w).
System capture is hard to reverse.	Focus on prevention, not just punishment.

5. Conclusion

This study demonstrates that semigroup theory offers a structured approach to modeling corruption in government procurement. By formalizing corruption elements and their interactions within an algebraic system, we uncover how corruption escalates and how specific intervention-especially whistle-blowing and monitoring can disrupt the process.

While simplified, the model highlights strategic points for intervention and reform. Future research can expand this model using probalistic weights, non-commutative structures, or simulations of evolving procurement environments.

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