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RESTRAINED GEO CHROMATIC NUMBER OF A SOME GRAPHS

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Abstract

A set S of vertices is a geodetic set if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the geodetic number of G and is denoted by $g(G)$. A set S of vertices in G is a restrained geodetic set if S is a geodetic set and the subgraph $G[V\setminus S]$ induced be $V\setminus S$ has no isolated vertex. A subset $S \subseteq V(G)$ is said to be a geo chromatic set if S is both a geodetic and a chromatic set of G. A subset $S \subseteq V(G)$ is a restrained geo chromatic set of G if S is a geo chromatic set and the subgraph $G[V\setminus S]$ induced by $V\setminus S$ has no isolated vertex. The minimum cardinality of a restrained geo chromatic number of G and is denoted by $\chi_{g_r}(G)$. In this paper, we determine bounds for it and characterize graphs which realize these bounds for some known graphs.

Keywords: Geo Chromatic set – Geo Chromatic Number – Restrained Geo Chromatic set – Restrained Geo Chromatic Number.

Mathematics Subject Classification (2010) : 05C12

1. Introduction

By a graph $G = (V, E)$, we mean a simple undirected connected graph. The order and size of G are denoted by m and n, respectively. For basic theoretic terminology, we refer to Harary [1,8]. For any two vertices u and v in a connected graph G, the distance $d(x, y)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(x, y)$ is called $u - v$ geodesic P if v is a vertex of P including the vertices u and v. For any vertex u of G, the eccentricity of u is defined as $e(u) = \max\{d(u, v); v \in V(G)\}$. The radius $r(G)$ and diameter $diam(G)$of G are defined as $r(G) = \min\{e(v); v \in V(G)\}$ and $diam(G) = \max\{e(v); v \in V(G)\}$ respectively. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v of G is called an extreme vertex of G if the subgraph induced by its neighbours is complete.

The closed interval $I[x, y]$ consists of all vertices lying on some $u - v$ geodesic of G while for $S \subseteq V(G), I[S] = \cup_{x,y\in S} I[x, y]$, A set S of vertices of G is a geodetic set of G if $I[S] = V(G)$, and the minimum cardinality of a geodetic set of G is the geodetic number $g(G)$ of G. The geodetic number of a graph and its variants have been studied several authors

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in [2,9,10]. A set S of vertices is called chromatic set if S contains all vertices of distinct colours in G. The chromatic number of G is the chromatic sets in G and is denoted by $\chi(G)$. A set S of vertices in G is called a restrained geodetic set of G if S is a geodetic set and the subgraph $G[V\backslash S]$ induced by $V\backslash S$ has no isolated vertex. The minimum cardinality among all the restrained geodetic set of G is the restrained geodetic number of G and is denoted by $g_r(G)$. A set $S \subseteq V(G)$ is called a geo chromatic set of G if S is both a geodetic set and a chromatic set of G. It was introduced and studied in [1].

2. Restrained Geo Chromatic Number of Some Graphs

Definition 2.1 [12]

A set S of geo chromatic set of G is a restrained geo chromatic set of G such that the subgraph induced by $V\backslash S$ has no isolated vertices. The minimum cardinality of a restrained geo chromatic set of G is the restrained geo chromatic number of G and is denoted by $\chi_{g_r}(G)$.

Example 2.2

For the graph G given in Figure 2.1, $S = \{v_1, v_2, v_3, v_5\}$ is a minimum geo chromatic set of G and so $\chi_{g_r}(G) = 4$.

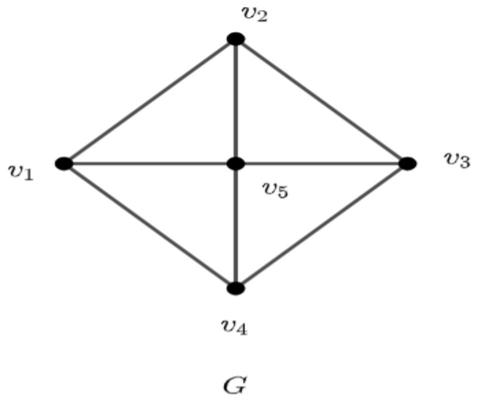

Figure 2.1

But $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$ is a minimum restrained geo chromatic set of G and so $\chi_{g_r}(G) = 5$.

Remark 2.3

The minimum restrained geo chromatic set of G need not be unique. For the graph G given in Figure 2.2, $S = \{v_1, v_4\}$ is a chromatic set of G such that $\chi(G) = 2$. Also, $S_1 = \{v_4, v_6, v_8\}$ is a minimum restrained geodetic set of G and so $g_r(G) = 3$. But S is not a restrained geodetic set of G and S_1 is not a chromatic set of G.

Here $S_2 = \{v_1, v_4, v_6, v_8\}$ is a minimum restrained geo chromatic set of G and so $\chi_{g_r}(G) = 4$.

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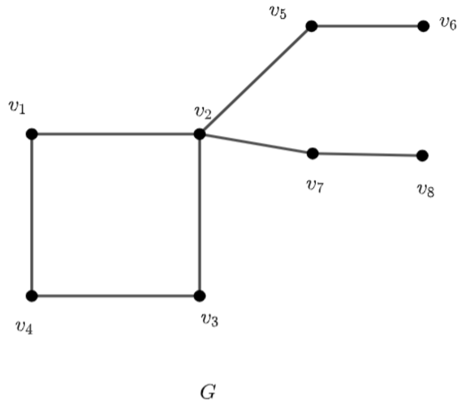

Figure 2.2

It is clear from the figure 2.2, $S_3 = \{v_4, v_5, v_6, v_8\}, S_4 = \{v_4, v_6, v_7, v_8\}, S_5 = \{v_3, v_4, v_6, v_6\}$ are also minimum restrained geo chromatic sets of G.

Theorem 2.4

Let G be a connected graph of order n. Then $2 \leq g_r(G) \leq \chi_{g_r}(G) \leq n$.

Proof

A restrained geodetic set needs at least two vertices and so $g_r(G) \geq 2$. Since every restrained geo chromatic set is a restrained geodetic set of G, $g_r(G) \leq \chi_{g_r}(G)$. Moreover, it is clear that the set of all vertices of G is a geo chromatic set of G and so $\chi_{g_r}(G) \leq n$. Hence, $2 \leq g_r(G) \leq \chi_{g_r}(G) \leq n$.

Remark 2.5

All the inequalities in Theorem 2.4 are strict. For the graph G given in Figure 2.2, $g_r(G) = 3, \chi_{g_r}(G) = 4$ and $n = 8$. Therefore $2 < g_r(G) < \chi_{g_r}(G) < n$.

Corollary 2.6

Let G be a connected graph of order n. If $g_r(G) = n$, then $\chi_{g_r}(G) = n$.

Proof

This follows from Theorem 2.4.

Theorem 2.7 [12]

Each extreme vertices of a connected graph G belongs to every restrained geo chromatic set of G.

Theorem 2.8

For any complete graph $K_n (n \geq 2), \chi_{g_r}(K_n) = n$.

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Proof

All the vertices are extreme vertices of the complete graph K_n. Therefore, by Theorem 2.7, the restrained geo chromatic set of K_n must contain all the vertices of K_n and hence $\chi_{g_r}(K_n) = n$.

Theorem 2.9

For any star graph $K_{1,n-1}(n \geq 2), \chi_{g_r}\left(K_{1,n-1}\right) = n$.

Proof

For $n = 2, K_{1,n-1} = K_{1,1} \simeq K_2$ and by Theorem 2.7, $\chi_{g_r}\left(K_{1,1}\right) = 2$. Now assume $n \geq 3$. Let $V\left(K_{1,n-1}\right) = \{x, x_1, x_2, \dots, x_{n-1}\}$, where x is the only vertex of degree $n-1$ and each x_i are the end vertices adjacent to x. By Theorem 2.7, $S = \{x_1, x_2, \dots, x_{n-1}\}$ is a geodetic set of G, such that $V(K_{1,n-1})\backslash S$ has an isolated vertex. Therefore, that S is not a restrained geodetic set of $K_{1,n-1}$. Also it is clear that S is not a chromatic set of $K_{1,n-1}$. Thus that S itself is not a restrained geo chromatic set $K_{1,n-1}$ and so $\chi_{g_r}\left(K_{1,n-1}\right) > |S| = n-1$. By Theorem 2.4, $\chi_{g_r}\left(K_{1,n-1}\right) \leq n$. Hence, we conclude that $\chi_{g_r}\left(K_{1,n-1}\right) = n$.

Observation 2.10

(i) Every restrained geo chromatic set of G is a chromatic set of G.
(ii) Every restrained geo chromatic set of G is a restrained geodetic set of G.

Theorem 2.11

Let G be a connected graph of order dn, then $2 \leq g_r(G)\backslash\chi_{g_r}(G) \leq n$.

Proof

Every restrained geodetic set of G contains at least two vertices and so $g_r(G) \geq 2$. By Observation 2.10, we see that every restrained geo chromatic set is a restrained geodetic set of G. Since G is connected, that $V(G)$ us a restrained geo chromatic set of G and so $\chi_{g_r} \leq |V(G)| = n$. Hence, $2 \leq g_r(G) \leq \chi_{g_r}(G) \leq n$.

Remark 2.12

All the inequalities in Theorem 2.11 are sharp. For the complete graph $K_n, (n \geq 2)$ by Theorem 2.8, $\chi_{g_r}(K_n) = n$ and $g_r(K_n) = n$. Also, all the inequalities in Theorem 2.11 are strict. Consider the graph G given in Figure 2.2. It is clear that, $g_r(G) = 3, \chi_{g_r}(G) = 4$ and $n = 8$. Thus, $2 < g_r(G) < \chi_{g_r}(G) < n$.

Theorem 2.13

Let G be a connected graph of order $n \geq 2$, then $2 \leq \chi(G) \leq \chi_{g_r}(G) \leq n$.

Proof

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Every chromatic set of G contains at least two vertices and so $\chi(G) \geq n$, because $n \geq 2$. By observation 2.10, we see that every restrained geo chromatic set of G is a chromatic set of G. Thus $\chi(G) \leq \chi_{g_r}(G)$. Moreover, by Theorem 2.11, we conclude that $\chi_{g_r}(G) \leq n$. Hence, $2 \leq \chi(G) \leq \chi_{g_r}(G) \leq n$.

Remark 2.13

All the inequalities in Theorem 2.13 are sharp. For the complete graph $G = K_n (n \geq 2), \chi(G) = \chi_{g_r}(G) = n$. Also, all the inequalities in Theorem 2.13 are strict. Consider the graph G, given in Figure 2.3.

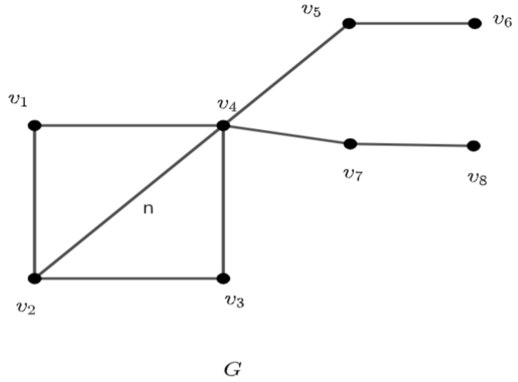

Figure 2.3

Here, $n = 8$ and $S = \{v_1, v_2, v_4\}$ is a minimum chromatic set of G and so $\chi(G) = 3$. Also it is clear that $S_1 = \{v_1, v_3, v_6, v_8, v_2\}$ is a minimum restrained geo chromatic set of G and so $\chi_{g_r}(G) = 5$. Hence, $2 < \chi(G) < \chi_{g_r}(G) < n$.

Theorem 2.14

For the path graph $P_n (n \geq 2), \chi_{g_r}(P_n) = \begin{cases} 2 \; if \; n \; is \; even \\ 3 \; if \; n \; is \; odd \end{cases}$.

Proof

Let $G = P_n: v_1 v_2 \ldots v_n$ be a path graph of order $n \geq 2$. We consider the following cases:

Case (i) n is even.

Consider $S = \{v_1, v_n\}$ restrained geodetic set of G. Also that S is a chromatic set of G. Thus S is a restrained geo chromatic set of G with minimum cardinality. Therefore, $\chi_{g_r}(G) = 2$.

Case (ii) n is odd

Consider $S = \{v_1, v_n\}$. Clearly, $S = \{v_1, v_n\}$ is a restrained geodetic set with minimum cardinality. But that S is not a chromatic set of G. Because we need minimum three

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colours to be proper colouring of G. Now consider, $S_1 = S \cup \{v_{n-1}\}$. Then it is clear that S_1 is a chromatic set of G. Therefore S_1 is a restrained geo chromatic set of G with minimum cardinality. Hence, $\chi_{g_r}(G) = 3$.

$$\text{Thus, } \chi_{g_r}(P_n) = \begin{cases} 2 \text{ } if \text{ } n \text{ } is \text{ } even \\ 3 \text{ } if \text{ } n \text{ } is \text{ } odd \end{cases}.$$

Theorem 2.15

$$\text{For the cycle graph } C_n (n \geq 3), \chi_{g_r}(C_n) = \begin{cases} n \text{ } if \text{ } n = 3,4,5 \\ 2 \text{ } if \text{ } n \equiv 2 (mod\ 4) \\ 3 \text{ } if \text{ } n \equiv 0 (mod 4) \\ 4 \text{ } if \text{ } n \equiv 1,3 (mod\ 4) \end{cases}.$$

Proof

Let $G = C_n: v_1 v_2 \ldots v_n v_1$ be a cycle graph of order $n \geq 3$. We consider for cases:

Case (i) Suppose $n = 3,4,5$.

If $n = 3$, then $C_3 \cong K_3$ and so by Theorem 2.8, $\chi_{g_r}(G) = 3$.

If $n = 4$, any two non-adjacent vertices only form a geodetic set of G. So the remaining vertices must be isolate. Therefore for $n = 4, V(G)$ is the only restrained geo chromatic set of G and so $\chi_{g_r}(G) \geq 4$. By Theorem 2.4, $\chi_{g_r}(G) \leq 4$. Thus, $\chi_{g_r}(G) = 4$.

If $n = 5, S = \{v_1, v_2, v_4\}$ is a chromatic set of G as well as geodetic set of G with minimum cardinality. But the subgraph induced by $V(G) \backslash S$ has two isolate vertices. Therefore, that S is not a restrained geo chromatic set of G. Now, take $S_1 = S \cup \{v_3, v_5\}$. Clearly S_1 is a chromatic as well as restrained geodetic set of minimum cardinality. Hence, $\chi_{g_r}(G) = |S_1| = 5$.

Case (ii) $n \equiv 2 (mod\ 4)$

Consider the set $S = \left\{v_1, v_{\frac{n+2}{2}}\right\}$. Clearly S is a chromatic set of G and a restrained geodetic set of G with minimum cardinality. Thus that S is a minimum restrained geo chromatic set of G and hence, $\chi_{g_r}(G) = 2$.

Case (iii) $n \equiv 0 (mod\ 4)$

Consider the set $S = \left\{v_1, v_{\frac{n+2}{2}}\right\}$. It is clear that S is a restrained geodetic set of G with minimum cardinality. But that S is not a chromatic set of G. Consider $S_1 = S \cup \left\{a_{\frac{n+2}{2}+1}\right\}$. Now it is clear that S_1 is a chromatic set of minimum cardinality and $S \subseteq S_1$. So that S_1 is a minimum restrained geo chromatic set of G and so $\chi_{g_r}(G) = |S_1| = 3$.

Case (iv) $n \equiv 1,3 \ (mod\ 4)$

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Consider the set $S = \left\{v_1, v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}\right\}$. Clearly, S is a restrained geodetic set of G. But that S is not a chromatic set of G. So S is not a restrained geo chromatic set of G. Now consider $S_1 = S \cup \{v_n\}$. It can be easily verified that S_1 is a minimum restrained geo chromatic set of G and hence, $\chi_{g_r}(G) = |S_1| = 4$.

$$\text{Thus, } \chi_{g_r}(C_n) = \begin{cases} n \; if \; n = 3,4,5 \\ 2 \; if \; n \equiv 2(mod\,4) \\ 3 \; if \; n \equiv 0(mod 4) \\ 4 \; if \; n \equiv 1,3(mod\,4) \end{cases}.$$

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