International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Initial Value Problem Involving Hadamard Fractional Derivative via Monotone Method

Jagdish A. Nanware[1], Vikas K. Wakode[2] and Savata S. Phulsagar[3]

[1,2]Department of PG Studies and Research in Mathematics,
Shrikrishna Mahavidyalaya,
Gunjoti, Dist.Dharashiv-413606(MS).India
1.jag_skmg91@rediffmail.com
2.vikaswakode08@gmail.com

[3]Department of Mathematics,
Ramkrishna Paramhansa Mahavidyalaya,
Dharashiv-413501(MS).India
3.ssp.maths@yahoo.co.in

Abstract

In this paper, we investigate the existence and uniqueness of solution of fractional differential equations involving Hadamard fractional derivative with initial conditions. Monotone method is constructed using lower and upper solutions. Two monotone convergent sequences are obtained converging to minimal and maximal solutions of the problem.

Keyword: Fractional differential equations, Existence and uniqueness, Monotone method, Upper and lower solutions.

Mathematics Subject Classification(2020): 34A08, 34A12 , 26A33.

1 Introduction

The fractional calculus was developed during nineteenth century [15,25]. The idea of fractional calculus has been a topic of interest not only among mathematicians, but also among physicists and engineers. Fractional calculus is being applied in

Received: October 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

various fields of engineering and science, rheology, economics, mechanics, chemistry, physics, viscoelasticity, finance, aerodynamics, electrodynamics of complex medium, etc. [9,12,13,14,21,22,27]. The study of theory of differential equations of fractional order parallel to the well known theory of ordinary differential equations has been growing independently since last four decades [16,17,18,19,20]. An effective technique for discussing the existence of solutions of differential equations is the monotone iterative technique combined with the lower and upper solutions method. The method of upper and lower solutions combined with the monotone iterative technique not only guarantees the interval of existence but also provides a constructive way to obtain the solution. Wang and Xie [29] developed monotone method and obtained existence and uniqueness of solution of fractional differential equation with integral boundary conditions. Vasundhara Devi developed [10] the general monotone method for periodic boundary value problem of Caputo fractional differential equation when the function is sum of nondecreasing and nonincreasing function. Nanware and Dhaigude developed [24] monotone method for Riemann-Liouville fractional differential equation with integral boundary conditions when the sum of nondecreasing and nonincreasing function. Some other contributions to the theory can be seen in [3,5,7,8,11,21,26,28,30]. K. Balachandran et al. [6] studied the existence results for fractional differential equations with Hadamard type fractional derivative by using the Banach fixed point and the Schauder fixed point theorems. Ahmad and Ntouyas [2,4] studied the existence and uniqueness of solutions for fractional integral boundary value problems involving Hadamard fractional operator by applying some standard fixed point theorems.
In this paper, we consider the Hadamard fractional differential equation with initial condition and develop monotone method for Hadamard fractional differential equation with initial condition and obtained existence and uniqueness of solution of the problem. So far, existence and uniqueness of solutions has not been studied involving Hadamard fractional derivative.
The rest of paper is organized in the following manner:
In section 2, we consider some definitions and lemmas required for the study. In section 3, the main results for the monotone method are developed for the problem. As an application of the method existence and uniqueness results for Hadamard fractional differential equation with initial condition are obtained. In section 4, we provide an example to illustrate our main results. In section 5, at the end conclusion section is added.

2 Preliminaries

We consider the following Hadamard fractional differential equation with initial condition

$$\mathcal{D}^q u(t) = f(t, u(t)), \quad t \in [1, T] \tag{2.1}$$

$$u(1) = u_0 \tag{2.2}$$

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

where $f \in C(J \times \mathbb{R}, \mathbb{R})$, $u_0 \in \mathbb{R}$, $J = [1, T]$.
Here and throughout, the notation $\mathcal{D}^q$ denotes the Hadamard fractional derivative of order q $(0 < q < 1)$.
Define $C(J, \mathbb{R}) = \{u(t) | u(t) \text{ is continuous on J}\}$.

Definition 2.1 *[1] The Hadamard integral of u(t) of fractional order q is defined as*

$$\mathcal{I}^q u(t) = \frac{1}{\Gamma(q)} \int_1^t \left(log \frac{t}{s} \right)^{q-1} \frac{u(s)}{s} ds,$$

where $\mathcal{I}^q$ denotes the Hadamard fractional integral of order q.

Definition 2.2 *[1] The Hadamard derivative of u(t) of fractional order q is defined as*

$$\mathcal{D}^q u(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(log \frac{t}{s} \right)^{n-q-1} \frac{u(s)}{s} ds,$$

where n=[q]+1, [q] denotes the integer part of the real number q.

Definition 2.3 *A pair of functions v(t) and w(t) in $C(J, \mathbb{R})$ are said to be lower and upper solutions of the problem (2.1)-(2.2) if*

$$\mathcal{D}^q v(t) \leq f(t, v(t)), \quad \text{for all } t \in J, \quad v(1) \leq v_0,$$
$$\mathcal{D}^q w(t) \geq f(t, w(t)), \quad \text{for all } t \in J, \quad w(1) \geq w_0.$$

Lemma 2.1 *Let $m \in C(J, \mathbb{R})$ be continuous function and for any $t_1 \in (1, T]$, we have $m(t_1) = 0$, and $m(t) > 0$ for $1 \leq t \leq t_1$ then it follows that $\mathcal{D}^q m(t_1) < 0$.*

Proof: We know that

$$\mathcal{D}^q m(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(log \frac{t}{s} \right)^{n-q-1} \frac{m(s)}{s} ds$$

and let

$$H(t) = \int_1^t \left(log \frac{t}{s} \right)^{n-q-1} \frac{m(s)}{s} ds.$$

Consider for $h > 0$,

$$\begin{aligned} H(t_1) - H(t_1 - h) &= \int_1^{t_1 - h} \left[\left(log \frac{t_1}{s} \right)^{n-q-1} - \left(log \frac{t_1 - h}{s} \right)^{n-q-1} \right] \frac{m(s)}{s} ds \\ &\quad + \int_{t_1 - h}^{t_1} \left(log \frac{t_1}{s} \right)^{n-q-1} \frac{m(s)}{s} ds. \\ &= I_1 + I_2, \quad say. \end{aligned}$$

Since $\left(log \frac{t_1}{s} \right) > \left(log \frac{t_1 - h}{s} \right)$ and $n - q - 1 < 0$, we have $\left[\left(log \frac{t_1}{s} \right)^{n-q-1} - \left(log \frac{t_1 - h}{s} \right)^{n-q-1} \right] < 0$. Thus it follows that $I_1 < 0$.

Received: October 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Also $H(t_1) - H(t_1 - h) \leq \int_{t_1-h}^{t_1} \left(log\frac{t_1}{s}\right)^{n-q-1} \frac{m(s)}{s} ds = I_2$.
Since $m(t_1) = 0$ and $m(s) > 0$ for $1 < t_1 < T$. Thus, we have $H(t_1) - H(t_1 - h) < I_2$.
Dividing by h and taking limits $h \to 0$ we have

$$\lim_{h \to 0} \left[\frac{H(t_1) - H(t_1 - h)}{h}\right] < 0.$$

Since $0 < q < 1$ we conclude that $H'(t_1) < 0$,
which implies that $\mathcal{D}^q m(t_1) = \frac{1}{\Gamma(n-q)} t_1 H'(t_1)$.
That is $\mathcal{D}^q m(t_1) < 0$. Thus the lemma.

Lemma 2.2 *Let $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $v(t), w(t) \in C^1(J, \mathbb{R})$ are lower and upper solutions of (2.1) and there exists $M \in (1, q^q)$ such that $f(t, x) - f(t, y) \geq -M(x - y)$ for $x \geq y$. Then $v(1) \leq w(1)$ implies $v(t) \leq w(t)$.*

Proof: Consider $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $v(t), w(t) \in C^1(J, \mathbb{R})$ are lower and upper solutions of (2.1), we have

$$(i)\mathcal{D}^q v(t) \leq f(t, v(t)), \qquad (ii)\mathcal{D}^q w(t) \geq f(t, w(t)), \quad 1 \leq t \leq T$$

one of the inequalities being strict. Then

$$v(1) < w(1)$$

implies

$$v(t) < w(t). \tag{2.3}$$

Suppose that the inequality (2.3) is not true. Let us suppose that the inequality (ii) is strict. Then setting $m(t) = v(t) - w(t)$, $1 \leq t \leq t_1$, we find that $m(t) > 0$, $1 \leq t \leq t_1$ and $m(t_1) = 0$. Then by Lemma 2.1, we get $\mathcal{D}^q m(t_1) < 0$, which yields,

$$\begin{gathered} m(t_1) = v(t_1) - w(t_1) \\ \mathcal{D}^q m(t_1) = \mathcal{D}^q v(t_1) - \mathcal{D}^q w(t_1) \\ \mathcal{D}^q v(t_1) > \mathcal{D}^q w(t_1) \end{gathered}$$

and

$$\begin{gathered} m(t_1) = v(t_1) - w(t_1) = 0. \\ v(t_1) = w(t_1). \end{gathered}$$

using (i) and (ii), and the definition of $m(t)$, we have

$$f(t_1, v(t_1)) \geq \mathcal{D}^q v(t_1) \geq \mathcal{D}^q w(t_1) > f(t_1, w(t_1)).$$

This is a contradiction since $v(t_1) = w(t_1)$. Hence the inequality (2.3) is valid, therefore $v(1) \leq w(1)$ implies $v(t) \leq w(t)$ and the proof is complete.

Received: October 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 2.1 *Assume that :*
(i) The functions $v(t)$ and $w(t)$ in $C(J,\mathbb{R})$ are lower and upper solutions of (2.1)-(2.2).
(ii) The function f(t,u(t)) satisfy one-sided Lipschitz condition

$$f(t,x) - f(t,y) \leq M(x-y), \quad M \geq 1.$$

Then $v(1) \leq w(1)$ implies that $v(t) \leq w(t)$, $1 \leq t \leq T$.

Proof: For small $\epsilon > 0$, setting

$$w_\epsilon(t) = w(t) + \epsilon t^q. \tag{2.4}$$

We have $w_\epsilon(1) > w(1)$ and $w_\epsilon(t) > w(t)$, $1 \leq t \leq T$.
Taking fractional derivative of (2.4), we get

$$\begin{aligned}
\mathcal{D}^q w_\epsilon(t) &= \mathcal{D}^q[w(t) + \epsilon t^q] \\
&= \mathcal{D}^q w(t) + \epsilon \mathcal{D}^q t^q \\
&\geq f(t, w(t)) + \epsilon q^q t^q \\
&\geq f(t, w_\epsilon(t)) + M(w - w_\epsilon) + \epsilon q^q t^q \\
&\geq f(t, w_\epsilon(t)) + q^q(-\epsilon t^q) + \epsilon q^q t^q \\
&\geq f(t, w_\epsilon(t)).
\end{aligned}$$

Thus it follows that $w_\epsilon(t)$ is upper solution of (2.1)-(2.2). Applying Lemma 2.2 for lower and upper solutions of (2.1)-(2.2) we get $v(t) \leq w_\epsilon(t)$, $1 \leq t \leq T$. Since ϵ is arbitrary, we conclude that $v(t) \leq w(t)$, $1 \leq t \leq T$.

3 Main Results

In this section we develop monotone method for Hadamard fractional differential equation (2.1) with initial condition (2.2). Also obtained existence and uniqueness of solution of Hadamard fractional differential equation (2.1)-(2.2), using monotone method.

Theorem 3.1 *Assume that :*
(i) f(t,u(t)) in $C[J \times \mathbb{R}, \mathbb{R}]$ is nondecreasing in u for each t.
(ii) $v_0(t)$ and $w_0(t)$ in $C(J,\mathbb{R})$ are lower and upper solutions of (2.1)-(2.2) such that $v_0(t) \leq w_0(t)$ on J=[1,T].
(iii) f(t,u) satisfy one-sided Lipschitz condition

$$f(t,u) - f(t,v) \leq -M(u-v), \quad M \geq 1, \quad \textit{for all } u \geq v, \quad t \in J.$$

Then there exists monotone sequences $v_n(t)$ and $w_n(t)$ in $C(J,\mathbb{R})$ such that

$$v_n(t) \to v(t) \quad \textit{and} \quad w_n(t) \to w(t) \quad \textit{as} \quad n \to \infty,$$

Received: October 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$v(t)$ and $w(t)$ are minimal and maximal solutions of (2.1)-(2.2) respectively satisfying

$$\mathcal{D}^q v(t) = f(t, v(t))$$

$$\mathcal{D}^q w(t) = f(t, w(t))$$

on $[1, T]$.

Proof: For any $\phi(t)$ and $\psi(t)$ in $C(J, \mathbb{R})$ such that for $v_0(1) \leq \phi(t)$ and $w_0(1) \leq \psi(t)$ on J, consider the following linear fractional differential equation

$$\mathcal{D}^q u(t) + Mu(t) = f(t, \phi(t)) + M\phi(t), \quad u(1) = u_0. \tag{3.1}$$

We first show that the uniqueness of solution of linear fractional differential equation (3.1), For this, let u_1 and u_2 be two solutions of (3.1). Then we have

$$\mathcal{D}^q u_1(t) + Mu_1(t) = f(t, \phi(t)) + M\phi(t), \quad u_1(1) = u_{0_1}$$

$$\mathcal{D}^q u_2(t) + Mu_2(t) = f(t, \phi(t)) + M\phi(t), \quad u_2(1) = u_{0_2}$$

where $\phi(t) \in C(J, \mathbb{R})$. Hence

$$\mathcal{D}^q (u_1(t) - u_2(t)) = -M(u_1(t) - u_2(t))$$

and

$$u_1(1) - u_2(1) = u_{0_1} - u_{0_2} = 0.$$

This implies $u_1(t) = u_2(t)$.
Define the sequences as follows:

$$\mathcal{D}^q v_{n+1}(t) = f(t, v_n) - M(v_{n+1} - v_n), \quad v_{n+1}(1) = v_{0_{(n+1)}}.$$
$$\mathcal{D}^q w_{n+1}(t) = f(t, w_n) - M(w_{n+1} - w_n), \quad w_{n+1}(1) = w_{0_{(n+1)}}.$$

Now

$$\mathcal{D}^q v_{n+1}(t) + Mv_{n+1}(t) = f(t, v_n) + Mv_n(t), \quad v_{n+1}(1) = v_{0_{(n+1)}}. \tag{3.2}$$
$$\mathcal{D}^q w_{n+1}(t) + Mw_{n+1}(t) = f(t, w_n) + Mw_n(t), \quad w_{n+1}(1) = w_{0_{(n+1)}}.$$

It follows that exist unique solution $v_{n+1}(t)$ and $w_{n+1}(t)$ for above equation. Putting $n = 0$ in (3.2) the existence of solutions of $v_1(t)$ and $w_1(t)$ is clear. Next we prove that $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Setting $p(t) = v_0(t) - v_1(t)$ we have

$$\begin{aligned}
\mathcal{D}^q p(t) &= \mathcal{D}^q v_0(t) - \mathcal{D}^q v_1(t) \\
&\leq f(t, v_0) - [f(t, v_0) - M(v_1 - v_0)] \\
&\leq f(t, v_0) - f(t, v_0) + M(v_1 - v_0) \\
&\leq -M(v_0 - v_1) \\
&\leq -Mp(t)
\end{aligned}$$

and $p(1) = v_0(1) - v_1(1) \leq v_{0_{(0)}} - v_{0_{(1)}} = p = 0$.
Thus we have $\mathcal{D}^q p(t) \leq -Mp(t)$ and $p = 0$. Hence $v_0(t) \leq v_1(t)$.

Received: October 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Similarly, we can show that $w_0(t) \geq w_1(t)$ and $v_1(t) \leq w_1(t)$. Thus $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$.
Assume that for some $k > 1$, $v_{k-1}(t) \leq v_k(t) \leq w_k(t) \leq w_{k-1}(t)$. We claim that $v_k(t) \leq v_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on J. To prove this, set $p(t) = v_k(t) - v_{k+1}(t)$. Since $f(t,u) + Mu$ is nondecreasing in u, we have

$$\begin{aligned}
\mathcal{D}^q p(t) &= \mathcal{D}^q v_k(t) - \mathcal{D}^q v_{k+1}(t) \\
&= f(t, v_{k-1}) - M(v_k - v_{k-1}) - [f(t, v_k) - M(v_{k+1} - v_k)] \\
&\leq f(t, v_{k-1}) - f(t, v_k) - M(v_k - v_{k-1}) + M(v_{k+1} - v_k) \\
&\leq -M(v_k - v_{k+1}) \\
&\leq -Mp(t)
\end{aligned}$$

and $p(1) = v_k(1) - v_{k+1}(1) = v_{0_{(k)}} - v_{0_{(k+1)}} = p = 0$.
Thus we have $\mathcal{D}^q p(t) \leq -Mp(t)$ and $p = 0$. Therefore we have $v_k \leq v_{k+1}$.
Similarly we prove that $v_{k+1}(t) \leq w_{k+1}(t)$.
Utilizing the corresponding fractional Volterra integral equations

$$\begin{aligned}
v_{n+1}(t) &= v_0 + \frac{1}{\Gamma(q)} \int_1^T (log\frac{t}{s})^{q-1} \frac{[f(s, v_{0_{(n+1)}}) - M(v_{n+1} - v_n)]}{s} ds, \\
w_{n+1}(t) &= w_0 + \frac{1}{\Gamma(q)} \int_1^T (log\frac{t}{s})^{q-1} \frac{[f(s, w_{0_{(n+1)}}) - M(w_{n+1} - w_n)]}{s} ds.
\end{aligned}$$

it follows that $v(t)$ and $w(t)$ are solutions (3.1).
Next we claim that $v(t)$ and $w(t)$ are the minimal and maximal solution of (3.2). Let $u(t)$ be any solution of (2.1)-(2.2) different from $v(t)$ and $w(t)$, so that there exists k such that $v_k(t) \leq u_k(t) \leq w_k(t)$ on J and set $p(t) = v_{k+1}(t) - u_k(t)$ so that

$$\begin{aligned}
\mathcal{D}^q p(t) &= \mathcal{D}^q v_{k+1}(t) - \mathcal{D}^q u_k(t) \\
&= f(t, v_k) - M(v_{k+1} - v_k) - f(t, u_k) \\
&\leq f(t, v_k) - f(t, u_k) - M(v_{k+1} - v_k) \\
&\leq -M(v_{k+1} - u_k) \\
&\leq -Mp(t)
\end{aligned}$$

and $p(1) = v_{k+1}(1) - u_k(1) = v_{0_{(k+1)}} - u_{0_{(k)}} = p = 0$.
Thus $v_{k+1} \leq u_k$ on J. Since $v_0(t) \leq u_0(t)$ on J, by induction it follows that $v_k(t) \leq u_k(t)$ for all k. Similarly we can show that $u_k(t) \leq w_k(t)$ for all k on J. Thus $v_k(t) \leq u_k(t) \leq w_k(t)$ on J. Taking limit as $n \to \infty$, it follows that $v(t) \leq u(t) \leq w(t)$ on J.

Next we establish the uniqueness of solution of (2.1)-(2.2) in the following.

Theorem 3.2 *Assume that :*
(i) f(t,u(t)) in $C[J \times \mathbb{R}, \mathbb{R}]$ *is nondecreasing in u for each t.*
(ii) $v_0(t)$ *and* $w_0(t)$ *in* $C(J, \mathbb{R})$ *are lower and upper solutions of (2.1)-(2.2) such that*

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$v_0(t) \leq w_0(t)$ on J=[1,T].
(iii) f(t,u) satisfy Lipschitz condition,

$$|f(t,u) - f(t,v)| \leq M|(u-v)|, \quad M \geq 1.$$

Proof: We have already prove $v(t) \leq w(t)$, we need to prove $v(t) \geq w(t)$. Consider $p(t) = w(t) - v(t)$ we find that

$$\begin{aligned}\mathcal{D}^q p(t) &= \mathcal{D}^q w(t) - \mathcal{D}^q v(t)\\ &= f(t,w) - f(t,v)\\ &\leq M[w-v]\\ &\leq -Mp(t)\end{aligned}$$

and

$$p(1) = w(1) - v(1) = 0.$$

This gives $w(t) \leq v(t)$. Hence $v(t) = w(t)$ is the unique solution of (2.1)-(2.2) on $[1,T]$.

4 Example

Example 1. Let us consider the following initial value problem

$$\begin{aligned}\mathcal{D}^q u(t) &= t^q u(t)\\ u(1) &= 1\end{aligned} \tag{4.1}$$

where $0 < q < 1$.
Solution: Choose $v_0(t) = 0, \quad w_0(t) = 3(\ln t)^q + 1, \quad t \in (1,T]$ are lower and upper solutions of equation (4.1),
such that

$$v_0(t) \leq w_0(t).$$

We observe that

$$0 \leq 3(\ln t)^q + 1.$$

Since $v_0(t)$ is constant, therefore $\mathcal{D}^q v_0(t) = 0$.
Also

$$\mathcal{D}^q w_0(t) = 3\Gamma(q+1),$$

Since

$$\mathcal{D}^q w_0(t) \geq t^q w_0(t).$$

Taking $t = 1$, $q = \frac{1}{2}$

$$2.658 > 1.$$

Now we verify the one-sided Lipschitz condition.
We have

$$f(t,u) - f(t,v) \leq -M(u-v).$$

Received: October 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Inequality becomes,

$$t^q(u-v) \leq -M(u-v),$$

which implies

$$M \geq t^q$$
$$M = 1.$$

The solution $u(t)$,

$$u(t) = 1 + \frac{(\ln t)^q}{\Gamma(q+1)}.$$

The lower solution sequence $\{v_n(t)\}$ of equation (4.1), is

$$\begin{aligned}
v_0(t) &= 0,\\
v_1(t) &= 1,\\
v_2(t) &= 1 + \frac{(\ln t)^q}{\Gamma(q+1)},\\
v_3(t) &= 1 + \frac{(\ln t)^q}{\Gamma(q+1)} + \frac{(\ln t)^{2q}}{\Gamma(2q+1)},\\
v_4(t) &= 1 + \frac{(\ln t)^q}{\Gamma(q+1)} + \frac{(\ln t)^{2q}}{\Gamma(2q+1)} + \frac{(\ln t)^{3q}}{\Gamma(3q+1)},\\
v_5(t) &= 1 + \frac{(\ln t)^q}{\Gamma(q+1)} + \frac{(\ln t)^{2q}}{\Gamma(2q+1)} + \frac{(\ln t)^{3q}}{\Gamma(3q+1)} + \frac{(\ln t)^{4q}}{\Gamma(4q+1)},\\
&\vdots\\
&\vdots\\
v_n(t) &= 1 + \sum_{k=1}^{n-1} \frac{(\ln t)^{kq}}{\Gamma(kq+1)}.
\end{aligned}$$

Taking the limit as $n \to \infty$, the sequence of lower solutions converges to $v(t)$. The upper solution sequence $\{w_n(t)\}$ of equation (4.1), is

$$\begin{aligned}
w_0(t) &= 3(\ln t)^q + 1,\\
w_1(t) &= 1 + \frac{(\ln t)^q}{\Gamma(q+1)} + 3\frac{(\ln t)^{2q}\Gamma(q+1)}{\Gamma(2q+1)}\\
w_2(t) &= 1 + \frac{(\ln t)^q}{\Gamma(q+1)} + \frac{(\ln t)^{2q}}{\Gamma(2q+1)} + \frac{(\ln t)^{3q}}{\Gamma(3q+1)} + 3\frac{(\ln t)^{4q}\Gamma(q+1)}{\Gamma(4q+1)},\\
w_3(t) &= 1 + \frac{(\ln t)^q}{\Gamma(q+1)} + ... + \frac{(\ln t)^{5q}}{\Gamma(5q+1)} + 3\frac{(\ln t)^{6q}\Gamma(q+1)}{\Gamma(6q+1)},\\
&\vdots\\
&\vdots\\
w_n(t) &= 1 + \sum_{k=1}^{n-1} \frac{(\ln t)^{kq}}{\Gamma(kq+1)} + 3\sum_{k=1}^{n-1} \frac{(\ln t)^{2kq}\Gamma(q+1)}{\Gamma(2kq+1)},
\end{aligned}$$

Received: October 22, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Taking the limit as $n \to \infty$, the sequence of upper solutions converges to $w(t)$.

The lower and upper solutions are shown graphically in the following figure.

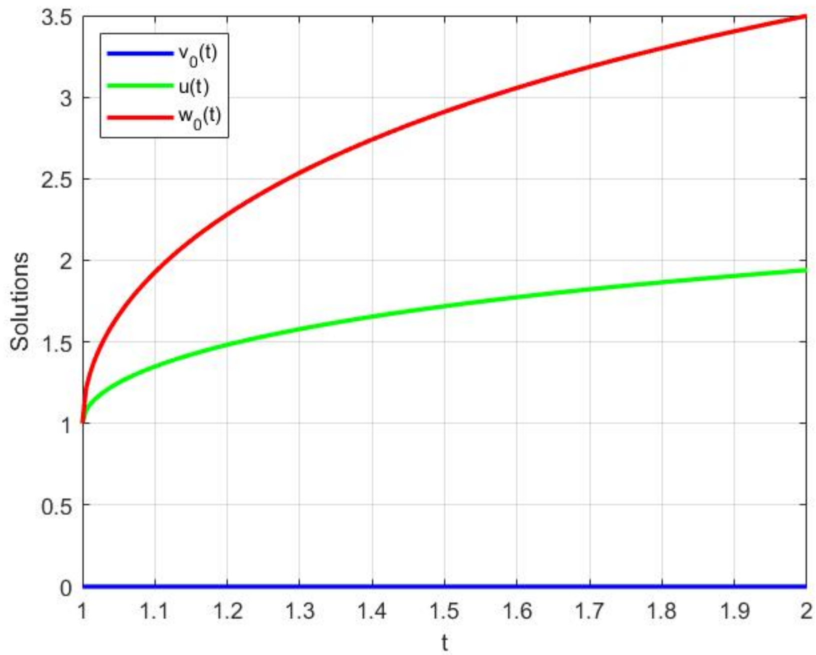

Figure 1: lower and upper solutions of (4.1).

5 Conclusion

Investigated the existence and uniqueness of solution of fractional differential equation involving the Hadamard fractional derivative by using monotone method with lower and upper solutions. It is shown that two monotone sequences converge to the minimal and maximal solutions of the problem (2.1)-(2.2). The obtained results are illustrate by suitable example. The graph of the solution obtained is plotted using matlab.

References

[1] Ahmad, B., Alsaedi, A., Ntouyas, S. K. & Tariboon, J., (2017). Hadamard-type fractional differential equations, inclusions and inequalities, Springer, Cham.

[2] B. Ahmad, J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58**(9) (2009), 1838–1843.

[3] B. Ahmad, S. K. Ntouyas, A. Alsaedi, New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions, *Bound. Value Probl.* **2013** (2013), 275, 1–14.

[4] B. Ahmad, S. K. Ntouyas, A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, *Fract. Calc. Appl. Anal.* **17** (2014), 348–360.

[5] D. Araya, C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal. Theory, Methods Appl.* **69**(11) (2008), 3692–3705.

[6] K. Balachandran, M. Matar, N. Annapoorani, D. Prabu, Hadamard functional fractional integrals and derivatives and fractional differential equations, *Filomat* **38**(3) (2024), 779–792.

[7] B. Bonilla, M. Rivero, L. Rodriguez-Germa, J. J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, *Appl. Math. Comput.* **187**(1) (2007), 79–88.

[8] Y. K. Chang, J. J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions, *Math. Comput. Model.* **49**(3–4) (2009), 605–609.

[9] S. Corlay, J. Lebovits, J. L. Vehel, Multifractional stochastic volatility models, *Math. Finance* **24** (2014), 364–402.

[10] J. V. Devi, Generalized monotone method for periodic boundary value problems of Caputo fractional differential equations, *Commun. Appl. Anal.* **12**(4) (2008), 399–410.

[11] K. Diethelm, N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265**(2) (2002), 229–248.

[12] M. S. El-Khatib, A. A. K. Abu Hany, M. M. Matar, M. A. Alqudah, T. Abdeljawad, On Cerone's and Bellman's generalization of Steffensen's integral inequality via conformable sense, *AIMS Math.* **8** (2023), 2062–2082.

[13] C. Q. Fang, H. Y. Sun, J. P. Gu, Application of fractional calculus methods to viscoelastic response of amorphous shape memory polymers, *J. Mech.* **31** (2015), 427–432.

[14] W. G. Glockle, T. F. Nonnenmacher, A fractional calculus approach to self-similar protein dynamics, *Biophys. J.* **68** (1995), 46–53.

[15] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[16] G. S. Ladde, V. Lakshmikantham, A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman Advanced Publishing Program, London, 1985.

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[17] V. Lakshmikantham, S. Leela, J. V. Devi, *Theory and Applications of Fractional Dynamic Systems*, Cambridge Scientific Publishers Ltd., 2009.

[18] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities Vol. I*, Academic Press, New York, 1969.

[19] V. Lakshmikantham, A. S. Vatsala, Theory of fractional differential inequalities and applications, *Commun. Appl. Anal.* **11** (2007), 395–402.

[20] V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations and applications, *Nonlinear Anal.* **69** (2008), 2677–2682.

[21] R. L. Magin, C. Ingo, L. Colon-Perez, W. Triplett, T. H. Mareci, Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives and entropy, *Microporous Mesoporous Mater.* **178** (2013), 39–43.

[22] F. A. McRae, Monotone iterative technique and existence results for fractional differential equations, *Nonlinear Anal. Theory, Methods Appl.* **71**(12) (2009), 6093–6096.

[23] M. M. Matar, J. Alzabut, M. I. Abbas, M. M. Awadallah, N. I. Mahmudov, On qualitative analysis for time-dependent semi-linear fractional differential systems, *Prog. Fract. Differ. Appl.* **8** (2022), 525–544.

[24] J. A. Nanware, D. B. Dhaigude, Existence and uniqueness of solution of Riemann-Liouville fractional differential equations with integral boundary conditions, *Int. J. Nonlinear Sci.* **14**(4) (2012), 410–415.

[25] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[26] A. L. Shi, S. Q. Zhang, Upper and lower solutions method and a fractional differential equation boundary value problem, *Electron. J. Qual. Theory Differ. Equ.* (2009), 1–13.

[27] I. Suwan, M. Abdo, T. Abdeljawad, M. Matar, A. Boutiara, M. Almalahi, Existence theorems for ψ-fractional hybrid systems with periodic boundary conditions, *AIMS Math.* **7** (2022), 171–186.

[28] G. T. Wang, Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments, *J. Comput. Appl. Math.* **236** (2012), 2425–2430.

[29] T. Wang, F. Xie, Existence and uniqueness of fractional differential equations with integral boundary conditions, *J. Nonlinear Sci. Appl.* **1** (2008), 206–212.

[30] S. Zhang, Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives, *Nonlinear Anal. Theory, Methods Appl.* **71**(5–6) (2009), 2087–2093.

Received: October 22, 2025