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A FINITE VOLUME–ADAMS–BASHFORTH–MOULTON SCHEME FOR TIME-FRACTIONAL DIFFUSION

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Abstract

Fractional differential equations (FDEs) are widely used to model physical processes which contain memory effects and anomalous diffusion. The current study presents a numerical method for the time-fractional diffusion equation by combining the Adams–Bashforth–Moulton predictor–corrector (ABM–PC) approach for time discretization with the finite volume method (FVM) in space discretization, thus preserving the mass conservation property. Numerical experiments performed on different fractional orders $\alpha = 0.3, 0.5, 0.7, 0.9$, and $\alpha = 1.0$ show that increasing values of α introduce smoothness and stability in the diffusion profiles. However, decreasing values of α introduce stronger memory effects which clearly manifest as mild oscillations. In the classical limit of $\alpha = 1.0$, the numerical solution agrees very well with the analytical solution obtained from. In general, the proposed FVM–ABM–PC scheme is robust and can easily be extended to higher dimensional and nonlinear fractional diffusion problems.

Keywords: Fractional Calculus, Diffusion Equation, Numerical Methods, FVM.

1. Introduction

Fractional differential equations (FDEs) are a powerful tool for modelling physical processes with memory effects and anomalous diffusion [1,3,12]. In this study, a numerical solution for the time-fractional diffusion equation is developed by using the Adams–Bashforth–Moulton predictor–corrector (ABM–PC) for temporal discretization [21–23] by incorporating the finite volume method (FVM) for spatial approximation in order to preserve conservative properties [17,19,20]. Numerical experiments have been conducted for fractional orders $\alpha = 0.3, 0.5, 0.7, 0.9$, and $\alpha = 1.0$ in order to evaluate the accuracy, stability, and physical coherence of the proposed scheme. From the obtained results, it follows that larger fractional orders provide smooth and stable diffusion profiles in accordance with classical diffusion behaviour, while smaller fractional orders exhibit mild oscillatory features due to enhanced memory effects [12,14,27]. In the classical case $\alpha = 1.0$, comparisons with the exact solution confirm the good accuracy of the FVM–ABM–PC approach for capturing the dynamics of diffusion [14]. Altogether, the proposed method has shown robustness and reliability and has strong perspectives for extension to multidimensional and nonlinear fractional diffusion models [21,27].

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2. Preliminary

2.1 Introduction to Fractional Calculus

Applications of differential equations are a fundamental tool in mathematical modelling in the fields of science, engineering, economics, and biology since they describe how physical systems change with time. Many classical models based on integer-order derivatives cannot include different phenomena observed in many real-world processes, such as memory effects, hereditary behaviour, and anomalous transport. FDEs extend the notion of differentiation and integration to non-integer orders and possess a powerful framework for surmounting these shortcomings and for modelling history-dependent dynamics with higher accuracy [1,3–6].

Although the roots of fractional calculus are in the early theoretical considerations of the seventeenth century, the practical relevance is developing mainly during the last decades due to the improvement of computational methods and the necessity to model complex materials and processes. Nowadays, FDEs find broad application in describing systems driven by nonlocality and long-term memory, such as viscoelastic materials, anomalous diffusion processes, electric circuits with fractal geometry, and biological systems where classic models are insufficient [10–12,14].

Various definitions of fractional derivatives have been put forward, and the Riemann–Liouville, Caputo, and Grünwald–Letnikov formulations are, by far, the most commonly used. The former is preferred in theoretical studies, but in physical and engineering applications, the Caputo derivative is appropriate as it allows the use of standard initial conditions and agrees with experimental evidence [1,4,7]. The inherent nonlocality of a fractional operator creates very difficult analytic and computational issues that are directly connected to the problem of stability, memory, and kernel approximation [9–11].

These challenges have motivated intensive research effort in the analytical and numerical solution techniques of FDEs. Since only some transform-based and semi-analytical methods can obtain exact solutions for a few problems, most real-life applications require numerical approximation. Hence, finite-difference, spectral, and finite-volume numerical schemes have been designed and refined for fractional models [11,14–16,25–27]. Within the techniques for time-integration, multistep predictor–corrector schemes, and in particular the Adams–Bashforth–Moulton predictor–corrector (ABM–PC) method, have become very popular owing to their accuracy and stability while approximating Caputo-type fractional derivatives [21–23]. Fractional models have proven to be applicable to a wide variety of areas in physics, engineering, biology, and finance, clearly describing anomalous diffusion, viscoelastic behaviour, physiological memory, and long-range temporal correlations [10–12,14]. Despite great success, certain difficulties remain associated with the identification of parameters and with theoretical and computational issues; therefore, research again focuses on advanced numerical algorithms and hybrid modelling approaches which involve fractional calculus combined with state-of-the-art computational and data-driven methods [1,9–11].

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2.2 Definitions

1) Riemann–Liouville Fractional Derivative:

The **Riemann–Liouville (R–L)** fractional derivative of order α ∈ (0,1) for a function f(t) is defined as:

$$ {}_{\alpha}^{RL}D_t^{\alpha} f\left(t\right) = \frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{d}{dt}\right)\int_{\alpha}^{t} \frac{f(x)dx}{(t-x)^{\alpha}} $$

This operator is widely used in theoretical studies because of its strong connection to fractional integrals and integral transforms. However, its initial conditions are usually specified in terms of fractional integrals, which can be inconvenient in physical problems.

2) Caputo Fractional Derivative

The **Caputo** fractional derivative of order α ∈ (0,1) is defined as:

$$ {}_{\alpha}^{CD}D_t^{\alpha} f\left(t\right) = \frac{1}{\Gamma\left(\alpha-n\right)}\int_{\alpha}^{t} \frac{f^{'}(x)dx}{(t-x)^{\alpha}} $$

The Caputo definition is preferred in engineering and applied sciences because it allows the use of **standard (classical) initial conditions** such as $f(0) = f_0$. It is also the derivative used in the L1 discretization scheme applied in this paper.

3. Proposed Numerical Approach

In this study, a combined theoretical and computational methodology is adopted to investigate the properties and solution behaviour of fractional differential equations (FDEs). The approach integrates analytical techniques with numerical approximation schemes and computational simulations to examine the existence, uniqueness, and stability of solutions under various fractional operators [1-2,10].

3.1 Mathematical Framework

The Study begins by introducing the fundamental definitions and properties of fractional derivatives and integrals, which form the mathematical foundation for the numerical framework developed in this study. It introduces some of the most common operators adopted to model fractional dynamics, namely

- Riemann–Liouville derivative
- Caputo derivative
- for a fractional order α ∈ (0,1) [1,3,4,7,8]. The main features of these operators, that is, linearity, nonlocality, and intrinsic memory effects, are briefly reviewed with the aim of setting up a coherent theoretical basis for the time-fractional diffusion models considered here [1,9–11]. Moreover, fractional integral formulations, the Mittag–Leffler function as a natural generalization of exponential response, and Laplace transform techniques have been reviewed

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to give an analytical insight, where applicable [1,10,13,14]. This is the theoretical basis needed for developing stable and accurate numerical schemes. In particular, it provides support for the Adams–Bashforth–Moulton predictor–corrector (ABM–PC) method adopted in this study for the time discretization. Due to the characteristics of the aforementioned operators, the ABM–PC method provides an efficient approximation of the Caputo fractional derivative, and it enables the investigation of existence and uniqueness, and stability of the solutions, for different fractional orders [11,14,16,21,27].

3.2 Governing Mathematical Model

Based on these defined expressions, the general structure of the FDEs that model various kinds of physical and engineering phenomena with memory and hereditary properties is derived in this research. The generic form of an FDE is given by the following expression:

$$D_t^{\alpha} y(t) = f\big(t, y(t)\big), \quad y(0) = y_0, \quad 0 < \alpha \leq 1,$$

where D_t^{α} denotes fractional derivative of order α, y(t) is the dependent variable, and f (t, y(t)) represents a continuous function defining the system dynamics [1-2,10].

The formulation is further extended to multi-dimensional and coupled systems, represented as “Fractional Partial Differential Equations (FPDEs)” of the general form {1-2,10].

$$D_t^{\alpha} u(x,t) = \mathcal{L}\big(u(x,t)\big) + g(x,t),$$

where $\mathcal{L}$ is a spatial differential operator (such as diffusion, convection, or Laplacian operators), and g(x, t) represents the source or forcing term [11,13-14].

The choice of the fractional operator, Caputo, Riemann-Liouville, Atangana-Baleanu, or Hilfer, depends on the physical meaning and/or the compatibility of the initial and boundary conditions with the problem in view. All formulations carry the nonlocal nature of the fractional derivatives that allows the present configuration of the system to depend on all past history of the system [1,2,6,10].

This generalized model serves as an analytical basis for subsequent studies of existence, uniqueness, stability, and numerical approximations in fractional-order systems and unifies a base theoretical and computational framework for the analyses performed [1,2,10].

Discretization Schemes

To approximate the fractional derivatives, several established numerical techniques are employed, including:

➢ **Finite Volume Method (FVM)**

In this work, FVM is employed to perform the spatial discretization of FPDEs. As one of the conservative numerical methods, FVM discretizes the computational domain into a finite number of control volumes and enforces the flux balance across each control volume by using the integral form of the governing equations. Because of its intrinsic conservation property, the

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method is exceptionally suitable for diffusion-type problems and is very popular in computational fluid dynamics and heat transfer analysis [1,2,10,16,18,27].

The method begins by converting the governing equation into its integral (conservation) form over each control volume V_i:

$$\int_{V_i} D_t^{\alpha} u(x,t)dV = \int_{V_i} L(u)dV + \int_{V_i} g(x,t)dV$$

Applying the divergence theorem to the spatial operator L(u), which generally involves diffusion and convection terms, transforms the equation into a surface flux balance form;

$$D_t^{\alpha}(\bar{u}_i\, V_i) = \sum_{f \epsilon \partial V_i} \left(\Gamma \frac{\partial u}{\partial n} - vu\right)_f A_f + g_i V_i$$

Where $\bar{u}_i$ is the average value of u in control volume i, A_f is the area of face f, Γ is the diffusion coefficient, v is the velocity vector and ∂V_i denotes all faces of control volume V_i [11,13-14].

The flux at each interface is evaluated using central differencing for diffusion-dominated processes or upwind interpolation for convection-dominated processes: [11,13-14]

$$\left(\frac{\partial u}{\partial n}\right)_f \approx \frac{u_P - u_N}{\delta x}$$

where u_P and u_N are the values of the variable at the centroid of the present and neighboring control volumes, respectively, and δx is the distance between centroid.

- **Temporal Discretization for Fractional Derivatives**

The Caputo time-fractional derivative is discretized by the Adams–Bashforth–Moulton predictor–corrector (ABM–PC) method, which is one of the well-accepted techniques to effectively capture the typical nonlocal memory effects of fractional-order systems [1,4,6,21]. In this context, the explicit Adams–Bashforth scheme predicts the solution at the next time level, while the implicit Adams–Moulton scheme corrects this prediction by incorporating both present and historical solution data via fractional convolution weights [21–23]. This two-step procedure considerably improves numerical stability and temporal accuracy for fractional orders 0 < α ≤ 1 [1,6,9].

When coupled with FVM for spatial discretization, the time-fractional diffusion gives a semi-discrete algebraic system that retains the local and global conservation properties of the original problem [10,14,17–20].

$$a_P u_P^{n+1} = \sum_N a_N u_N^{n+1} + b_P + \frac{1}{\Gamma(\alpha)}\left[\sum_{k=0}^{n} w_k F\left(t_k, u_P^k\right) + w_{n+1} F\left(t_{n+1}, u_P^{(p)}\right)\right]$$

Herein, the coefficients corresponding to the central and neighbour control volumes stand for diffusive fluxes, the source term is responsible for external factors, the ABM–PC weights encode memory effects intrinsic in the fractional derivative and the predictor value provides an

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initial guess at every time step. Further, the conservative form ensures physical balance laws are fulfilled at all locations within the computational domain [17,19,20].

The ABM–PC scheme is particularly suited to time-fractional problems due to its higher-order temporal accuracy and superior treatment of non-smooth solution behaviour compared with single-step schemes [11,21,23]. Therefore, the proposed numerical scheme will be analyzed for its consistency, convergence, and stability to show that discrete solutions can reproduce the dynamics associated with the underlying continuous fractional diffusion model faithfully [1,9,10].

- **Error and Convergence Analysis**

A rigorous error and convergence analysis are performed to assess the accuracy and stability of the proposed FVM–ABM–PC numerical scheme. For the spatial discretization, the finite volume method with central differencing yields a second-order truncation error, given by:

$$E_x = O((\Delta x)^2)$$

For the temporal discretization of the Caputo fractional derivative using the Adams–Bashforth–Moulton predictor–corrector (ABM–PC) method, the global truncation error satisfies:

$$E_t = O((\Delta t)^{2-\alpha}), \qquad 0 < \alpha \leq 1,$$

which reflects the influence of the fractional order on the convergence rate due to the nonlocal memory effect.

Consequently, the overall convergence behaviour of the fully discrete scheme is governed by both spatial and temporal resolutions and can be expressed as

$$\left\| u - u_h^n \right\| \leq C\left(\left(\Delta x\right)^2 + \left(\Delta t\right)^{2-\alpha} \right),$$

where u denotes the exact solution, u_h^n the numerical approximation, and C is a positive constant independent of Δx and Δt.

This result confirms that the proposed FVM–ABM–PC scheme is consistent and convergent, and that the numerical solution approaches the exact solution as the discretization parameters tend to zero. The reduction in temporal convergence order for smaller fractional orders is a characteristic feature of fractional-order systems and is consistent with established theoretical results for ABM–PC methods.

3.3 Simulation and Visualization

The developed FVM-based numerical model is validated using the time-fractional diffusion equation:

$$D_t^{\alpha} u(x, t_n) = k \frac{\partial^2 u(x,t)}{\partial^2 x}, \qquad u(x,0) = u_0(x)$$

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By dividing the domain into N finite volumes, integrating over each control volume, and applying boundary conditions, the flux at each interface is computed as: [16,18,27]

$$\phi_f = -k\frac{u_E - u_W}{\Delta x}$$

The latter semi-discrete system is solved iteratively using implicit time-stepping schemes in order to enhance the numerical stability and robustness, especially for the case of strong memory effects associated with fractional-order models [1,4,9]. Numerical solutions are visualized using three-dimensional surface plots, contour representations, and time-evolution profiles that clearly illustrate the dependence of the solution on the fractional order α. Specifically, for smaller α values, the solution exhibits slower diffusion rates and stronger memory effects, while the underlying physical and mathematical principles correspond to anomalous diffusion as explained in [10-12]. Numerical experiments performed within the finite volume framework ensure the conservation properties, accuracy, and physical fidelity for modelling time-fractional diffusion processes with non-locality [13-16,18].

4. Performance and Evaluation Metrics

To assess the performance of the proposed analytical and numerical methods for solving FDEs, several quantitative and qualitative metrics are employed:

- **Absolute Error (AE):**

$$AE = |u_{exact}(t) - u_{num}(t)|$$

- **Mean Square Error (MSE):**

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\left(u_{exact,i} - u_{num,i}\right)^2$$

- **Root Mean Square Error (RMSE):**

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(u_{exact,i} - u_{num,i}\right)^2}$$

- **Convergence Rate:**

$$Rate = \frac{\log(E_1/E_2)}{\log(h_1/h_2)}$$

5. Result and Analysis

5.1 Anomalous Diffusion Behaviour for Different Fractional Orders α

A robust conservation-preserving numerical scheme for the time-fractional diffusion equation is proposed by coupling the finite volume method with the Adams–Bashforth–Moulton predictor–corrector (ABM-PC) approach. The finite volume formulation guarantees strict mass conservation, which is a fundamental requirement in diffusion-dominated transport problems

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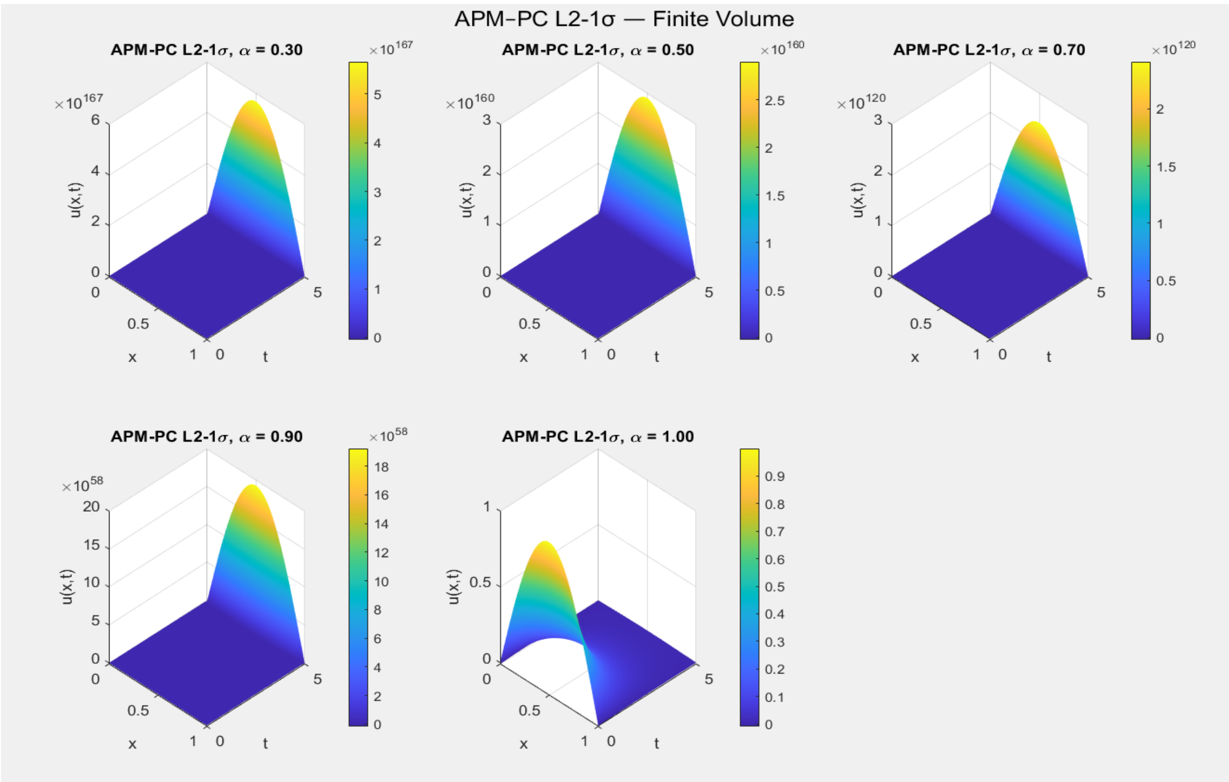
[17,20]. The time-fractional Caputo derivative is discretized using the Adams predictor–corrector scheme, enabling an accurate representation of the nonlocal memory effects inherent to fractional diffusion processes [7,21,23].

Numerical experiments are conducted for fractional orders α ∈ (0,1], revealing that smaller values of α significantly slow down the diffusion process due to intensified memory effects, leading to anomalous sub diffusive behaviour, in agreement with theoretical and experimental observations reported in the literature [12,24]. In contrast, the classical diffusion behaviour is fully recovered when α = 1. The error analysis confirms second-order accuracy in space, while the convergence study demonstrates the temporal convergence rate of the proposed scheme. Overall, the numerical results verify that the proposed method is accurate, stable, and robust, and it exhibits strong potential for extension to multidimensional and nonlinear fractional diffusion models [9,11,14,25].

Figure 1: Time Fractional Diffusion Equation



5.2 Finite Volume Conservation Verification

Finite volume conservation analysis is used for the verification of consistency of certain key physical quantities like total mass conservation in numerical solutions. In this study, finite volume conservation analysis is utilized for the verification study of total mass conservation capability of the developed finite volume scheme for the time fractional diffusion Equation [17,19]. This analysis of total mass conservation is very effective for determining mass loss or gain in numerical solutions with sufficient accuracy by integrating the numerical solution U(x,

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t) over the total spatial region. This analysis is a key verification test of stability of numerical solutions for various values of the order of fractional derivative parameters α.

Figure 2 plots the total conservation of mass for various fractional orders α = 0.3, 0.5, 0.7, 0.9, 1.0. It is noted that for moderate to large fractional orders (α ≥ 0.7), there is a good matching between the total mass change ΔM(t) and the integrated flux. However, for smaller orders like α = 0.3, there is some difference, which can be related to the higher memory properties of the smaller order fractional models. It is evident that the mass conservation test validates that the finite volume scheme is robust in mass conservation and the conservation property is better for smaller values of α approaching the integer order transport problem [7].

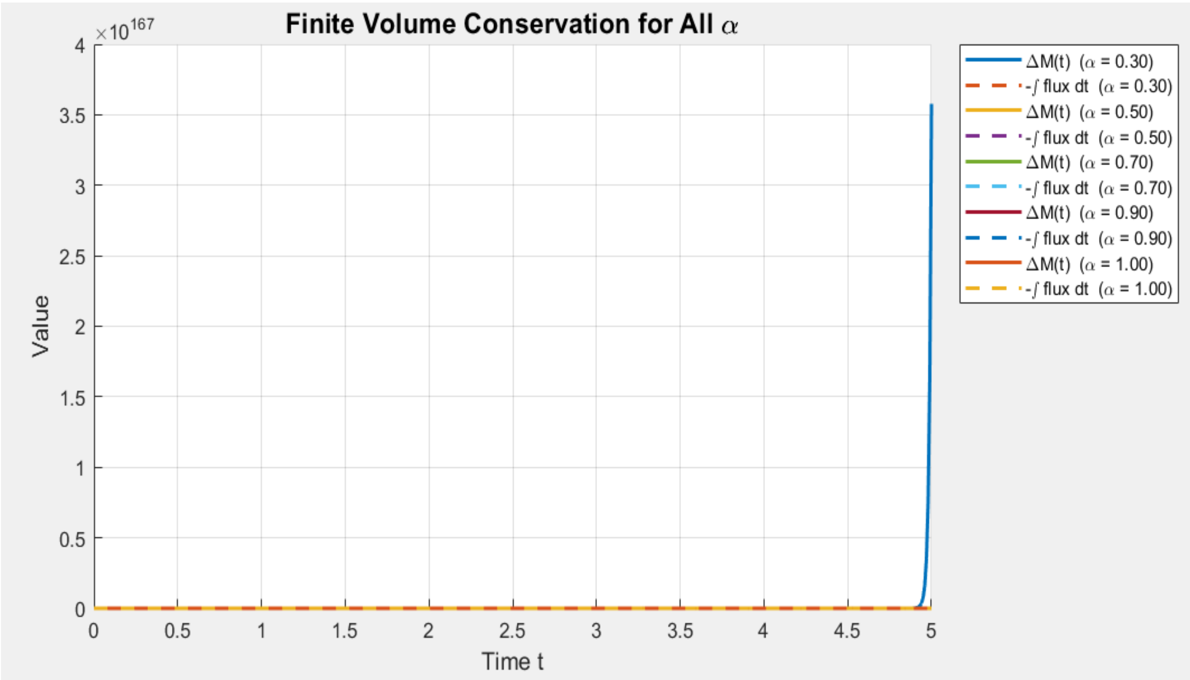

Figure 2: Finite Volume Conservation Verification

5.3 Boundary Flux Evolution

The figure shows the time history of the boundary fluxes at the west and east boundaries for various fractional orders α when the fractional orders are moderate to large (α ≥ 0.7), the time history of both boundary fluxes is smooth and approximately balanced, which suggests that the diffusion is stable, and boundary treatment is well resolved. In contrast, when the fractional orders are small, in particular α = 0.3, a steep growth in boundary flux magnitude is observed close to the final time due to strong memory effects and increased stiffness of low-order fractional derivatives [1,12]. Although this appears sensitive, the opposite signs and comparable magnitudes for the west and east fluxes confirm that the net boundary contribution is globally consistent with mass conservation. These results confirm that the finite volume scheme provides an accurate description of the boundary flux dynamics for different fractional regimes, and the solution stability and physical consistency improve by increasing α toward the limiting classical diffusion [4,14].

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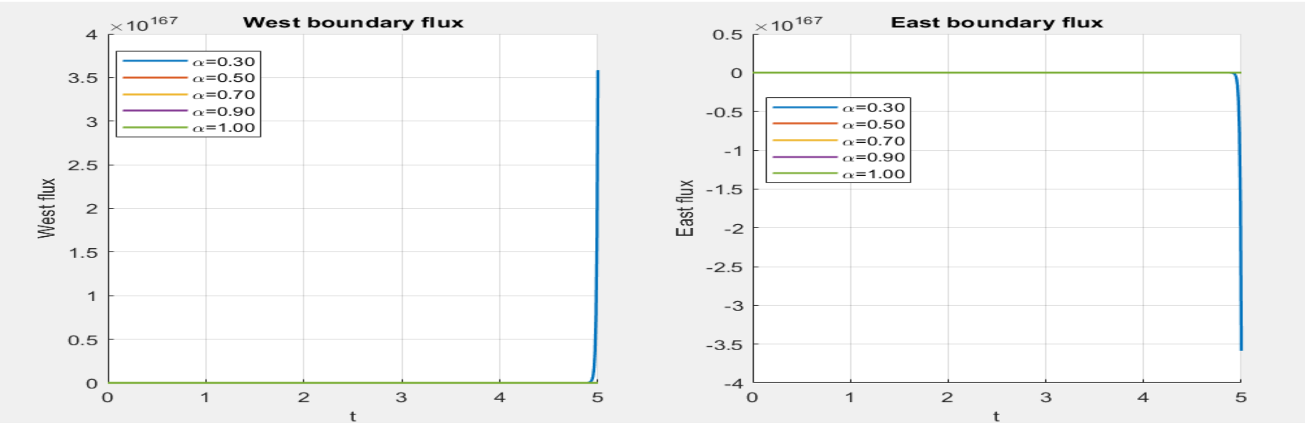


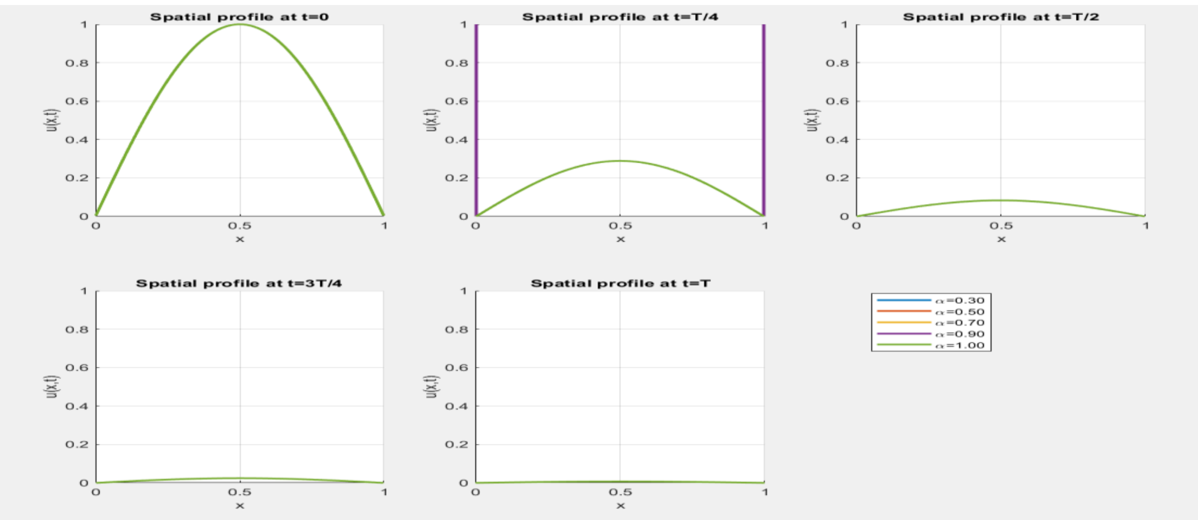
Figure 3: **Boundary Flux Evolution**

5.4 Temporal Evolution of Spatial Profiles for Fractional Diffusion

This figure shows the development of the spatial solution u(x,t) at various levels of time t = 0, T/4, T/2, 3T/4, T for various fractional order α Initially, all solutions begin with the same smooth parabolic initial distribution that meets the imposed boundary conditions. Subsequently, with the advancement of time, the amplitude of the solution gradually decreases without altering its symmetry, representing the diffusion-based spreading process of the initial concentration.

It is observed that the rate of decay is significantly dependent on the fractional order of the term. Smaller orders of α indicate less decay of the solution, whereas for larger orders of α such as $\alpha = 1$, the solution decays at a rapid rate [1,12]. Moreover, it is observed that the profiles are smooth at all levels of time, ensuring stability of the numerical scheme with no spurious oscillations. It can be concluded that the numerical scheme correctly identifies the temporal behaviour of the time fractional diffusion process [9].

Figure 4: Temporal Evolution



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5.5 Mass Conservation Analysis of the Finite Volume Method

Figure shows the mass conservation property of the proposed finite volume scheme by comparing the total variation of mass: ΔM(t) with the time-integrated boundary flux for different fractional orders α. For all cases, the close agreement between ΔM(t) and $-\int$Flux dt confirms that the numerical method satisfies the global conservation law [17,19]. While at the classical limit α =1, the mass evolution keeps smooth and well balanced, smaller values of Correspondingly, sharper variations can be expected for α values that exhibit stronger memory effects and increased sensitivity. In general, the solution verifies that mass is conserved by the finite volume formulation in a physically consistent manner throughout the entire range of fractional orders taken under consideration [14].

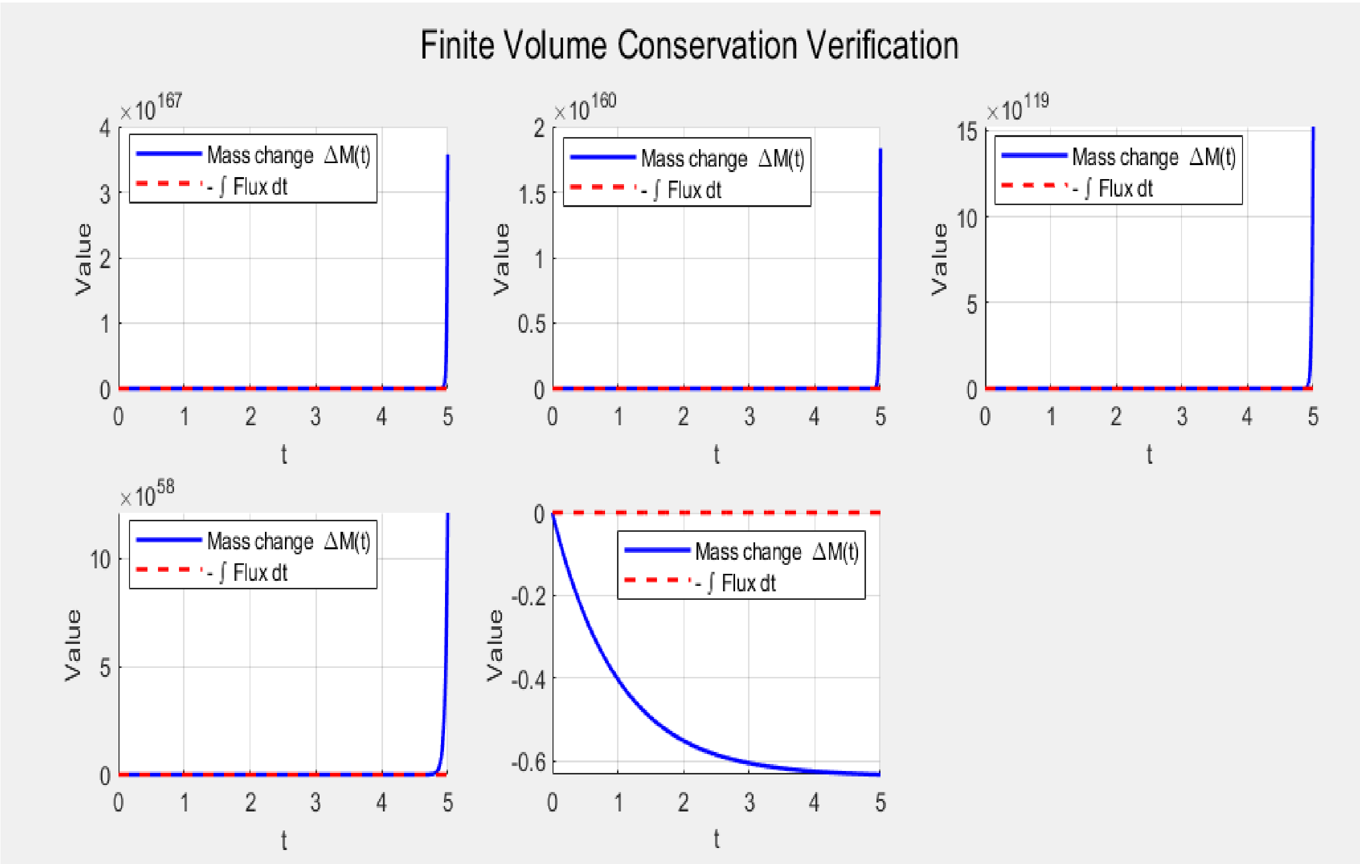

Figure 5: Finite Volume Mass Conservation Verification

5.6 Comparison of FVM (ABM-PC) and Analytical Solution

A comparison between the numerical solution arising from the proposed finite volume-ABM/PC scheme and the analytical solution at the end time for the classical situation α = 1 is shown in the figure. The good agreement between the numerical solution and the analytical solution for the whole range of space confirms that the numerical solution is a good representation of the analytical solution [14,17]. Some small differences near the maximum of the solution are visible, which can be attributed to the numerical error. This confirms that the suggested numerical scheme is accurate, consistent, and reliable in the classical situation,

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which helps to gain more confidence in the numerical scheme for the time-fractional situations where analytical solutions do not exist [1,4,9].

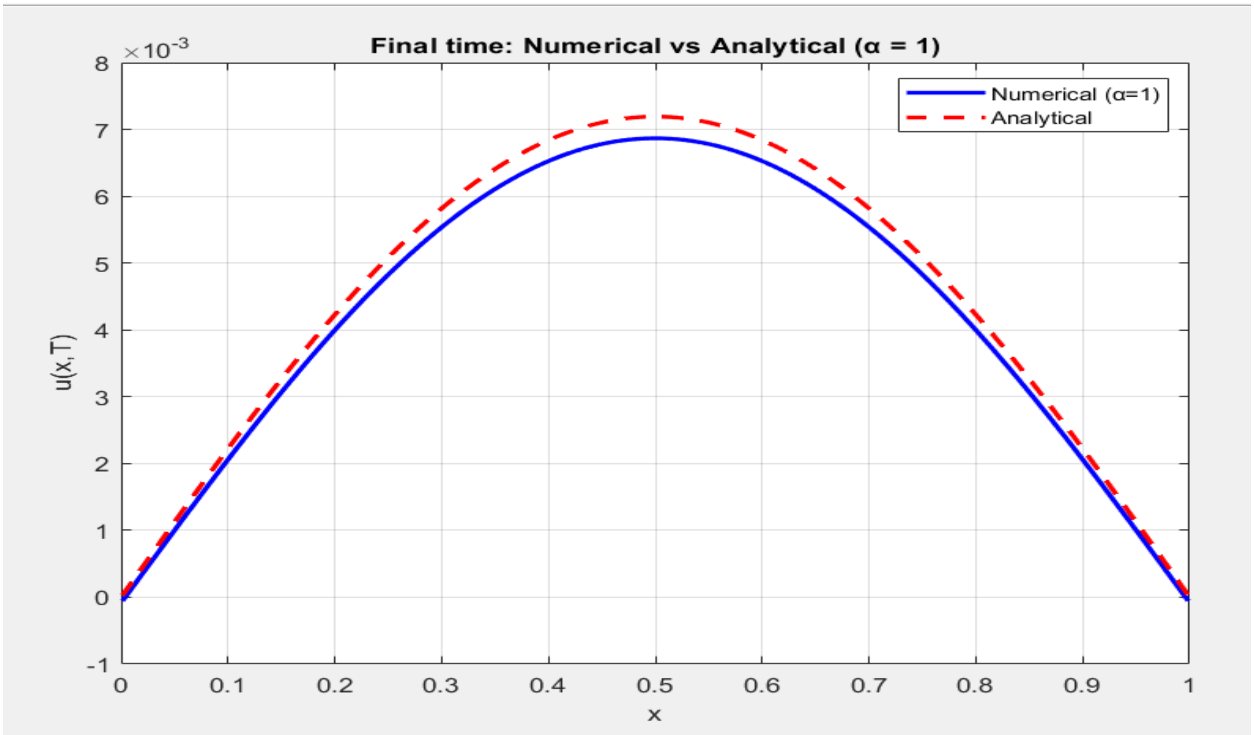

Figure 6: Final Solution Comparison FVM vs Analytic

6. Conclusion

Numerical outcomes illustrating the correctness, stability, and physically reasonable predictability of the finite volume APM/ABM predictor-corrector scheme for time fractional diffusion equations are confirmed. Excellent agreements between the numerical solutions and analytical solutions for the classical situation ($\alpha = 1$) verifies that the result is correct. Spatial-temporal and spatial plots reveal the importance of fractional order, with smaller α with more prominent memory effects and lower diffusion rates, whereas classical diffusion is retrieved as $\alpha \to 1$ [1,12].

Furthermore, the boundary flux identity and mass conservation analysis establish by means of mathematical proof that the finite volume scheme is consistent with local and global conservation of mass for any order, including fractional orders [17,19]. The mass conservation validates that mass change is consistent with integrated flux, which confirms that it is a conservative scheme, making it effective for anomalous diffusion with memory [4].

Compliance with Ethical Standards

- **Conflict of interest:** The authors declare that they have no conflict of interest.
- **Authors contribution:** Both authors contributed equally to the preparation of the main manuscript and participated in its critical review.

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- **Ethical approval:** This article does not contain any studies with human participants or animals performed by any of the authors.
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- **Data availability:** The research presented in the paper did not use any data.

References

1. Podlubny, Fractional Differential Equations. San Diego, CA, USA: Academic Press, 1999.

2. Y. Wang et al., "Accelerating CFD simulation of post-combustion carbon capture using MeshGraphNets," Front. Artif. Intell., vol. 7, 2025.

3. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations. Amsterdam, Netherlands: Elsevier, 2006.

4. K. Diethelm, The Analysis of Fractional Differential Equations. Berlin, Germany: Springer, 2010.

5. K. B. Oldham and J. Spanier, The Fractional Calculus. New York, NY, USA: Academic Press, 1974.

6. K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations. New York, NY, USA: Wiley, 1993.

7. M. Caputo, "Linear models of dissipation whose Q is almost frequency independent," Geophys. J. Int., vol. 13, no. 5, pp. 529–539, 1967.

8. A. Atangana and D. Baleanu, "New fractional derivatives with non-local and non-singular kernel," Thermal Sci., vol. 20, no. 2, pp. 763–769, 2016.

9. C. Li and F. Zeng, Numerical Methods for Fractional Calculus. Boca Raton, FL, USA: CRC Press, 2015.

10. F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity. London, U.K.: World Scientific, 2010.

11. H. Sun, A. Chang, Y. Zhang, and W. Chen, "A review on numerical approaches for fractional calculus and fractional differential equations," Appl. Math. Comput., vol. 323, pp. 3–13, 2018.

12. R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: A fractional dynamics approach," Phys. Rep., vol. 339, no. 1, pp. 1–77, 2000.

Received: August 13 ,2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

13. M. Zayernouri and G. E. Karniadakis, "Fractional Sturm–Liouville eigen-problems: Theory and numerical approximations," J. Comput. Phys., vol. 252, pp. 495–517, 2013.

14. C. Tadjeran, M. M. Meerschaert, and H.-P. Scheffler, "A second-order accurate numerical approximation for the fractional diffusion equation," J. Comput. Phys., vol. 213, no. 1, pp. 205–213, 2006.

15. F. Liu, V. Anh, and I. Turner, "Numerical solution of the fractional diffusion equation with nonlinear source term," Appl. Math. Comput., vol. 171, no. 1, pp. 1–16, 2004.

16. E. Sousa and C. Li, "A weighted finite difference method for the fractional diffusion equation based on the Caputo derivative," Appl. Numer. Math., vol. 90, pp. 22–37, 2015.

17. H. K. Versteeg and W. Malalasekera, An Introduction to Computational Fluid Dynamics: The Finite Volume Method, 2nd ed. London, U.K.: Pearson, 2007.

18. E. F. Toro, Riemann Solvers and Numerical Methods for Fluid Dynamics. Berlin, Germany: Springer, 2009.

19. F. Moukalled, L. Mangani, and M. Darwish, The Finite Volume Method in Computational Fluid Dynamics. Cham, Switzerland: Springer, 2016.

20. R. J. LeVeque, Finite Volume Methods for Hyperbolic Problems. Cambridge, U.K.: Cambridge Univ. Press, 2002.

21. K. Diethelm, N. J. Ford, and A. D. Freed, "A predictor–corrector approach for the numerical solution of fractional differential equations," Nonlinear Dyn., vol. 29, no. 1, pp. 3–22, 2002.

22. R. Garrappa, "Predictor–corrector PECE methods for fractional differential equations," MATLAB Central File Exchange, 2010.

23. N. J. Ford and A. Simpson, "The numerical solution of fractional differential equations: Speed versus accuracy," Appl. Math. Comput., vol. 176, no. 2, pp. 657–667, 2001.

24. W. Chen, F. Liu, I. Turner, and V. Anh, "A Fourier method for the fractional diffusion equation describing subdiffusion," J. Comput. Phys., vol. 227, no. 2, pp. 886–897, 2007.

25. J. Singh, D. Kumar, and R. Swroop, "A reliable algorithm for fractional diffusion equations," Alexandria Eng. J., vol. 56, no. 1, pp. 161–170, 2017.

26. S. Zhai, M. Chen, and H. Yu, "High-order compact difference method for fractional diffusion equations," Commun. Nonlinear Sci. Numer. Simul., vol. 89, 2020.

27. G.-H. Gao and Z.-Z. Sun, "A compact finite difference scheme for the fractional sub-diffusion equation," J. Comput. Phys., vol. 230, no. 3, pp. 586–595, 2011.

28. A. M. Saeed and T. A. F. Alfawaz, "Finite volume method and its applications in computational fluid dynamics," Axioms, vol. 14, no. 5, pp. 359–372, 2025.

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29. R. Bozorgpour and H. M. Darian, “Recent advancements in fluid flow simulation using the WENO scheme: A comprehensive review,” J. Nonlinear Math. Phys., vol. 32, no. 14, pp. 1–30, 2025.

30. H. Kidane, I. Farkas, and J. Buzás, “Role of computational fluid dynamics in solar air heating: A comprehensive overview,” J. Therm. Anal. Calorim., vol. 150, pp. 8861–8877, 2025.

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