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ASSOCIATION OF *ABC*-INDEX WITH OTHER TOPOLOGICAL INDICES

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Abstract:

A graph's links as well as verges determine its atom-bond-connectivity indicator. This study highlights key findings on the *Atom bond* indicator. The study explores the *Atom bond* indicator indicator's findings, paying particular attention to its degrees, verges, and links. Notably, relationships for the atom-bond-connectivity indicator are established by incorporating various Topological indicator such as the *Randic-* indicator (*RI*),, *Zagreb-* indicator *(ZI), Geometric-Arithmetic* indicator *(GI), Harmonic-* indicator *(AI).* The study explores the atom-bond-connectivity indicator's findings, paying particular attention to its degrees, verges, and links. In addition we establish that the bonds between methane and cyclopropane are strong

Keywords: *Atom-bond-connectivity* indicator (*ABI*), *Randic-* indicator(*RI*), *Zagreb-* indicator *(ZI), Geometric-Arithmetic* indicator *(GI), Harmonic-* indicator *(HI),* Cartesian product, Composition and Corona Graph.

Introduction

The topological indicators along with various molecular characterization tools are crucial for QSPR/QSAR investigations in mathematical chemistry. [1]. Chemistry has made use of many topological indicator [2]. For a basic undirected graph G, V(G) equals a number of verges and E(G) equals m links since these two variables describe the verges and links sets of the graph. A number associated with the graph G is its topological indicator. The idea that topology and geometry may be separated for the sake of both was originally recognized by Riemann. Poincare enters the picture as the father of topology and Riemann's logical heir.

It wasn't until 2008 that the mathematical world began paying much attention to the $\mathscr{ABI}$, which was first establish in 1998. Following the publication of numerous encouraging chemical applications in [10], this study altered the Atom-bond-connectivity indicator, igniting interest in it. Many articles have addressed the statistical features of the Atom-bond-

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connectivity indicator (ABI) since it was initially examined mathematically in a study [4]. These papers continue to be issued at a steady pace and frequently concentrate on extreme results or boundaries. Zagreb- indicator along with additional topographical descriptors [12–14], sum-connectivity broadens [22, 23], *Zagreb*- indicator [17], and *harmonic*- indicator [18, 19], have been investigated.

By representing chemical events mathematically using graph invariants, chemical graph theory is essential to mathematical chemistry. Topological descriptors, graph invariants obtained from molecular graph representations of chemical compounds, are used in QSPR and QSAR research. Based on connection cardinality, graphs can be analyzed using the Zagreb connection indices, which are topological descriptors. In 1972, they were initially employed to calculate the total electron energy of alternative hydrocarbons. Because they can give a more accurate correlation for the physicochemical properties of many substances than basic Zagreb indices, they have recently attracted renewed study [12-14].

Numerous structures for comprehending substances issues from a numerical perspective are offered by chemical graph theory. It is particularly important for developing new mathematical models for chemical properties using a variety of chemistry-related mathematical methodologies. The interaction between different chemical structures in a chemical graph that can flexibly represent hypothetical or actual chemical systems, such molecules, is the subject of this multidisciplinary research. In mathematical chemistry, molecular invariants are important, particularly when doing QSPR and QSAR studies. Atoms are represented as vertices & links between them as edges in a molecular graph.

Correlating graphs with numbers requires the use of graph-theoretical invariants, such as local vertex invariants, such as vertex degree, distance sum, etc. since chemical structures are discrete graphs whereas physical attributes or bioactivities are given in numerical form. To date, hundreds of topological indexes have been proposed. We can distinguish three categories of topological indices based on the invariant, which is essential to the definition: indices that are based on degrees, distances, and spectrums. The Randi´c index, Zagreb index, connective eccentricity index, and here are some more instances of degree-based indicators. The Wiener index, Wiener polarity index, Szeged index, ABC index, Harary index, and others are distance-based indices.

Since shortest paths are commonly employed to characterize the distances between vertices in a graph, these indices account for their lengths. The Wiener index is the most well-known and computes the total of all shortest path lengths between vertex pairs. Other examples are the Hyper Wiener index and the Harary index.

A function that takes a molecule's topology and assigns it a numerical value that stays constant is called a topological index. Under isomorphism of graphs. This index, which contains certain physicochemical characteristics of molecules, can be evaluated by workers using computer methods, saving time and money on expensive and time-consuming laboratory research. In 1947, The idea of a topological index was initially presented.. The

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discipline of chemo informatics has benefited from the substantial number of topological indices that have since been created based on different graph properties.

Chemical graph theory uses the Zagreb indices, a pair of topological indices, to describe molecular structures. A graph's 1st Zagreb index (M1) is the sum of the squares of its vertex degrees, and its 2nd Zagreb index (M2) is the sum of the products of its neighboring vertices' degrees. The boiling points and other physicochemical properties of molecules are among the many attributes that can be predicted using these indices.

Molecular graphs are characterized by the geometric-arithmetic index (GA index), a topological index in mathematical chemistry that is the sum of the contributions from each edge, where the contribution is the ratio of the geometric mean to the arithmetic mean of the degrees of the connected vertices. This measure was created in an effort to enhance the Randić index's prediction power and has demonstrated promise in quantitative structure-property interactions and related domains.

In electrical engineering, harmonic indices—which mostly refer to a graph's harmonic index and total harmonic distortion, or THD—are mathematical ideas used to describe graphs and assess power quality in electrical systems. The mass 2/(d(u) + d(v)) for each edge uv in a graph G are added together to determine the graph's harmonic index, or H(G), Here d(u) and d(v) stand for the degrees of vertices u and v, respectively.

To learn more about these indices, The articles [1], [2], [3], [4], [5], [6], and [7] are recommended reading. *Atom-bond-connectivity* indicator (*ABI*)*, Randic-* indicator(*RI*)*, Zagreb-* indicator *(ZI), Geometric-Arithmetic* indicator *(GI), Harmonic-* indicator *(HI).*

2 Preliminaries

The entire graph, cycle, star, and route on n verges are represented by the symbols $\mathfrak{Cy}_n, \mathfrak{St}_n, \mathfrak{Pa}_n$ and $\mathfrak{Km}_n$ *appropriately.* Also study methene, Cyclopropane for join by Cartesian product, Composition and Corona product are calculated. *Moreover,* $\mathfrak{Km}_n$ *exposes the entire partitioned* $\mathfrak{G}$ *on n and m verges in its two segmentation.*

Some Important Results:

If $\mathrm{ABX}(\pi_2) = 0,\ \varphi o\rho\ \ \nu \geq 3$

Results 1: $\mathrm{ABX}(\pi_\nu) = (\nu - 1) * \frac{\sqrt{2}}{2}$

Results 2: $\mathrm{ABX}(\chi_\nu) = (\nu) * \left(\frac{\sqrt{2}}{2}\right)$

Results 3: $\mathrm{ABX}(\kappa_\nu) = (\nu) * \left(\sqrt{\nu - 2}\right)\left(\frac{\sqrt{2}}{2}\right)$

Definition 1.1: *Estrada et al. presented the* $\mathfrak{ABI}$ *in [14, 24, 27]:*

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$$\mathcal{ABI} = \sum_{\upsilon v \in verges(\mathbb{G})} (\delta_u + \delta_v - 2)^{1/2} \left(\frac{1}{(\delta_u + \delta_v)^{1/2}} \right)$$

Definition 1.2: Milan Randic proposed the $\mathbb{RI}$ in [28] :

$$\mathbb{RI} = \sum_{\upsilon v \in verges(\mathbb{G})} \sqrt{2\left(\frac{1}{\delta_u \delta_v} \right)}$$

Definition 1.3:*Fajtlowicz et al. suggest the harmonic-index in [17], [26]:*

$$\mathrm{HI} = \sum_{\upsilon v \in verges(\mathbb{G})} 2 \frac{1}{\delta_u + \delta_v}$$

Definition 1.4: The *Geometric-Arithmetic indicator (GI)* was offered by Vukicevic et al. in [31] as $\mathbb{GI} = \sum_{\upsilon v \in verges(\mathbb{G})} \frac{(2)\left(\sqrt{\delta_u \delta_v}\right)}{\left(\sqrt{\delta_u \delta_v}\right)\left(\sqrt{\delta_u \delta_v}\right)}$

Definition 1.5: Trinajestic and Gutman created the 1st Zagreb-indicator in [27] as

$$\mathbb{ZI}\text{-}1^{st} = \sum_{\upsilon v \in verges(\mathbb{G})} 1 * (\delta_u + \delta_v)$$

Definition 1.6: In [20], Gutman and Trinajestic propose the 2nd Zagreb- indicator $\mathbb{ZI}_2(\mathbb{G})$ as

$$\mathbb{ZI}\text{-}2^{nd} = \sum_{\upsilon v \in verges(\mathbb{G})} \delta_u \delta_v$$

3 ASSOCIATION OF *ATOM-BOND-CONNECTIVITY* INDICATOR WITH $\mathrm{HI}(\mathbb{G})$ $\mathbb{RI}(\mathbb{G})$, $\mathbb{GI}(\mathbb{G})$, $\mathbb{ZI}_1(\mathbb{G})$ & $\mathbb{ZI}_2(\mathbb{G})$

Theorem: 3.1a G

Let $\mathbb{G}$ be a graph with n verges & m links then $\mathcal{ABI}(\mathbb{G}) \leq \sqrt{\mathbb{RI}(\mathbb{G})\mathbb{Z}_2(\mathbb{G})(\mathbb{Z}_1(\mathbb{G}) - 2\mu)}$ when $\mathbb{G}$ is a complete graph, and then with equality[1-7].

Proof:

Let $\forall$ links $\varepsilon = \upsilon v \in \mathbf{v}erges(\mathbb{G})$, $\delta_\upsilon, \delta_v\, greater 1$ *then* $1 \leq (\delta_\upsilon \delta_v)^{1/2}$

$$\mathcal{ABI}(\mathbb{G}) = \sum_{\upsilon v \in verges(\mathbb{G})} 1 * \left(\frac{\sqrt{\delta_\upsilon + \delta_v - 2}}{\sqrt{\delta_\upsilon \delta_v}} \right)$$

Using Cauchy-Schwarz inequality

$$\mathcal{ABI}(\mathbb{G}) \leq \sum_{\upsilon v \in verges(\mathbb{G})} 1 * \left(\sqrt{\delta_\upsilon + \delta_v - 2} \right)$$

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$$\mathcal{ABI}(\mathbb{G}) \leq \sum_{\upsilon v\in verges(\mathbb{G})} \left(\frac{\delta_\upsilon \delta_v}{\delta_\upsilon \delta_v}\right)^{1/2} \left(\delta_\upsilon + \delta_v - 2\right)^{1/2}$$

$$\mathcal{ABI}(\mathbb{G}) \leq \sqrt{\mathbb{RI}(\mathbb{G})\mathbb{Z}_2(\mathbb{G})(\mathbb{Z}_1(\mathbb{G}) - 2\mu)}$$

The equivalence occurs above $\Leftrightarrow\ \forall\ \varepsilon = \upsilon v$

$\therefore \mathbb{G} \cong K_{\mathfrak{p}}$

Theorem: 3.1 b

Let $\mathbb{G}$ be a graph with n verges and m links then $\mathcal{ABI} \leq \sqrt{\mathbb{RI}(\mathbb{G})(\mathbb{Z}_1(\mathbb{G}) - 2\mu)}$ when $\mathbb{G}$ is a complete graph, and then with equality [1-5].

Proof:

Let $\forall$ links $\varepsilon = \upsilon v \in \mathrm{verges}(\mathbb{G})$ **&** $\delta_\upsilon, \delta_v \geq 1$ then $1 \leq \sqrt{\delta_\upsilon \delta_v}\sqrt{\delta_\upsilon \delta_v}$

$$\mathcal{ABI} = \sum_{\upsilon v\ \in \mathrm{verges}(\mathbb{G})} \frac{\sqrt{\delta_\upsilon + \delta_v - 2}}{\sqrt{\delta_\upsilon \delta_v}}$$

Using Cauchy-Schwarz inequality

$$\mathcal{ABI} \leq \sum_{\upsilon v\in \mathrm{verges}(\mathbb{G})} \left(\sqrt{\delta_\upsilon \delta_v}\right) * \left(\sqrt{\delta_\upsilon + \delta_v - 2}\right)$$

$$\mathcal{ABI} \leq \sqrt{\mathbb{RI}(\mathbb{G})(\mathbb{Z}_1(\mathbb{G}) - 2\mu)}$$

The equivalence occurs above $\Leftrightarrow\ \forall\ \varepsilon = \upsilon v$

$\therefore \mathbb{G} \cong K_{\mathfrak{p}}$

Theorem: 3.1c

Let $\mathbb{G}$ be a graph with n verges and m links then $\mathcal{ABI} \leq \sqrt{\mu\left(\mathbb{Z}_1(\mathbb{G}) - 2\mu\right)}$ when $\mathbb{G}$ is a complete graph, and then with equality[1-6].

Proof:

Let $\forall$ links $\varepsilon = \upsilon v \in \mathrm{verges}(\mathbb{G})$ $\delta_\upsilon, \delta_v greater 1$ then $\frac{1}{\delta_\upsilon \delta_v} \leq 1$

$$\left[\mathcal{ABI}\right]^2 = \left[\sum_{\upsilon v\in verges(\mathbb{G})} \frac{\sqrt{\delta_\upsilon + \delta_v - 2}}{\sqrt{\delta_\upsilon \delta_v}}\right]^2$$

Using Cauchy-Schwarz inequality

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$$[\mathbf{ABI}]^2 \le \sum_{\upsilon v \in \mathbf{verges(G)}} \mathbf{1}.[\boldsymbol{\delta}_\upsilon + \boldsymbol{\delta}_v - \mathbf{2}]$$

$$[\mathcal{ABI}]^2 \le \sum_{\upsilon v \in verges(\mathbb{G})} \mu.(\delta_\upsilon + \delta_v - 2)$$

$$\mathcal{ABI} \le \sqrt{\mu(\mathbb{Z}_1(\mathbb{G}) - 2\mu)}$$

The equivalence occurs above $\Leftrightarrow \ \forall \ \varepsilon = \upsilon v$

$\therefore \mathbb{G} \cong \mathrm{K}_{\mathfrak{n}}$

Theorem:3.1d

Let $\mathbb{G}$ be a graph with n verges and m links then $\mathcal{ABI} \le \sqrt{\mu(Z_1(\mathbb{G}) - 2\mu}$ then the upper bound does not happen.

Proof:

Let $\forall$ links $\varepsilon = \upsilon v \in \mathrm{verges(G)}$, $\delta_\upsilon, \delta_v$ greater 1 & $\delta_u + \delta_v - 2 < \delta_\upsilon \delta_v$

$$[\mathbf{ABI}]^2 = \left[\sum_{\upsilon v \in \mathbf{verges(G)}} \frac{\sqrt{\boldsymbol{\delta}_\upsilon + \boldsymbol{\delta}_v - \mathbf{2}}}{\sqrt{\boldsymbol{\delta}_\upsilon \boldsymbol{\delta}_v}} \right]^2$$

Using Cauchy-Schwarz inequality,

We get

$$[\mathcal{ABI}]^2 = \sum_{\upsilon v \in verges(\mathbb{G})} \frac{\delta_\upsilon + \delta_v - 2}{\delta_\upsilon \delta_v}$$

$$[\mathcal{ABI}]^2 < \sum_{\upsilon v \in verges(\mathbb{G})} \frac{\delta_\upsilon \delta_v}{\delta_\upsilon \delta_v}$$

$$\mathcal{ABI} \le \sqrt{\mu(\mathbb{Z}_1(\mathbb{G}) - 2\mu}$$

The equivalence occurs above $\Leftrightarrow \ \forall \ \varepsilon = \upsilon v$

$\therefore \mathbb{G} \cong \mathrm{K}_{\mathfrak{n}}$

4 ASSOCIATION OF *ABC*- INDICATOR WITH GEOMETRIC-ARITHMETIC INDICATOR

In graph theory, topological indices known as the atom-bond connectivity (ABC) index and the geometric arithmetic (GA) index are used to quantify the connectivity and structure

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of molecular graphs. To calculate these indices, the degrees of nearby verges connected by links are utilized. The GA index considers the geometric arithmetic means of the degrees of neighboring atoms, whereas the ABC index focuses on the connectivity of atoms [32], [36], and [37]. For a graph on $n \geq 3$ verges, we derive a relationship between the ABC- indicator and geometric-arithmetic indicator in the subsequent subsection

Theorem: 4.1

Let $\mathbb{G}$ be a associated graph on $n \geq 3$ that has low degree verges $\rho \geq \alpha$ and *greatest* degree $\sigma \leq \beta$ where $1 \leq \alpha \leq \beta \leq n-1$ and $\beta \leq 1$ Then

(i) $\frac{\sqrt{2\beta-2}}{\beta}\mathbb{GI}(\mathbb{G}) \leq \mathcal{ABI}$ with equality $\Leftrightarrow$ G is a graph

(ii) If $\beta < 2\alpha - 3 + \sqrt{5\alpha^2 - 14\alpha + 9}$ then $\mathcal{ABI}(\mathbb{G}) \leq \frac{\sqrt{2\alpha-2}}{\alpha}\mathbb{GI}(\mathbb{G})$

(iii) If $\beta < 2\alpha - 3 + \sqrt{5\alpha^2 - 14\alpha + 9}$ then $\mathcal{ABI}(\mathbb{G}) \leq \frac{(\alpha+\beta)\sqrt{\alpha+\beta-2}}{2\alpha\beta}\mathbb{GI}(\mathbb{G})$

For every edge of $\mathbb{G}$, there is equivalence $\Leftrightarrow$ one verge has a degree and the others have b degrees.

Proof:

Let $\forall$ links $\varepsilon = \upsilon v$, $\alpha \leq \delta_\upsilon \leq \delta_v \leq \beta$ .Since $\mathbb{G}$ is a associated graph with $n \geq 3$ verges and $\delta_v \geq 2$

Let's explore the function

$\mathbf{f(\delta_u,\delta_v) = \frac{(\delta_u+\delta_v)^2(\delta_u+\delta_v-2)}{4\delta_u{}^2\delta_v{}^2}}$. Here $\alpha \leq \delta_\upsilon \leq \delta_v \leq \beta$ and α_v *greater*2

$$\mathbf{\frac{\partial f(\delta_u,\delta_v)}{\partial \delta_u} = \frac{(\delta_u+\delta_v)^2(\delta_u+\delta_v-2)}{4\delta_u{}^2\delta_v{}^2}}$$

$$\mathbf{\frac{\partial f(\delta_u,\delta_v)}{\partial \delta_u} = \frac{(\delta_u+\delta_v)[\delta_u(\delta_u-\delta_v)+2\delta_v(2-\delta_v)]}{4\delta_u{}^3\delta_v{}^2}}$$

$$\mathbf{\frac{\partial\, f(\delta_u,\delta_v)}{\partial \delta_u} \leq 0}$$

We are aware that there is no answer to the Diophantine solution x+y-2=xy in a collection of natural numbers. Therefore $f(\delta_u,\delta_v)$ is purely systematically declining in α_u.

Thus, the min value of $\mathbf{f(\delta_u,\delta_v)}$ is $\mathbf{f(\delta_v,\delta_v)}$, $\forall \boldsymbol{\mu\alpha\xi\{\alpha,2\}} \leq \boldsymbol{\delta_v} \leq \boldsymbol{\beta}$

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The max value of $\mathbf{f(\delta_u,\delta_v)}$ is $\mathbf{f(\delta(\delta_v)}$, $\forall \mathbf{\mu\alpha\xi\{\alpha,2\}} \le \mathbf{\delta_v} \le \mathbf{\beta}$

$$\mathbf{f(\delta_u,\delta_v) = \frac{2\delta_v - 2}{\delta_v^{\,2}} \;\&\; \frac{\partial f(\delta_u,\delta_v)}{\partial \delta^v} = \frac{4-2\delta_v}{\delta_v^{\,3}} \le 0}$$ So

Therefore $\mathbf{\varphi(\delta_u,\delta_v)}$ is purely systematically declining in α_v.

Since the least value value of $\mathbf{\varphi(\alpha, \delta_v)}$ is $\mathbf{\frac{\sqrt{2\beta-2}}{\beta}}$

$$\therefore \mathbf{\frac{\sqrt{2\beta-2}}{\beta} \le \frac{ABI(G)}{GI(G)}}$$

On the other hand,

$$\mathbf{\varphi(\alpha,\delta_v) = \frac{\left[(\alpha+\delta_v)^2\right]\times\left[(\alpha+\delta_v-2)\right]}{4\alpha^2\delta_v^{\,2}}}$$. Here $\mathbf{\alpha \le \delta_\upsilon \le \delta_v \le \beta}$ **and** $\mathbf{\delta_v}$**greater2**

$$\mathbf{\frac{\partial\varphi(\alpha,\delta_v)}{\partial\delta_v} = \frac{\left[(\alpha+\delta_v)^2\right]\left[\left(\alpha^2+\alpha\delta_v\right)-(2\alpha^2-4\alpha\alpha\right]}{4\alpha^2\delta_v^{\,3}}}$$

If $\mathbf{\alpha = 1}$ **or** $\mathbf{\alpha = 2}$**, then** $\mathbf{\frac{\partial\varphi(\alpha,\delta_v)}{\partial\delta_v} \ge 0}$

Therefore $\phi(\alpha,\delta_v)$ is purely monotonically expanding in α_v.

If $\mathbf{\alpha \ge 3}$ **and** $\mathbf{\beta \le \frac{\alpha+\sqrt{9\alpha^2-16\alpha}}{2}}$ **then** $\mathbf{\frac{\partial\varphi(\alpha,\delta_v)}{\partial\delta_v} \le 0}$

If $a \ge 3$ *and* $b \le \frac{a+\sqrt{9a^2-16a}}{2}$ *then* $\frac{\partial f(a,\alpha_v)}{\partial_v} \le 0$ and $\phi(\alpha,\delta_v)$ is purely monotonically expanding in α_v.

We determine that the largest value of $\phi(\alpha,\delta_v)$ is $\mathbf{\mu\alpha\xi\{\alpha,2\} \le \delta_v \le \beta}$, $\mu\alpha\xi$ $\mathbf{\varphi(\alpha,\delta_v),\varphi(}\alpha,\alpha$).

Since $\mathbf{\varphi(\alpha,\delta_v) - \varphi(\alpha,\alpha\;) = \left(\frac{(\alpha+\beta)^2(\alpha-\beta+2)}{4\alpha^2\beta^2}\right) - \left(\frac{2\alpha-2}{\alpha^2}\right)}$

$$\mathbf{\varphi(\alpha,\delta_v) - \varphi(\alpha,\alpha\;) = \frac{\beta^3-(5\alpha-6)\beta^2+(3\alpha^2-4\alpha\alpha)+(\alpha^3-2\alpha^2)}{4\alpha^2\beta^2}}$$

$$\mathbf{\varphi(\alpha,\delta_v) - \varphi(\alpha,\alpha\;) = \frac{\left|\beta-(2\alpha-3-\sqrt{5\alpha^2-14\alpha+9}\right| \left|(\alpha\alpha\_\beta\right|\left|\beta-92\alpha-3+\sqrt{5\alpha^2-14\alpha+9}\right|}{4\alpha^2\beta^2}}$$

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We have

$$\max \boldsymbol{\varphi}(\boldsymbol{\alpha},\boldsymbol{\alpha}_{v}),\boldsymbol{\varphi}(\boldsymbol{\alpha},\boldsymbol{\alpha}\ )=\begin{cases}\boldsymbol{\varphi}(\boldsymbol{\alpha},\boldsymbol{\alpha}\ )=\dfrac{\mathbf{2\alpha-2}}{\boldsymbol{\alpha}^{2}} & \mathbf{,if}\ \boldsymbol{\beta}<\mathbf{2\alpha-3}+\sqrt{\mathbf{5\alpha^{2}-14\alpha+9}}\\ \boldsymbol{\varphi}(\boldsymbol{\alpha},\boldsymbol{\delta}_{v})=\dfrac{(\boldsymbol{\alpha}+\boldsymbol{\beta})^{2}(\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{2})}{\mathbf{4\alpha^{2}\beta^{2}}} & \mathbf{,if}\ \boldsymbol{\beta}\geq\mathbf{2\alpha-3}+\sqrt{\mathbf{5\alpha^{2}-14\alpha+9}}\end{cases}$$

if $\boldsymbol{\beta}<\mathbf{2\alpha-3}+\sqrt{\mathbf{5\alpha^{2}-14\alpha+9}}$ then $\dfrac{\mathbf{ABI(G)}}{\mathbf{GI(G)}}\leq\dfrac{\sqrt{\mathbf{2\alpha-2}}}{\boldsymbol{\alpha}}$

Basic similarity is permitted to take possible provided $\boldsymbol{\varphi}(\boldsymbol{\alpha},\boldsymbol{\delta}_{v})=\boldsymbol{\varphi}(\boldsymbol{\alpha},\boldsymbol{\alpha}\ )$ $\forall$ links $\varepsilon=\upsilon v$

if $\boldsymbol{\beta}<\mathbf{2\alpha-3}+\sqrt{\mathbf{5\alpha^{2}-14\alpha+9}}$ then $\dfrac{\mathbf{ABI(G)}}{\mathbf{GI(G)}}\leq\dfrac{(\boldsymbol{\alpha}+\boldsymbol{\beta})\sqrt{\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{2}}}{\mathbf{2\alpha\alpha}}$

Basic similarity is permitted to take possible provided $\boldsymbol{\varphi}(\boldsymbol{\delta}_{\upsilon},\boldsymbol{\delta}_{v})=\boldsymbol{\varphi}(\boldsymbol{\alpha},\boldsymbol{\beta}\ )$ $\varepsilon=\upsilon v$

Hence proved the theorem.

Theorem:4.2

Assume $\mathbb{G}$ be a basic associated graph with verges, then $\dfrac{\sqrt{2n-4}}{n-1}\mathcal{GI}(\mathbb{G})\leq\mathcal{ABI}(\mathbb{G})\leq\dfrac{n\sqrt{n-2}}{2(n-1)}\mathcal{GI}(\mathbb{G})$ The limitation bound is achieved $\Leftrightarrow\therefore\mathbb{G}\cong K_{\mathfrak{n}}$ and the higher limit is achieved $\Leftrightarrow.\therefore\mathbb{G}\cong\mathfrak{St}_{\mathfrak{n}}$

Theorem: 4.3

Let $\mathbb{G}$ be a linked graph with low degree verges $\rho\geq\alpha$ and g*reatest* degree $\sigma\leq\beta$ where $1\leq\alpha\leq\beta\leq n-1$ and $\beta\geq1$ If $\beta\leq4\alpha^{2}-3\alpha+1$ then $\mathcal{ABI}(\mathbb{G})\leq\mathcal{GI}(\mathbb{G})$

5 main results

Theorem: 5.1

Suppose that α and β are arbitrary graphs then prove that atom bond index disjoint vertex sets are[1-4]

$$\mathbf{ABI}(\boldsymbol{\alpha}+\boldsymbol{\beta})\leq\left(\mathbf{ABI}(\boldsymbol{\alpha}\mathbf{B}\frac{\boldsymbol{\psi}_{\alpha}}{(\boldsymbol{\delta}_{\alpha}+|\boldsymbol{\beta}|)}+\frac{\mathbf{2}*|\boldsymbol{\beta}|}{(\boldsymbol{\delta}_{\alpha}+|\boldsymbol{\beta}|)}\right)+\left(\mathbf{ABI}(\boldsymbol{\beta}\mathbf{B}\frac{\boldsymbol{\psi}_{\beta}}{(\boldsymbol{\delta}_{\beta}+|\boldsymbol{\alpha}|)}+\frac{\mathbf{2}*|\boldsymbol{\alpha}|}{(\boldsymbol{\delta}_{\beta}+|\boldsymbol{\alpha}|)}\right)$$

$$+\left\||\boldsymbol{\alpha}|*|\boldsymbol{\beta}|\right\|\left(\sqrt{\frac{\boldsymbol{\psi}_{\alpha}+\boldsymbol{\psi}_{\beta}+|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-\mathbf{2}}{(\boldsymbol{\delta}_{\alpha}+|\boldsymbol{\beta}|)(\boldsymbol{\delta}_{\beta}+|\boldsymbol{\alpha}|)}}\right)$$

Proof:

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Let us Consider that $(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)\leq 1\ *(\delta_\alpha+|\beta|)^2$

$$\frac{1}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}\leq\frac{1}{(\delta_\alpha+|\beta|)^2}$$

$$ABI(\alpha+\beta)=\sum_{e=uv}\sqrt{\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}}$$

$$ABI(\alpha+\beta)=\sum_{\substack{e=uv\\ u,v\in V(\alpha(}}1*\sqrt{\left(\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}\right)}+\sum_{\substack{e=uv\\ u,v\in V(\beta(}}1*\sqrt{\left(\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}\right)}+\sum_{\substack{e=uv\\ u\in V(\alpha(\ ,v\in V(\beta(}}1*\sqrt{\left(\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}\right)}$$

Suppose that if $u\in V(\alpha)$ then $\alpha_u=\Delta\gamma_\alpha(u)+|\beta|$ and $\alpha_v=\Delta\gamma_\alpha(v)+|\beta|$

$$\sum_{\substack{e=uv\\ u,v\in V(\alpha(}}1*\sqrt{\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}}=\sum_{\substack{e=uv\\ u,v\in V(\alpha(}}1*\left(\sqrt{\frac{\Delta\gamma_\alpha(u)+|\beta|+\Delta\gamma_\alpha(v)+|\beta|-2}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}}\right)$$

$$\sum_{\substack{e=uv\\ u,v\in V(\alpha(}}\frac{\Delta\gamma_\alpha(u)+|\beta|+\Delta\gamma_\alpha(v)+|\beta|-2}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}=\frac{\Delta\gamma_\alpha(u)+\Delta\gamma_\alpha(v)-2}{\Delta\gamma_\alpha(u)*\Delta\gamma_\alpha(v)}\frac{\Delta\gamma_\alpha(u)*\Delta\gamma_\alpha(v)}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}$$
$$+\frac{2|\beta|}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}$$

Using Cauchy's Schwarz inequality,

Since $\left(\frac{\Delta\gamma_\alpha(u)*\Delta\gamma_\alpha(v)}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}\right)\leq\left(\frac{\psi_\alpha^2}{(\delta_\alpha+|\beta|)^2}\right)$ and

$$\frac{1}{(\Delta\Delta_\alpha(u)+|\beta|)(\Delta\gamma_\alpha(v)+|\beta|)}\leq\frac{1}{(\delta_\alpha+|\beta|)^2}$$

$$\frac{\psi_\alpha}{(\delta_\alpha+|\beta|)}+\frac{2|\beta|}{(\delta_\alpha+|\beta|)}$$

Similarly

$$\sum_{\substack{e=uv\\ u,v\in V(\alpha(}}\sqrt{\left(\frac{\Delta\gamma_\beta(u)+|\alpha|+\Delta\gamma_\beta(v)+|\alpha|-2}{(\Delta\Delta_\beta(u)+|\alpha|)*(\Delta\gamma_\beta(v)+|\alpha|)}\right)}\leq\left(\frac{\Delta\gamma_\beta(u)+\Delta\gamma_\beta(v)-2}{\Delta\gamma_\beta(u)*\Delta\gamma_\beta(v)}\right)\frac{\psi_\beta}{(\delta_\beta+|\alpha|)}+\frac{2|\alpha|}{(\delta_\beta+|\alpha|)}$$

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$$\mathbf{ABI}(\alpha+\beta) \le \sum_{\substack{e=uv \\ u,v\in V(\alpha(}} \left(\sqrt{\left(\frac{\Delta\gamma_\alpha(u)+\Delta\gamma_\alpha(v)-2}{\Delta\gamma_\alpha(u)*\Delta\gamma_\alpha(v)}\right)} * \left(\frac{\psi_\alpha}{(\delta_\alpha+|\beta|)}\right) + \frac{2|\beta|}{(\delta_\alpha+|\beta|)} \right)$$

$$+ \sum_{\substack{e=uv \\ u,v\in V(\beta(}} \left(\sqrt{\left(\frac{\Delta\gamma_\beta(u)+\Delta\gamma_\beta(v)-2}{\Delta\gamma_\beta(u)*\Delta\gamma_\beta(v)}\right)} * \left(\frac{\psi_\beta}{(\delta_\beta+|\alpha|)}\right) + \left(\frac{2|\alpha|}{(\delta_\beta+|\alpha|)}\right) \right)$$

$$+ \sum_{\substack{e=uv \\ u\in V(\alpha(\alpha v\in V(\beta(}} \left(\sqrt{\left(\frac{\psi_\alpha+\psi_\beta+|\alpha|+|\beta|-2}{(\delta_\alpha+|\beta|)(\delta_\beta+|\alpha|)}\right)} \right)$$

$$\mathcal{A}\mathbf{BI}(\alpha+\beta) \le \left(\mathbf{ABI}(\alpha B\frac{\psi_\alpha}{(\delta_\alpha+|\beta|)} + \mathbf{E}(\alpha(\frac{2|\beta|}{(\delta_\alpha+|\beta|)}) \right)$$

$$+ \left(\mathbf{ABI}(\beta B\frac{\psi_\beta}{(\delta_\beta+|\alpha|)} + \mathbf{E}(\beta(\frac{2|\alpha|}{(\delta_\beta+|\alpha|)} \right)$$

$$+ \| \alpha| * |\beta\| \left(\sqrt{\frac{\psi_\alpha+\psi_\beta+|\alpha|+|\beta|-2}{(\delta_\alpha+|\beta|)*(\delta_\beta+|\alpha|)}} \right)$$

Theorem: 5.2

Suppose that α and β are arbitrary $\mathcal{G}$ then the Cartesian product of $\mathcal{ABI}$ [1-5]

$$\mathcal{ABI}(\alpha*\beta) \le \mathcal{ABI}(\alpha)*|\beta| \left(\frac{\psi_\alpha}{(\delta_\alpha+\delta_\beta)}\right) + (\beta)*|\alpha| \left(\frac{\psi_\beta}{(\delta_\alpha+\delta_\beta)}\right) + \frac{2}{(\delta_\alpha+\delta_\beta)} +$$

$$|\alpha|*|\beta| \left(\frac{2}{(\delta_\alpha+\delta_\beta)}\right).$$

Proof:

Suppose *u= (a, b) and v= (c, d)* are vertices of atom bond index of $\alpha\times\beta$

Suppose that if u *belongs to* $V(\alpha)$ then $\delta_u = \Delta\gamma_\alpha(u)+|\beta|$ and $\delta_v = \Delta\gamma_\alpha(v)+|\beta|$

$$\mathbf{ABI}(\alpha+\beta) = \sum_{\substack{e=uv \\ u,v\in V(\alpha(}} \left(\sqrt{\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}} \right) \left(\frac{\delta_u+\delta_v-2}{\delta_u\delta_v} \right)$$

$$= \frac{((\Delta\Delta_\alpha(a)+\Delta\gamma_\beta(b)) + ((\Delta\Delta_\alpha(c)+\Delta\gamma_\beta(d)) - 2}{((\Delta\Delta_\alpha(a)+\Delta\gamma_\beta(b))*((\Delta\Delta_\alpha(c)+\Delta\gamma_\beta(d))}$$

$$= \frac{((\Delta\Delta_\alpha(a)+\Delta\gamma_\alpha(c)-2) + ((\Delta\Delta_\beta(b)+\Delta\gamma_\beta(d)-2) + 2}{((\Delta\Delta_\alpha(a)+\Delta\gamma_\beta(b))*((\Delta\Delta_\alpha(c)+\Delta\gamma_\beta(d))}$$

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$$= \frac{\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\alpha(c) - 2\right)}{\Delta\gamma_\alpha(a) * \Delta\gamma_\alpha(c)} \frac{\Delta\gamma_\alpha(a) * \Delta\gamma_\alpha(c)}{\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\beta(b)\right) * \left((\Delta\Delta_\alpha(c) + \Delta\gamma_\beta(d)\right)}$$

$$+ \frac{\left((\Delta\Delta_\beta(b) + \Delta\gamma_\beta(d) - 2\right)}{\Delta\gamma_\beta(b) * \Delta\gamma_\beta(d)} \frac{\Delta\gamma_\beta(b) * \Delta\gamma_\beta(d)}{\left(\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\beta(b)\right) * \left((\Delta\Delta_\alpha(c) + \Delta\gamma_\beta(d)\right)\right)}$$

$$+ \frac{2}{\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\beta(b)\right) * \left((\Delta\Delta_\alpha(c) + \Delta\gamma_\beta(d)\right)}$$

Using Cauchy's Schwarz inequality,

Since $\frac{\Delta\gamma_\alpha(a) * \Delta\gamma_\alpha(c)}{\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\beta(b)\right) * \left((\Delta\Delta_\alpha(c) + \Delta\gamma_\beta(d)\right)} \leq \left(\frac{\psi_\alpha^2}{(\delta_\alpha + \delta_\beta)^2}\right)$ and

$$\frac{\Delta\gamma_\beta(b)\Delta b_\beta(d)}{\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\beta(b)\right)\left((\Delta\Delta_\alpha(c) + \Delta\gamma_\beta(d)\right)} \leq \frac{\psi_\beta^2}{(\delta_\alpha + \delta_\beta)^2}$$

$$\mathbf{ABI}(\alpha * \beta) \leq \frac{\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\alpha(c) - 2\right)}{\Delta\gamma_\alpha(a) * \Delta\gamma_\alpha(c)} \left(\frac{\psi_\alpha^2}{(\delta_\alpha + \delta_\beta)^2}\right) + \frac{\left((\Delta\Delta_\beta(b) + \Delta\gamma_\beta(d) - 2\right)}{\Delta\gamma_\beta(b) * \Delta\gamma_\beta(d)} \frac{\psi_\beta^2}{(\delta_\alpha + \delta_\beta)^2} + $$

$$+ \frac{2}{(\delta_\alpha + \delta_\beta)^2}$$

Hence

$$\sqrt{\mathbf{ABI}(\alpha * \beta)} \leq \sqrt{\frac{\left((\Delta\Delta_\alpha(a) + \Delta\gamma_\alpha(c) - 2\right)}{\Delta\gamma_\alpha(a) * \Delta\gamma_\alpha(c)}} \left(\frac{\psi_\alpha^2}{(\delta_\alpha + \delta_\beta)^2}\right) + \sqrt{\frac{\left((\Delta\Delta_\beta(b) + \Delta\gamma_\beta(d) - 2\right)}{\Delta\gamma_\beta(b) * \Delta\gamma_\beta(d)}} \frac{\psi_\beta^2}{(\delta_\alpha + \delta_\beta)^2}$$

$$+ \frac{\sqrt{2}}{(\delta_\alpha + \delta_\beta)^2}$$

$$\sqrt{\mathbf{ABI}(\alpha * \beta)} \leq \sqrt{\mathbf{ABI}(\alpha B^* |\beta|} \frac{\psi_\alpha^2}{(\delta_\alpha + \delta_\beta)^2} + \sqrt{\mathbf{ABI}(\beta B^* |\alpha|} \left(\frac{\psi_\beta^2}{(\delta_\alpha + \delta_\beta)^2}\right) + \frac{\sqrt{2}}{(\delta_\alpha + \delta_\beta)^2}$$

$$+ \sqrt{|\alpha| * |\beta|} \frac{\sqrt{2}}{(\delta_\alpha + \delta_\beta)^2}$$

$$\mathbf{ABI}(\alpha * \beta) \leq \mathbf{ABI}(\alpha B^* |\beta| + \mathbf{ABI}(\beta B^* |\alpha| \left(\frac{\psi_\beta}{(\delta_\alpha + \delta_\beta)}\right) + \frac{2}{(\delta_\alpha + \delta_\beta)} + \quad |\alpha||\beta| \frac{2}{(\delta_\alpha + \delta_\beta)}$$

Theorem: 5.3

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Suppose that α and β are arbitrary 𝔊 then the Corona graphs of atom bond index is

$$\mathcal{ABI}\sqrt{(\alpha\times\beta)} \le \mathcal{ABI}(\alpha)|\beta| + A\mathcal{ABI}(\beta)|\alpha| \frac{\psi_\beta^2}{(\delta_\alpha+\delta_\beta)^2} + \frac{\sqrt{2}}{(\delta_\alpha+\delta_\beta)^2} + \quad |\alpha||\beta| \frac{\sqrt{2}}{(\delta_\alpha+\delta_\beta)^2}$$

Proof:

$$\mathbf{ABI}(\boldsymbol{\alpha}\circ\boldsymbol{\beta}) = \sum_{\substack{e=uv\\ u,v\in V(\alpha(}} \left(1*\sqrt{\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}}\right) + \sum_{\substack{e=uv\\ u,v\in V(\beta(}} \left(1*\sqrt{\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}}\right) + \sum_{\substack{e=uv\\ u\in V(\alpha(\alpha v\in V(\beta(}} \left(1*\sqrt{\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}}\right)$$

Suppose that if $u,v\in V(\alpha)$ then $\alpha_u = \Delta\gamma_\alpha(u)+|\beta|$ and $\alpha_v = \Delta\gamma_\alpha(v)+|\beta|$

$$\sum_{\substack{e=uv\\ u,v\in V(\alpha(}} \left(1*\sqrt{\frac{\delta_u+\delta_v-2}{\delta_u\delta_v}}\right) = \sum_{\substack{e=uv\\ u,v\in V(\alpha(}} 1*\left(\sqrt{\frac{\Delta\gamma_\alpha(u)+|\beta|+\Delta\gamma_\alpha(v)+|\beta|-2}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}}\right)$$

$$= \sum_{\substack{e=uv\\ u,v\in V(\alpha(}} 1*\left(\sqrt{\frac{\Delta\gamma_\alpha(u)+|\beta|+\Delta\gamma_\alpha(v)+|\beta|-2}{\Delta\gamma_\alpha(u)*\Delta\gamma_\alpha(v)}}\right)*\left(\sqrt{\frac{\Delta\gamma_\alpha(u)*\Delta\gamma_\alpha(v)}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}}\right)$$

$$+ \sum_{\substack{e=uv\\ u,v\in V(\alpha(}} \sqrt{\frac{2|\beta|}{(\Delta\Delta_\alpha(u)+|\beta|)*(\Delta\gamma_\alpha(v)+|\beta|)}}$$

$$\le \sum_{\substack{e=uv\\ u,v\in V(\alpha(}} \left(\sqrt{\frac{\Delta\gamma_\alpha(u)+\Delta\gamma_\alpha(v)-2}{\Delta\gamma_\alpha(u)*\Delta\gamma_\alpha(v)}}\left(\frac{\psi_\alpha}{(\delta_\alpha+|\beta|)}\right)+\frac{2|\beta|}{(\delta_\alpha+|\beta|)}\right)$$

If $u,v\in V(\beta)$ then $\alpha_u = \Delta\gamma_\beta(u)+1$ and $\alpha_v = \Delta\gamma_\beta(v)+1$

$$\frac{\alpha_u+\alpha_v-2}{\alpha_u\alpha_v} = \frac{\Delta\gamma_\beta(u)+1+\Delta\gamma_\beta(v)+1-2}{(\Delta\gamma_\beta(u)+1)*(\Delta\gamma_\beta(v)+1)}$$

$$=\left(\frac{\Delta\gamma_\beta(u)+\Delta\gamma_\beta(v)-2}{\Delta\gamma_\beta(u)*\Delta\gamma_\beta(v)}\right)*\left(\frac{\Delta\gamma_\beta(u)*\Delta\gamma_\beta(v)}{(\Delta\gamma_\beta(u)+1)*(\Delta\gamma_\beta(v)+1)}\right)+\left(\frac{2}{(\Delta\gamma_\beta(u)+1)*(\Delta\gamma_\varphi(v)+1)}\right)$$

$$\le\left(\sqrt{\frac{\Delta\gamma_\beta(u)+\Delta\gamma_\beta(v)-2}{\Delta\gamma_\beta(u)*\Delta\gamma_\beta(v)}}\right)\left(\frac{\psi_\beta^2}{(\delta_\beta+1)^2}\right)+\frac{2}{(\delta_\beta+1)^2}$$

Suppose that if $u\in V(\alpha)$ and $\alpha_u = \Delta\gamma_\alpha(u)+|\beta|$, $v\in V(\beta)$, $\alpha_v = \Delta\gamma_\beta(v)+1$

$$\frac{\alpha_u+\alpha_v-2}{\alpha_u\alpha_v} = \left(\frac{\Delta\gamma_\alpha(v)+1+\Delta\gamma_\alpha(u)+|\beta|}{(\Delta\gamma_\alpha(u)+|\beta|)*(\Delta\gamma_\beta(v)+1)}\right)$$

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$$\leq \frac{\psi_\alpha + \psi_\beta + |\beta| - 1}{(\delta_\alpha + |\beta|) * (\delta_\beta + 1)}$$

$$\mathcal{ABI}(\alpha \circ \beta) \leq \sum_{\substack{e=uv \\ u,v \in V(\alpha(}} \left(\left(\sqrt{\frac{\Delta\gamma_\alpha(u) + \Delta\gamma_\alpha(v) - 2}{\Delta\gamma_\alpha(u)\Delta u_\alpha(v)}} \right) * \left(\frac{\psi_\alpha}{(\delta_\alpha + |\beta|)} \right) + \frac{2|\beta|}{(\delta_\alpha + |\beta|)} \right)$$

$$+ \sqrt{\left(\frac{\Delta\gamma_\beta(u) + \Delta\gamma_\beta(v) - 2}{\Delta\gamma_\beta(u) * \Delta\gamma_\beta(v)} \right)} \left(\frac{\psi_\beta}{(\delta_\beta + 1)} \right) + \left(\frac{2}{(\delta_\beta + 1)} \right) + \sqrt{\frac{\psi_\alpha + \psi_\beta + |\beta| - 1}{(\delta_\alpha + |\beta|) * (\delta_\beta + 1)}}$$

$$\leq \left(\mathbf{ABI}(\alpha B \frac{\psi_\alpha}{(\delta_\alpha + |\beta|)} + \mathbf{E}(\alpha(\frac{\sqrt{2|\beta|}}{(\delta_\alpha + |\beta|)} \right) + \mathbf{ABI}(\beta B \frac{\psi_\beta}{(\delta_\beta + 1)} + \mathbf{E}(\beta(\frac{\sqrt{2}}{(\delta_\beta + 1)}$$

$$+ |\alpha| \, |\beta| \sqrt{\frac{\psi_\alpha + \psi_\beta + |\beta| - 1}{(\delta_\alpha + |\beta|) * (\delta_\beta + 1)}}.$$

Deciphering Visual Patterns

The calculated outcomes and accompanying charts show a number of significant correlations between the ABC index and other topological indices, including:

1. **ABC vs Randić Index**

- ❖ The scatter plot confirmed the theoretical bounds (2 R(G) ≤ ABC(G) ≤ Δ R(G) 2) by displaying a significant near-linear trend.
- ❖ The points of regular structures, such as cycles and complete graphs, are nearly on a straight line.
- ❖ For simple molecular graphs, this implies that the Randić and ABC indices are highly associated and both capture degree balance.

2. ABC vs GA Index

- ❖ The values of the GA index and the ABC index were constantly near each other.
- ❖ Given their similarities, GA is a good predictor of ABC since both indices highlight harmonic-mean-like correlations between vertex degrees.

3. ABC vs Harmonic Index (H)

- ❖ The scatter plot showed a dispersed and nonlinear pattern.

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❖ When the maximum degree (stars, full graphs) increased, divergence increased, yet ABC and H were moderately aligned for low-degree graphs like pathways.

❖ Since ABC is less susceptible to degree imbalance than H, it is more stable across a wider range of graph classes.

4. ABC vs Zagreb Indices (M1, M2)

❖ Scatter plots for ABC vs. M1/M2 revealed curving and more dispersed associations than Randić or GA.

❖ Since Zagreb indices amplify high-degree contributions, while ABC moderates them with a square-root function, the relationship is **nonlinear and graph-dependent**. Comparing Zagreb indices to ABC, this suggests that the former overstate high-degree impacts.

5. **Bar Chart Insights**

❖ For a single graph, when comparing indices

❖ The magnitude of the M1 and M2 values was significantly greater.

❖ Comparative interpretability was supported by the closer scale of the ABC, Randić, GA, and Harmonic indices.

❖ For graphs with several degrees, degree-based but scaled indices (ABC, R, GA, and H) are therefore more helpful in QSPR/QSAR than Zagreb indices.

The following important insights are highlighted by the results:

Strong Associations

- There is a high theoretical and visual link between the Randić and GA indices and the ABC index. Because of their similar magnitudes, they appear to measure structural balance similarly.

Moderate Association

- As the maximum degree rises, there is a divergence between the Harmonic index and the ABC index. This demonstrates ABC's greater stability across high degree variable graphs.

Weak/Nonlinear Association

- Since ABC moderates high-degree contributions and Zagreb indices augment them, the two have a nonlinear connection (M1, M2).

Comparative Magnitudes

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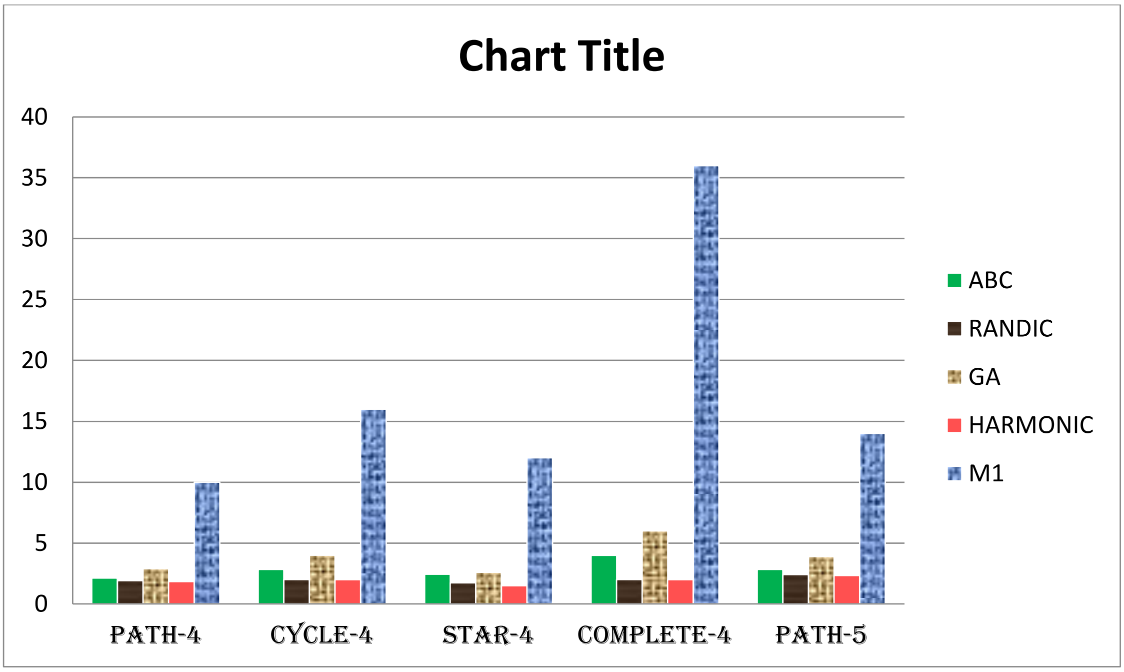
- M1 and M2 values are significantly bigger than those of ABC, Randić, GA, and Harmonic, as can be seen from bar chart analysis. This makes them less useful for direct comparisons in QSPR/QSAR investigations.

A visual comparison of the scaling of various indices for the same structure.

Graphic	ABC	Randic	GA	Harmonic	M1	M2
Path-4	2.121	1.914	2.886	1.833	10	8
Cycle-4	2.828	2.000	4.000	2.000	16	16
Star-4	2.500	1.732	2.598	1.500	12	9
Complete-4	4.000	2.000	6.000	2.000	36	54
Path-5	2.828	2.414	2.886	2.333	14	12

The following bar charts show how the ABC index compares to other topological indices (Randić, GA, Harmonic, M1, and M2) on various short graphs:



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These parallels unequivocally demonstrate:

- ❖ Randić and GA are the closest to ABC in terms of value.
- ❖ Harmonic and ABC deviate from each other somewhat.
- ❖ The Zagreb indices (M1, M2), which increase more quickly with increasing vertex degrees, are significantly larger than ABC.
- ❖ A grouped bar chart displays each graph's topological indices (ABC, Randić, GA, Harmonic, M1, and M2) next to each other.

6 Conclusion

In this paper, the association of *Atom-bond-connectivity* indicator (*ABI*), *Randic*-indicator (*RI*), *Zagreb-* indicator *(ZI), Geometric-Arithmetic* indicator *(GI), Harmonic*-indicator *(AI)*is *established* Using Cauchy-Schwarz inequality. The results in this paper can be extended to find the correlation between the *topological indices*. This paper deduces disconsistency for *Atom-bond-connectivity* indicator (*ABI*) of $\mathcal{G}$, which encourages research into the medicinal uses of methane stability and cyclopropane strain energy.

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