International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Fuzzy $\mathcal{B}^*$-open sets and Fuzzy $\mathcal{B}^*$-continuity in $\hat{S}$ostak's fuzzy topological spaces

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Abstract

The main objective of this article is to introduce and develop the concepts of r-fuzzy $\mathcal{B}^*$-open sets and r-fuzzy $\mathcal{B}^*$-continuous functions within a topological framework. These notions are investigated in detail, and their relationships with existing classes of open sets and continuous functions are examined. Furthermore, various fundamental properties and illustrative results concerning these newly introduced concepts are established, contributing to a deeper understanding of their role in general topology.

Keywords and phrases: r-fuzzy $\mathcal{B}^*$-open, r-fuzzy $\mathcal{B}^*$-continuous functions and r-fuzzy $(\mathcal{B}^*, \sigma)$-graph.

AMS (2000) subject classification: 54A40, 54C05, 54C08, 54D10 ,54D65.

1 Introduction and preliminaries

Over the past several decades, numerous generalizations of open sets and continuity have been proposed in order to deepen the understanding of topological structures and to extend classical results to broader settings. Among these developments, Abd El-Monsef et al. [1] introduced the concept of β-open sets along with the corresponding notion of β-continuity in topological spaces.

Ramadan proposed a related framework through the introduction of smooth topological spaces, which generalize classical topological structures. This notion has since been extensively investigated and expanded in various directions by several researchers [3, 4, 12, 8, 18]. A notable development in this area was made in 2004 by

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Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

S. E. Abbas [1], who introduced the concept of r-fuzzy β-open-open sets, thereby providing a broader class of generalized open sets within fuzzy topology. This concept played an important role in refining the interaction between fuzziness and topological openness. Subsequently, motivated by the study of generalized openness in fuzzy settings, Vadivel and co-authors, during 2017 and 2018, introduced and systematically examined the classes of e-open and e^*- sets in the sense of $\hat{S}$ostak fuzzy topology. These notions further extended the hierarchy of generalized open sets and contributed to a deeper understanding of structural properties and continuity in fuzzy topological spaces.

Throughout this article, let $\mathcal{P}$ be a non-empty set, $\mathcal{I} = [0,1], \mathcal{I}_0 = (0,1]$. A fuzzy set ψ of $\mathcal{P}$ is defined as a function $\psi : \mathcal{P} \to \mathcal{I}$, and the collection of all fuzzy subsets of $\mathcal{P}$ is denoted by $\mathcal{I}^X$.
For a fuzzy subset ψ of a space $(\mathcal{P}, \sigma)$, the symbols $\mathcal{C}_\sigma(\psi, r)$, $\mathcal{I}_\sigma(\psi, r)$ and $\overline{1} - \psi$ represent the r-fuzzy closure, r-fuzzy interior, and the complement of ψ, respectively. For $\psi \in \mathcal{I}^X$, the value $\psi(x)$ denotes the degree of membership of $p \in \mathcal{P}$.
For any $p \in \mathcal{P}$ and $t \in \mathcal{I}_0$, a fuzzy point x_t is defined by

$$P_t(q) = \begin{cases} t & \text{if } q = p; \\ 0 & \text{if } q \neq p. \end{cases}$$

Let $Pt(X)$ is the collection of all fuzzy points in $\mathcal{P}$. Unless stated otherwise, standrd terminology and notation from fuzzy set theory are used throughout this paper.

Definition 1.1. [11] A mapping $\sigma : \mathcal{I}^X \to \mathcal{I}$ is called a smooth topology space(STs) on $\mathcal{P}$ the below statement are holds:

(1) $\sigma(\overline{0}) = \sigma(\overline{1}) = 1$,

(2) $\sigma(\bigvee_{i\in\Gamma} \omega_i) \geq \bigwedge_{i\in\Gamma} \sigma(\omega_i)$, for every family $\{\omega\}_{i\in\Gamma} \subset \mathcal{I}^X$,

(3) $\sigma(\omega_1 \wedge \omega_2) \geq \sigma(\omega_1) \wedge \sigma(\omega_2)$, for all $\omega_1, \omega_2 \in \mathcal{I}^X$

The ordered pair $(\mathcal{P},\ \sigma)$ is called a smooth topological space (brifly. STs). A fuzzy subset ψ of $\mathcal{P}$ is said to be r-fuzzy open (briefly, r-fOP) if $\sigma(\psi) \geq r$. Similarly ψ is called r-fuzzy closed (briefly,r-fCl) if $\sigma(\overline{1} - \psi) \geq r$.

Theorem 1.1. [2] Let $(\mathcal{P}, \sigma)$ be a STs. For every $\psi \in \mathcal{I}^X$ & $r \in \mathcal{I}_0$, define the mapping $\mathcal{C}_\sigma : \mathcal{I}^X \times \mathcal{I}_0 \to \mathcal{I}^X$ as follows: $\mathcal{C}_\sigma(\psi,\ r) = \wedge\{\omega \in \mathcal{I}^X : \psi \leq \omega, \sigma(\overline{1} - \omega) \geq r\}$. For any $\psi, \omega \in \mathcal{I}^X$ & $r, s \in \mathcal{I}_0$, *the operator* $\mathcal{C}_\sigma$ holds the following properties:

(Ci) $\mathcal{C}_\sigma(0^-,\ r) = 0^-$,

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

(Cii) $\psi \leq \mathcal{C}_\sigma(\psi,\ r)$,

(Ciii) $\mathcal{C}_\sigma(\psi,\ r) \vee \mathcal{C}_\sigma(\omega,\ r) = \mathcal{C}_\sigma(\psi \vee \omega,\ r)$,

(Civ) $\mathcal{C}_\sigma(\psi,\ r) \leq \mathcal{C}_\sigma(\psi,\ s)$ if $r \leq s$,

(Cv) $\mathcal{C}_\sigma(\mathcal{C}_\sigma(\psi,\ r),\ r) = \mathcal{C}_\sigma(\psi,\ r)$.

Theorem 1.2. [2] Let $(\mathcal{P}, \sigma)$ be a STs. For every $\psi \in \mathcal{I}^X$ & $r \in \mathcal{I}_0$, define the mapping $\mathcal{I}_\sigma : \mathcal{I}^X \times \mathcal{I}_0 \to \mathcal{I}^X$ by:
$\mathcal{I}_\sigma(\psi, r) = \vee\{\omega \in \mathcal{I}^X : \omega \leq \psi, \sigma(\omega) \geq r\}$.
For $\psi, \omega \in \mathcal{I}^X$ & $r, s \in \mathcal{I}_0$, the operator $\mathcal{I}_\sigma$ satisfies the below conditions:

(Ii) $\mathcal{I}_\sigma(\overline{1}, r) = \overline{1}$,

(Iii) $\mathcal{I}_\sigma(\psi, r) \leq \psi$,

(Iiii) $\mathcal{I}_\sigma(\psi, r) \wedge \mathcal{I}_\sigma(\omega, r) = \mathcal{I}_\sigma(\psi \wedge \omega, r)$,

(Iiv) $\mathcal{I}_\sigma(\psi, r) \leq \mathcal{I}_\sigma(\psi, s)$ if $\leq r$,

(Iv) $\mathcal{I}_\sigma(\mathcal{I}_\sigma(\psi, r), r) = \mathcal{I}_\sigma(\psi, r)$,

(Ivi) $\mathcal{I}_\sigma(1^- - \psi, r) = 1^- - \mathcal{C}_\sigma(\psi, r)$ & $\mathcal{C}_\sigma(1^- - \psi, r) = 1^- - \mathcal{I}_\sigma(\psi, r)$.

Definition 1.2. [7] A pt $p \in \mathcal{P}$ is said to be an r-fuzzy δ-cluster point of ψ if $\mathcal{I}_\sigma(\mathcal{C}_\sigma(\mathcal{U}, r), r) \wedge \psi = \phi$, for every open set $\mathcal{U} \in X$ containing x. The collection of all r-fuzzy δ-cluster points of ψ is said to be r-fuzzy δ-closure of ψ and is indicate $f\delta\mathcal{C}_\sigma(\psi)$.

Definition 1.3. [7] A set ψ is r-fuzzy δ-closed iff $\psi = \delta\mathcal{C}_\sigma(\psi, r)$. The complement of a r-fuzzy δ-closed set is called an r-fuzzy δ-open set. The r-fuzzy δ-interior of a fuzzy subset ψ of X is the union of all r-fuzzy δ -open sets of X contained in ψ.

Definition 1.4. let ψ be a fuzzy subset of a space X. Then ψ is called:

(1) r-fuzzy a-open [16] if $\psi \leq \mathcal{I}_\sigma(\mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(\psi, r), r), r)$

(2) r-fuzzy α-open [14] if $\psi \leq \mathcal{I}_\sigma(\mathcal{C}_\sigma(\mathcal{I}_\sigma(\psi, r), r), r)$,

(3) r-fuzzy preopen [14] if $\psi \leq \mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi, r), r)$,

(4) r-fuzzy δ-preopen [15] $\psi \leq \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi, r), r)$,

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

(5) r-fuzzy δ-semiopen [15] if $\psi \leq \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(\psi,r),r)$,

(6) r-fuzzy Z-open [9] if if $\psi \leq \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(\psi,r),r) \vee \mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r)$,

(7) r-fuzzy γ-open [9] if $\psi \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\psi,r),r) \vee \mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r)$,

(8) r-fuzzy e-open [16] if $\psi \leq \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(\psi,r),r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r)$,

(9) r-fuzzy Z^*-open [9] if $\psi \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\psi,r),r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r)$,

(10) r-fuzzy β-open [1] or semi-preopen [2]if $\psi = \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r),r)$,

(11) r-fuzzy e^*-open [10] if $\psi = \mathcal{C}_\sigma(\mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r),r)$,.

The complement of a r-fuzzy a-open (respectively, r-fuzzy α-open, r-fuzzy δ-semiopen, r-fuzzy δ-preopen, r-fuzzy Z-open, r-fuzzy γ- open, r- fuzzy e-open, r-fuzzy Z^*-open, r-fuzzy β-open, r-fuzzy e^*-open) sets is called r-fuzzy a-closed [16] (resp. r-fuzzy α-closed [14], r-fuzzy δ-semiclosed [15], r-fuzzy δ-preclosed [15], r-fuzzy Z-closed [9], r-fuzzy γ-closed [13], r-fuzzy e-closed [16], r-fuzzy Z^*-closed [9], r- fuzzy β- closed [1], r-fuzzy e^*-closed [10]).
The intersection of all r-fuzzy δ-preclosed (respectively, r-fuzzy β-closed) set containing ψ is called the r-fuzzy δ-preclosure (respectively, r-fuzzy β-closure) of ψ and is indicated by

$$f\delta p\mathcal{C}_\sigma(\psi,r) \text{ (resp. } f\beta\mathcal{C}_\sigma(\psi,r)\text{).}$$

Similarly, The union of all r-fuzzy δ- preopen (respectively, r-fuzzy β-open)sets contained in ψ is called the r-fuzzy δ- preinterior (resp. r-fuzzy β-interior) of ψ and is indicate by

$$f\delta p\mathcal{I}_\sigma(\psi,r) \text{ (respectively, } f\beta\mathcal{I}_\sigma(\psi,r)\text{).}$$

Furthermore, the family of all r-fuzzy δ-open (resp. r-fuzzy δ- semiopen, r-fuzzy δ- preopen, r-fuzzy Z^*- open, r-fuzzy β-open, r-fuzzy e^*-open) sets is indicate by $r\text{-}f\delta O(X)$ (respectively, $r\text{-}f\delta SO(X)$, $r\text{-}f\delta PO(X)$, $r\text{-}f\delta Z^*O(X)$, $r\text{-}f\beta O(X)$, $r\text{-}fe^*O(X)$).

Lemma 1.1. [15] Let ψ be a fuzzy subset of a fuzzy topological space (X,σ). Then, the following properties hold:

(1) $f\delta p\mathcal{I}_\sigma(\psi,r) = \psi \wedge \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r))$ and $f\delta p\mathcal{C}_\sigma(\psi,r) = \psi \vee \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(\psi,r))$

(2) $f\beta\mathcal{I}_\sigma(\psi,r) = \psi \wedge \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r,),r)$ and $f\beta\mathcal{C}_\sigma(\psi,r) = \psi \vee \mathcal{I}_\sigma(\mathcal{C}_\sigma(\mathcal{I}_\sigma(\psi,r),r),r)$.

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

2 r-fuzzy $\mathcal{B}^*$-open sets

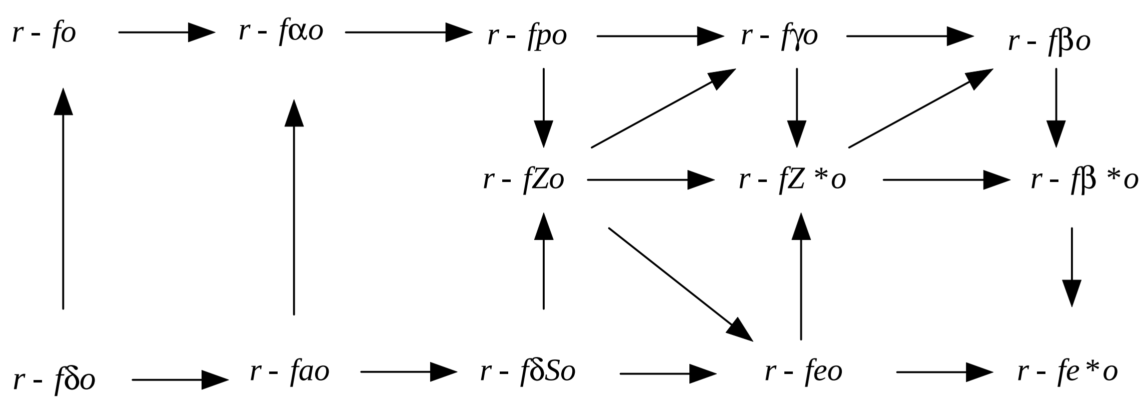
Definition 2.1. A topological space $(\mathcal{P}, \sigma)$. A subset ψ is called:

(1) a r-fuzzy $\mathcal{B}^*$-Open set (r-fy. $\mathcal{B}^*$-Op) if $\psi \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi, r), r), r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi, r), r)$,

(2) a r-fuzzy $\mathcal{B}^*$-closed set if $\mathcal{I}_\sigma(\mathcal{C}_\sigma(\mathcal{I}_\sigma(\psi, r), r), r) \wedge \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(\psi, r), r) \leq \psi$.

The set of all r-fuzzy $\mathcal{B}^*$-open (resp. r-fuzzy $\mathcal{B}^*$-closed) fuzzy subsets of a space (X, σ) will be as always mentioned $f\mathcal{B}^*O(X)$ (resp. $f\mathcal{B}^*\mathcal{C}(X)$).

Remark 2.1. For every fuzzy subset $\psi \in \mathcal{I}^X$, the following implication diagram holds among various classes of r-fuzzy open sets:



None of these indication can reversible in general, as illustrated by the examples given below. Additional counter examples may be found in [[1],[3],[4],[7],[10],[11],[13]-[18]].

Example 2.1 Let $\mathcal{P} = \{a_0,\ b_0,\ c_0\}$ and $\omega_1, \omega_2, \gamma \in \mathcal{I}^X$ be fuzzy subsets defined by

$$\omega_1(a_0) = 0.4, \omega_1(a_0) = 0.2, \omega_1(a_0) = 0.8,$$
$$\omega_2(b_0) = 0.9, \omega_2(b_0) = 0.8, \omega_2(b_0) = 0.3,$$

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\gamma(a_0) = 0.9, \gamma(b_0) = 0.6, \gamma(c_0) = 0.5$$

. Define a fuzzy topology $\sigma : \mathcal{I}^X \to \mathcal{I}$ as follows:

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{3} & if\ \psi = \omega_1, \\ \frac{1}{2} & if\ \psi = \omega_2, \\ \frac{2}{3} & if\ \psi = \omega_1 \vee \omega_2, \\ \frac{2}{3} & if\ \psi = \omega_1 \wedge \omega_2, \\ 0 & if\ otherwise. \end{cases}$$

In $(\mathcal{P},\ \sigma)$ fuzzy topological spaces with $0 < r \leq \frac{1}{3},\ \gamma$ is r-fuzzy $\mathcal{B}^*$-open set form

$$\psi \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi, r), r), r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi, r), r) = \underline{1}$$

For $0 < r \leq \frac{1}{3}$, γ is not r-fuzzy β -open set from

$$\psi \neq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi, r), r), r)$$

Example 2.2 If $\mathcal{P} = \{a_0,\ b_0,\ c_0\}$ and $\omega_1, \omega_2, \omega_3,\ \gamma \in \mathcal{I}^X$ be fuzzy subsets defined by

$$\omega_1(a_0) = 0.3, \omega_1(a_0) = 0.4, \omega_1(a_0) = 0.5,$$
$$\omega_2(b_0) = 0.6, \omega_2(b_0) = 0.5, \omega_2(b_0) = 0.5,$$
$$\omega_3(c_0) = 0.6, \omega_3(c_0) = 0.5, \omega_3(c_0) = 0.4,$$
$$\gamma(a_0) = 0.6, \gamma(b_0) = 0.4, \gamma(c_0) = 0.4$$

. Define a fuzzy topology $\sigma : \mathcal{I}^X \to \mathcal{I}$ by

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi = \omega_1, \\ \frac{1}{2} & if\ \psi = \omega_2, \\ \frac{1}{2} & if\ \psi = \omega_3, \\ \frac{1}{2} & if\ \psi = \gamma, \\ 0 & if\ otherwise. \end{cases}$$

In $(\mathcal{P},\ \sigma)$ fuzzy topological spaces with $0 < r \leq \frac{1}{2},\ \gamma$ is r-fuzzy e^*-open set form

$$\psi \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi, r), r), r)r) = \underline{1}$$

For $0 < r \leq \frac{1}{2}$, γ is not r-fuzzy $\mathcal{B}^*$ -open set from

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$\psi \nleq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r),r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r)$

Example 2.3 Let $\mathcal{P}=\{a_0,\ b_0,\ c_0\}$ and $\omega_1,\omega_2,\omega_3\gamma \in \mathcal{I}^X$ be fuzzy subsets defined by

$$\omega_1(a_0)=0.4, \omega_1(a_0)=0.5, \omega_1(a_0)=0.6,$$
$$\omega_2(b_0)=0.3, \omega_2(b_0)=0.5, \omega_2(b_0)=0.6,$$
$$\omega_3(c_0)=0.6, \omega_3(c_0)=0.5, \omega_3(c_0)=0.5$$

. Define a fuzzy topology $\sigma : \mathcal{I}^X \to \mathcal{I}$ by

$$\sigma(\psi)=\begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi=\omega_1, \\ \frac{1}{2} & if\ \psi=\omega_2, \\ \frac{1}{2} & if\ \psi=\omega_1 \vee \omega_2, \\ \frac{1}{2} & if\ \psi=\omega_1 \wedge \omega_2, \\ 0 & if\ otherwise. \end{cases}$$

In $(\mathcal{P},\ \sigma)$ fuzzy topological spaces with $0<r\le \frac{1}{2}$, γ is r-fuzzy $\mathcal{B}^*$-open set form

$\psi \le \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r),r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r)$

For $0<r\le \frac{1}{2}$, γ is not r-fuzzy Z^* -open set from

$\psi \nleq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\psi,r),r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r)$

Theorem 2.1. Let $(\mathcal{P},\sigma)$ be a FTs and $r \in \mathcal{I}_0$. Then the following assertions hold.

(a) The union of arbitrary set r-fuzzy $\mathcal{B}^*$-open sets is again an r-fuzzy $\mathcal{B}^*$-open,

(b) The intersection of any collection of r-fuzzy $\mathcal{B}^*$-closed sets is r-fuzzy $\mathcal{B}^*$-closed.

Proof. (a) Let $\{\psi_i, i \in \mathcal{I}\}$ be a family of r-fuzzy $\mathcal{B}^*$-open sets. By definition each ψ_i, satisfies

$$\psi_i \le \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi_i,r),r),r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi_i,r),r).$$

Taking the supremum over all $i \in I$, we obtain

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\bigvee_i \psi_i \leq [\bigvee_i (\mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi_i, r), r), r)) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi_i, r), r)].$$

Using the monotonicity properties of the operators C_σ and I_σ, we have

$$\bigvee_i \psi_i \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\bigvee_i \psi_i, r))) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\bigvee_i \psi_i, r), r)\,.$$

Hence $\bigvee_i \psi_i$ satisfies the defining condition of an r-fuzzy $\mathcal{B}^*$-open. Therefore, the union of an arbitrary family of r-fuzzy $\mathcal{B}^*$-open set is again r-fuzzy $\mathcal{B}^*$-open set.
(b) The proof of the second statement follows directly from part (1) by taking complement and using the duality between r-fuzzy $\mathcal{B}^*$-open and r-fuzzy $\mathcal{B}^*$-closed sets.

Remark 2.2. The following example shows that the intersection of two r-$\mathcal{B}^*$-fuzzy open sets need not be a r-fuzzy $\mathcal{B}^*$-open.

Example 2.4 Let $\mathcal{P} = \{a_0,\ b_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \in \mathcal{I}^X$ defined as follows:
$\omega_1(a_0) = 0.6,\ \omega_1(b_0) = 0.4,$
$\omega_2(a_0) = 0.3,\ \omega_2(b_0) = 0.6,$
$\omega_3(a_0) = 0.6,\ \omega_3(b_0) = 0.8,$
$\omega_4(a_0) = 0.4,\ \omega_4(b_0) = 0.6,$
$\omega_5(a_0) = 0.4,\ \omega_5(b_0) = 0.2,$
and let $\gamma_1,\ \gamma_2 \in \mathcal{I}^X$ be given by
$\gamma_1(a_0) = 0.4,\ \gamma_1(b_0) = 0.8,$
$\gamma_2(a_0) = 0.6,\ \gamma_2(b_0) = 0.6\,.$
Define a fuzzy topology $\sigma : \mathcal{I}^X \to \mathcal{I}$ by

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_1,\ \omega_1 \vee \omega_2\}, \\ \frac{2}{3} & if\ \psi = \{\omega_2,\ \omega_1 \wedge \omega_2,\ \underline{1} - \omega_1 \wedge \omega_2\}, \\ 0 & \ if\ otherwise. \end{cases}$$

(1) It follows that γ_1 and γ_2 are $\frac{1}{2}$-fuzzy $\mathcal{B}^*$-open sets in $(\mathcal{P},\ \sigma)$. However, their intersection satisfies $\gamma_1 \wedge \gamma_2 = \omega_4$, which is not a $\frac{1}{2}$-fuzzy $\mathcal{B}^*$-open sets.

Definition 2.2. Let $(\mathcal{P},\ \sigma)$ be a fuzzy topological space. Then:

(i) The union of all r-fuzzy $\mathcal{B}^*$-Op sets of contained in ψ is called the r-fuzzy $\mathcal{B}^*$-int. of ψ & is mentioned $f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r)$,

(ii) The intersection of all r-fuzzy $\mathcal{B}^*$-Cl sets of X containing ψ is called the r-fuzzy $\mathcal{B}^*$-clos. of ψ & is indicated by $f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$.

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 2.2. Let $(\mathcal{P},\ \sigma)$ a FTs & $\psi, \omega \in \mathcal{I}^X, r \in \mathcal{I}_0$. Then the below assertions are hold:

(i) $f\mathcal{B}^*\mathcal{C}_\sigma(\underline{1}, r) = \underline{1}$ and $f\mathcal{B}^*\mathcal{C}_\sigma(\underline{0}, r) = \underline{0}$,

(ii) $\psi \leq f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$,

(iii) If $\psi \leq \omega$, then $f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r) \leq f\mathcal{B}^*\mathcal{C}_\sigma(\omega, r)$,

(iv) $x \in f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$ iff for every a r-fuzzy $\mathcal{B}^*$-Op set $\mathcal{U}$ containing x, $\mathcal{U} \wedge \psi \neq \phi$,

(v) ψ is r-fuzzy $\mathcal{B}^*$-Cl set iff $\psi = f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$,

(vi) $f\mathcal{B}^*\mathcal{C}_\sigma\,(\,f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)) = f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$,

(vii) $f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r) \vee f\mathcal{B}^*\mathcal{C}_\sigma(\omega, r) \leq\ f\mathcal{B}^*\mathcal{C}_\sigma(\psi \vee \omega)$,

(viii) $f\mathcal{B}^*\mathcal{C}_\sigma(\psi \wedge \omega) \leq f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r) \wedge f\mathcal{B}^*\mathcal{C}_\sigma(\omega, r)$.

Proof. (vi) By using (ii) and $\psi \leq f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$, we have $f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r) \leq f\mathcal{B}^*\mathcal{C}_\sigma(f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r))$. Let $x \in f\mathcal{B}^*\mathcal{C}_\sigma(f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r))$. Then, for all r-fuzzy $\mathcal{B}^*$-Op set $\mathcal{V}$ containig x,
we have

$$\mathcal{V} \wedge f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r) \neq \phi.$$

Hence, there exists a point $y \in \mathcal{V} \wedge f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$. Then, for all r-fuzzy $\mathcal{B}^*$-Op set G containing y, it follows that $\psi \wedge \mathcal{G} \neq \phi$.

Since $\mathcal{V}$ is a r-fuzzy $\mathcal{B}^*$-open set, $y \in \mathcal{V}$ and $\psi \wedge \mathcal{V} \neq \phi$, then $x \in f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$.

Therefore, $f\mathcal{B}^*\mathcal{C}_\sigma(f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)) \leq f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$. Consequently, $f\mathcal{B}^*\mathcal{C}_\sigma(f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)) \leq f\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)$.

Theorem 2.3. Let $(\mathcal{P},\ \sigma)$ be a FTs, $\psi, \omega \in \mathcal{I}^X$, and $r \in \mathcal{I}_0$. Then the below properties hold:

(1) $f\mathcal{B}^*\mathcal{I}_\sigma(\underline{1}, r) = \underline{1}$ and $f\mathcal{B}^*\mathcal{I}_\sigma(\underline{0}, r) = \underline{0}$,

(2) $f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r) \leq \psi$,

(3) If $\psi \leq \omega$, then $f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r) \leq f\mathcal{B}^*\mathcal{I}_\sigma(\omega, r)$,

(4) $x \in f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r)$ if $\exists$ a r-fuzzy $\mathcal{B}^*$-OP set $U \ni x \in U \leq \psi$,

(5) ψ is r-fuzzy $\mathcal{B}^*$-open set iff $\psi = f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r)$,

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(6) $f\mathcal{B}^*\mathcal{I}_\sigma(f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r)) = f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r)$,

(7) $f\mathcal{B}^*\mathcal{I}_\sigma(\psi \wedge \omega, r) \leq f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r) \wedge f\mathcal{B}^*\mathcal{I}_\sigma(\omega, r)$,

(8) $f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r) \vee f\mathcal{B}^*\mathcal{I}_\sigma(\omega, r) \leq f\mathcal{B}^*\mathcal{I}_\sigma(\psi \vee \omega, r)$.

Remark 2.3. As the below example shows that the inclusion relations stated in parts parts (7) and (8) of Theorems 2.2. and 2.3. cannot, in general, be replaced by equality.

Example 2.5 Let $\mathcal{P} = \{a_0,\ b_0,\ c_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \in \mathcal{I}^X$ defined as follows:
$\omega_1(a_0) = 0.6,\ \omega_1(b_0) = 0.4,$
$\omega_2(a_0) = 0.3,\ \omega_2(b_0) = 0.6,$
$\omega_3(a_0) = 0.6,\ \omega_3(b_0) = 0.8,$
$\omega_4(a_0) = 0.4,\ \omega_4(b_0) = 0.6,$
$\omega_5(a_0) = 0.4,\ \omega_5(b_0) = 0.2,$
$\gamma_1(a_0) = 0.4,\ \gamma_1(b_0) = 0.8,$
$\gamma_2(a_0) = 0.6,\ \gamma_2(b_0) = 0.6\,,$
$\gamma_3(a_0) = 0.3,\ \gamma_3(b_0) = 0.2\,,$
$\gamma_4(a_0) = 0.2,\ \gamma_4(b_0) = 0.8\,.$
Define fuzzy topology $\sigma : \mathcal{I}^X \to \mathcal{I}$ as follows:

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_1,\ \omega_1 \vee \omega_2\}, \\ \frac{2}{3} & if\ \psi = \{\omega_2,\ \omega_1 \wedge \omega_2,\ \underline{1} - \omega_1 \wedge \omega_2\}, \\ 0 & \ if\ otherwise. \end{cases}$$

(i) If γ_1 and γ_2 are $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1, \frac{1}{2}) = \gamma_1$, $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_2, \frac{1}{2}) = \gamma_2$, and $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1 \vee \gamma_2, \frac{1}{2}) = \underline{1}$, $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1 \vee \gamma_2, \frac{1}{2}) \nleq \frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1, \frac{1}{2}) \vee \frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_2, \frac{1}{2})$,

(ii) If γ_1 and γ_3 then $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_3, \frac{1}{2}) = \underline{1}$, $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1, \frac{1}{2}) = \gamma_1$, $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1 \wedge \gamma_2, \frac{1}{2}) = \omega_4$, Thus and $\frac{1}{2}$-fuzzy $f\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1, \frac{1}{2}) \wedge \frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_3, \frac{1}{2}) \nleq \frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{C}_\sigma(\gamma_1 \wedge \gamma_3, \frac{1}{2})$,

(iii) If γ_3, γ_4 then $\gamma_3 \vee \gamma_4 = \gamma(a) = 0.3,\ \gamma(b) = 0.8$ and hence $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{I}_\sigma(\gamma_3, \frac{1}{2}) = \phi$, $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{I}_\sigma(\gamma_4, \frac{1}{2}) = \gamma_4$,and $\frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{I}_\sigma(\gamma_3 \vee \gamma_4, \frac{1}{2}) \nleq \frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{I}_\sigma(\gamma_3, \frac{1}{2}) \vee \frac{1}{2}$-fuzzy $\mathcal{B}^*\mathcal{I}_\sigma(\gamma_3 \vee \gamma_4, \frac{1}{2})$.

Theorem 2.4. Let a FTs $(\mathcal{P},\ \sigma)$ & $\psi \in \mathcal{I}^X, r \in \mathcal{I}_0$. The below identities hold:

(1) $f\mathcal{B}^*\mathcal{C}_\sigma(X \backslash (\psi, r)) = X \backslash f\mathcal{B}^*\mathcal{I}_\sigma(\psi, r)$,

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

(2) $f\mathcal{B}^*\mathcal{I}_\sigma(X\backslash(\psi,r)) = X\backslash f\mathcal{B}^*\mathcal{C}_\sigma(\psi,r)$.

Proof. These results follows from the fact the complement of r-fuzzy $\mathcal{B}^*$- open set is a r-fuzzy $\mathcal{B}^*$-closed set, together with the lattice identity

$$\bigwedge_i(\mathcal{P}\backslash\psi_i) = \mathcal{P}\backslash\bigvee_i\psi_i\,.$$

Hence, the statements are immediate.

Theorem 2.5. Let $(\mathcal{P},\ \sigma)$ be a FTs & $\psi \in \mathcal{I}^X, r \in \mathcal{I}_0$. Then the below conditions are equivalent:

(1) ψ is an r-fuzzy $\mathcal{B}^*$-Op set,

(2) $\psi = f\beta\mathcal{I}_\sigma(\psi,r) \vee f\delta pI_\sigma(\psi,r)$.

Proof. (1) $\Rightarrow$ (2) :
Assume that ψ be a r-fuzzy $\mathcal{B}^*$-open set. Then, by definition,

$\psi \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r),r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r)$ and hence by Lemma 1.1.,

$\psi \leq (\psi \wedge \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(\psi,r),r),r)) \vee (\psi \wedge \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(\psi,r),r))$.

Therefore

$$\psi \leq f\beta\mathcal{I}_\sigma(\psi,r) \vee f\delta p\mathcal{I}_\sigma(\psi,r) \leq \psi,$$

which implies $\psi = f\beta\mathcal{I}_\sigma(\psi,r) \vee f\delta p\mathcal{I}_\sigma(\psi,r)$
(2) $\Rightarrow$ (1):
This follows directly from the definition of 2.1..

Theorem 2.6. Let a FTs $(\mathcal{P},\ \sigma)$, and $r \in \mathcal{I}_0$.Then the below properties are equivalent:

(1) ψ is an r-fuzzy $\mathcal{B}^*$-Cl set,

(2) $\psi = f\beta\mathcal{C}_\sigma(\psi,r) \wedge f\delta p\mathcal{C}_\sigma(\psi,r)$.

Proof. The result follow immediately from Theorm 2.5. by taking complements and using the duality between r-fuzzy $\mathcal{B}^*$-OP set and r-fuzzy $\mathcal{B}^*$-Cl set.

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

3 r-fuzzy $\mathcal{B}^*$-Continuous functions

In this section, we introduce the concept of r-fuzzy $\mathcal{B}^*$-continuous mappings between FTs and investigate their fundamental properties. Several characterizations are established, and relationships with other types of fuzzy continuity arediscussed. Moreover, we study the behavior of these functions under composition,restriction, and product spaces, and examine their connections with separation axioms,compactness, and connectedness in the fuzzy setting

Definition 3.1. Let $h : (\mathcal{P}, \sigma) \to (\mathcal{Q}, \omega)$ be a mapping betweeen fuzzy topological spaces. The function h is said to be r-fuzzy $\mathcal{B}^*$-continuous (In short r-fy $\mathcal{B}^*$-Op) if for every $\mathcal{V} \in \sigma$, the inverse image $h^{-1}(\mathcal{V})$ is r-fuzzy $\mathcal{B}^*$-open in $\mathcal{P}$.

Definition 3.2. Let $h : (\mathcal{P}, \sigma) \to (\mathcal{Q}, \omega)$ be a mapping between fuzzy FTs. The function h is called :

- r-fuzzy super-continuous [9]
- r-fuzzy a-continuous [16],
- r-fuzzy α continuous[14],
- r-fuzzy pre continuous[14],
- r-fuzzy δ-semi-continuous[15],
- r-fuzzy Z -continuous[9],
- r-fuzzy γ-continuous[13],
- r-fuzzy e-continuous[16],
- r-fuzzy Z^*-continuous[9],
- r-fuzzy β-continuous[1],
- r-fuzzy e^*-continuous[10],

if, for every $V \in \phi$, the inverse image $h^{-1}(V)$ is respectively
r-fuzzy δ-open, r-fuzzy a-open, r-fuzzy α-open, r-fuzzy peropen, r-fuzzy δ-semiopen, r-fuzzy Z-open, r-fuzzy γ-open, r-fuzzy e-open, r-fuzzy Z^*-open, r-fuzzy β-open, r-fuzzy e^*-open in $\mathcal{P}$.

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Remark 3.1. The bwlow picture true for a mapping For a mapping $h : \mathcal{C} \to \mathcal{D}$, the following diagram illustrates the relationships among various types of r-fuzzy continuities:

$$\begin{tikzcd}
r-fc \arrow[r] & r-f\alpha c \arrow[r] & r-fpc \arrow[r] \arrow[d] & r-f\gamma c \arrow[r] \arrow[d] & r-f\beta c \arrow[d] \\
 & & r-fZc \arrow[r] \arrow[ur] \arrow[dr] & r-fZ^*c \arrow[r] \arrow[ur] & r-f\beta^*c \arrow[d] \\
r-f\delta c \arrow[r] \arrow[uu] & r-fac \arrow[r] \arrow[uu] & r-f\delta Sc \arrow[r] \arrow[u] & r-fec \arrow[r] \arrow[u] & r-fe^*c
\end{tikzcd}$$

These implications hold for any mapping $h : \mathcal{C} \to \mathcal{D}$. Furthermore, as shown in references [[1],[3],[4],[10],[11],[13]-[18]], together with the examples that follow, the reverse implications do not hold in general.

Example 3.1 Let $\mathcal{P} = \mathcal{Q} = \{a_0,\ b_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4, \in \mathcal{I}^X$ defined as follows:
$\omega_1(a_0) = 0.2,\ \omega_1(b_0) = 0.4,$
$\omega_2(a_0) = 0.6,\ \omega_2(b_0) = 0.5,$
$\omega_3(a_0) = 0.3,\ \omega_3(b_0) = 0.4,$
$\omega_4(a_0) = 0.6,\ \omega_4(b_0) = 0.7$.
Define fuzzy topology $\sigma, \gamma : \mathcal{I}^X \to \mathcal{I}$ as follows:

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_1\ or\ \ \underline{0.4}\}, \\ \frac{2}{3} & if\ \psi = \{\omega_2\ or\ \ \underline{0.7}\}, \\ 0 & if\ otherwise. \end{cases}$$

$$\gamma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_3\}, \\ \frac{2}{3} & if\ \psi = \{\omega_4\}, \\ 0 & if\ otherwise. \end{cases}$$

Consequently, the identity mapping $ID_X : (\mathcal{P}, \sigma) \to (\mathcal{Q}, \sigma)$ is $\frac{1}{2}$-fuzzy $\mathcal{B}^*$-continuous but it is not $\frac{1}{2}$-fuzzy β-continuous.

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Example 3.2 Let $\mathcal{P} = \mathcal{Q} = \{a_0,\ b_0,\ c_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4, \in \mathcal{I}^X$ defined as follows:
$\omega_1(a_0) = 0.4,\ \omega_1(b_0) = 0.5,\ \omega_1(c_0) = 0.6$
$\omega_2(a_0) = 0.4,\ \omega_2(b_0) = 0.5,\ \omega_2(c_0) = 0.4,$
$\omega_3(a_0) = 0.5,\ \omega_3(b_0) = 0.5,\ \omega_3(c_0) = 0.6,$
$\omega_4(a_0) = 0.4,\ \omega_4(b_0) = 0.5,\ \omega_4(c_0) = 0.6\,.$
Define fuzzy topology $\sigma, \gamma : \mathcal{I}^X \to \mathcal{I}$ as follows:

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_1\}, \\ \frac{1}{2} & if\ \psi = \{\omega_2\}, \\ 0 & if\ otherwise. \end{cases}$$

$$\gamma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_3\}, \\ \frac{1}{2} & if\ \psi = \{\omega_4\}, \\ 0 & if\ otherwise. \end{cases}$$

Consequently, the identity function $ID_X : (\mathcal{P}, \sigma) \to (\mathcal{Q}, \gamma)$ is $\frac{1}{2}$-fuzzy e^*-continuous but it is not $\frac{1}{2}$-fuzzy $\mathcal{B}^*$-continuous.

Example 3.3 Let $\mathcal{P} = \mathcal{Q} = \{a_0,\ b_0,\ c_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4, \in \mathcal{I}^X$ defined as follows:
$\omega_1(a_0) = 0.4,\ \omega_1(b_0) = 0.5,\ \omega_1(c_0) = 0.6$
$\omega_2(a_0) = 0.4,\ \omega_2(b_0) = 0.5,\ \omega_2(c_0) = 0.4,$
$\omega_3(a_0) = 0.6,\ \omega_3(b_0) = 0.5,\ \omega_3(c_0) = 0.4,$
$\omega_4(a_0) = 0.4,\ \omega_4(b_0) = 0.5,\ \omega_4(c_0) = 0.6\,.$
Define fuzzy topology $\sigma, \gamma : \mathcal{I}^X \to \mathcal{I}$ as follows:

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_1\}, \\ \frac{1}{2} & if\ \psi = \{\omega_2\}, \\ 0 & if\ otherwise. \end{cases}$$

$$\gamma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_3\}, \\ \frac{1}{2} & if\ \psi = \{\omega_4\}, \\ \frac{1}{2} & if\ \psi = \{\omega_3 \vee \omega_4\}, \\ \frac{1}{2} & if\ \psi = \{\omega_3 \wedge \omega_4\}, \\ 0 & if\ otherwise. \end{cases}$$

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Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

For the identity function $ID_X : (\mathcal{P}, \sigma) \to (\mathcal{Q}, \gamma)$ is $\frac{1}{2}$-fuzzy $\mathcal{B}^*$-continuous but it is not $\frac{1}{2}$-fuzzy Z^*-continuous.

Theorem 3.1. Let $(\mathcal{P},\ \sigma)$ and $(\mathcal{Q},\ \omega)$ be a fuzzy topoloical space & $h : (\mathcal{P},\ \sigma) \to (\mathcal{Q},\ \omega)$ be a map. Then the below condition are equivalent:

(i) h is r-fuzzy $\mathcal{B}^*$- continuous,

(ii) For each $c \in \mathcal{P}$ and every $\mathcal{V} \in \sigma$ with $h(x) \in \mathcal{V}$, there exists an r-fuzzy $\mathcal{B}^*$-open set $\mathcal{U} \in C$ such that $cin\mathcal{U}$ and $h(\mathcal{U})$

(iii) The inverse image of every r-fuzzy closed set in Y is r-fuzzy $\mathcal{B}^*$-closed in $\mathcal{P}$,

(iv) For every $\omega \leq Q$
$\mathcal{I}_\sigma(\mathcal{P}_\sigma(\mathcal{I}_\sigma(h^{-1}(\omega, r), r), r)) \wedge \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(h^{-1}(\omega, r), r)) \leq h^{-1}(\mathcal{C}_\sigma(\omega, r))$,

(v) For every $\omega \leq \mathcal{Q}$,
$h^{-1}(\mathcal{I}_\sigma(\omega, r)) \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(h^{-1}(\omega, r), r), r), r) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(h^{-1}(\omega, r), r), r)$,

(vi) For every $\omega \leq \mathcal{Q}$, $\mathcal{B}^*\mathcal{C}_\sigma(f^{-1}(\omega, r)) \leq h^{-1}(\mathcal{C}_\sigma(\omega, r))$,

(vii) For every $\psi \leq \mathcal{P}$, $h(\mathcal{B}^*\mathcal{C}_\sigma(\psi, r)) \leq \mathcal{C}_\sigma(h(\psi, r))$,

(viii) for every $\omega \leq \mathcal{Q}$. $h^{-1}(\mathcal{I}_\sigma(\omega, r)) \leq \mathcal{B}^*\mathcal{I}_\sigma(h^{-1}(\omega, r))$. for each $\omega \leq \mathcal{Q}$.

Proof. The equivalence of the above ststements is established as follows.
(i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii):
These follow directly from the definition of r- fuzzy $\mathcal{B}^*$-continuity.
(iii) $\Leftrightarrow$ (iv):
Let $\omega \leq \mathcal{P}$:

By (iii) $h^{-1}(\mathcal{C}_\sigma(\omega, r))$ is r- fuzzy $\mathcal{B}^*$-closed in $\mathcal{P}$.

This means

$h^{-1}(\mathcal{C}_\sigma(\omega, r)) \geq \mathcal{I}_\sigma(\mathcal{C}_\sigma(\mathcal{I}_\sigma(h^{-1}(\mathcal{C}_\sigma(\omega, r), r), r), r)) \wedge \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(h^{-1}(\mathcal{C}_\sigma(\omega, r), r), r)) \geq \mathcal{I}_\sigma(\mathcal{C}_\sigma(\mathcal{I}_\sigma(h^{-1}(\omega, r), r), r), r) \wedge \mathcal{C}_\sigma(\delta\mathcal{I}_\sigma(h^{-1}(\omega, r), r), r)$,
$(iv) \Rightarrow (v)$:
By replacing $\mathcal{Q}\backslash\omega$ instead of ω in (iv), we obtain
$h^{-1}(\mathcal{I}_\sigma(\omega, r)) \leq \mathcal{C}_\sigma(\mathcal{I}_\sigma(\mathcal{C}_\sigma(f^{-1}(\omega, r), r), r)) \vee \mathcal{I}_\sigma(\delta\mathcal{C}_\sigma(h^{-1}(\omega, r), r))$. Thus, (5) is holds
$(v) \Rightarrow (i)$:
This follows immediately from the definition of r- fuzzy $\mathcal{B}^*$-open sets,
$(iii) \Rightarrow (vi)$:
Let $\omega \leq \mathcal{Q}$. Since $h^{-1}(\mathcal{C}_\sigma(\omega, r))$ is r-fuzzy $\mathcal{B}^*$-closed,

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$\mathcal{B}^*\mathcal{C}_\sigma(h^{-1}(\omega,r),r) \leq h^{-1}(\mathcal{C}_\sigma(\omega,r)).\ \text{Hence, } (6) \text{ is satisfied.}$$

$(vi) \Rightarrow (vii)$:
Let $\psi \leq \mathcal{P}$. Then $h(\psi) \leq \mathcal{Q}$. By (vi),

$$h^{-1}(\mathcal{C}_\sigma(h(\psi,r),r)) \geq \mathcal{B}^*\mathcal{C}_\sigma(h^{-1}(f(\psi,r),r)) \geq \mathcal{B}^*\mathcal{C}_\sigma(\psi,r).$$

Therefore, Applying h, we obtain

$$\mathcal{C}(h(\psi,r)) \geq hh^{-1}(\mathcal{C}_\sigma(h(\psi,r),r)) \geq h(\mathcal{B}^*\mathcal{C}_\sigma(\psi,r)).$$

Thus, (vii) holds.
$(vii) \Rightarrow (iii)$: Let $\mathcal{F} \leq \mathcal{Q}$ be an r-fuzzy closed set. Then $\mathcal{F} = \mathcal{C}_\sigma(\mathcal{F},r)$ and $h^{-1}(\mathcal{F}) = h^{-1}(\mathcal{C}_\sigma(\mathcal{F},r))$. By (vii),

$$h(\mathcal{B}^*\mathcal{C}_\sigma(h^{-1}(\mathcal{F},r),r)) \leq \mathcal{C}_\sigma(\mathcal{F},r) = \mathcal{F},$$

Therefore, $\mathcal{B}^*\mathcal{C}_\sigma(h^{-1}(\mathcal{F},r),r) \leq h^{-1}(\mathcal{F})$, which implies that $h^{-1}(\mathcal{F})$ is r-fuzzy $\mathcal{B}^*$-closed. $(i) \Rightarrow (viii)$:
Assume that h is r-fuzzy $\mathcal{B}^*$- continuous. Then $h^{-1}(\mathcal{I}_\sigma(\omega,r))$ is r-fuzzy $\mathcal{B}^*$-open. Hence, $h^{-1}(\mathcal{I}_\sigma(\omega,r)) = \mathcal{B}^*\mathcal{I}_\sigma(h^{-1}(\mathcal{I}_\sigma(\omega,r),r)) \leq \mathcal{B}^*\mathcal{I}_\sigma(h^{-1}(\omega,r))$.
Therefore, $h^{-1}(\mathcal{I}_\sigma(\omega,r)) \leq \mathcal{B}^*\mathcal{I}_\sigma(h^{-1}(\omega,r))$.
$(viii) \Rightarrow (i)$: Let $\mathcal{U}$ be an r-fuzzy open set in $\mathcal{Y}$. Then

$$h^{-1}(\mathcal{U}) = h^{-1}(\mathcal{I}_\sigma(\mathcal{U})) \leq \mathcal{B}^*\mathcal{I}_\sigma(h^{-1}(\mathcal{U})).$$

Hence, $h^{-1}(\mathcal{U})$ is r-$\mathcal{B}^*$- fuzzy open in $\mathcal{P}$, which shows that h is r-fuzzy $\mathcal{B}^*$-continuous.
Therefore, all the statements $(i) - (viii)$ are equivalent.

Remark 3.2. The composition of two r-fuzzy $\mathcal{B}^*$-continuous functions need not be r-fuzzy $\mathcal{B}^*$- continuous as show by the following example.

Example 3.4 Let $\mathcal{P} = \mathcal{Q} = \mathcal{R} = \{a_0,\ b_0,\ c_0\}$ and $\omega_1, \omega_2, \in \mathcal{I}^X$, $\omega_3, \omega_4 \in \mathcal{I}^X$ and $\omega_5 \in \mathcal{I}^X$ defined as follows:
$\omega_1(a_0) = 0.4,\ \omega_1(b_0) = 0.5,\ \omega_1(c_0) = 0.6$
$\omega_2(a_0) = 0.4,\ \omega_2(b_0) = 0.5,\ \omega_2(c_0) = 0.4,$
$\omega_3(a_0) = 0.6,\ \omega_3(b_0) = 0.5,\ \omega_3(c_0) = 0.4,$
$\omega_4(a_0) = 0.4,\ \omega_4(b_0) = 0.5,\ \omega_4(c_0) = 0.6,$

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ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$\omega_5(a_0) = 0.6,\ \omega_5(b_0) = 0.5,\ \omega_5(c_0) = 0.4,$
Define fuzzy topology $\sigma, \gamma : \mathcal{I}^X \to \mathcal{I}$ as follows:

$$\sigma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_1\}, \\ \frac{1}{2} & if\ \psi = \{\omega_2\}, \\ 0 & if\ otherwise. \end{cases}$$

$$\gamma(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_3\}, \\ \frac{1}{2} & if\ \psi = \{\omega_4\}, \\ \frac{1}{2} & if\ \psi = \{\omega_3 \vee \omega_4\}, \\ \frac{1}{2} & if\ \psi = \{\omega_3 \wedge \omega_4\}, \\ 0 & if\ otherwise. \end{cases}$$

$$\eta(\psi) = \begin{cases} 1 & if\ \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{2} & if\ \psi \in \{\omega_5\}, \\ 0 & if\ otherwise. \end{cases}$$

The identity mappings $h = h_{ID} : (\mathcal{P},\ \sigma) \to (\mathcal{Q},\ \gamma)$ and $g = g_{ID} : (\mathcal{Q},\ \gamma) \to (\mathcal{R},\ \eta)$ are $\frac{1}{2}$-fuzzy $\mathcal{B}^*$-continous. However, their composition $g \circ f$ fail to be $\frac{1}{2}$-fuzzy $\mathcal{B}^*$-continous.

Theorem 3.2. Let $h : \mathcal{P} \to \mathcal{Q}$ be an r-fuzzy $\mathcal{B}^*$-continuous mapping and let $g : \mathcal{Q} \to \mathcal{R}$ be an r-fuzzy continuous mapping. Then the composition function $g \circ h : \mathcal{P} \to \mathcal{R}$ is r-fuzzy $\mathcal{B}^*$-continuous.

4 r-fuzzy separation axioms and properties

Lemma 4.1. If $\psi \in \delta f O(X)$ and $\omega \in f\mathcal{B}^*O(X),\ \Rightarrow\ \psi \wedge \omega \in f\mathcal{B}^*O(X),$

Lemma 4.2. A fuzzy topological space $(X,\ \sigma)$, let ψ & ω be two fuzzy subsets. If $\psi \in h\delta O(X)$ & $\omega \in h\mathcal{B}^*O(X),\ \Rightarrow\ \psi \wedge \omega \notin h\mathcal{B}^*O(\psi).$

Theorem 4.1. If $h : (\mathcal{X},\ \sigma) \to (\mathcal{Y},\ \sigma)$ is a r-fy $\mathcal{B}^*$- cont. function & ψ is r-fuzzy open in $\mathcal{X},\ \Rightarrow$ the restriction $h \backslash \psi : (\psi, \sigma_\psi) \to (Y, s)$ is r-fy $\mathcal{B}^*$-cont..

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Proof. Consider $\mathcal{V}$ be an r-fuzzy open set of $\mathcal{Y}$. Then by hypothesis $h^{-1}(\mathcal{V})$ is r-fuzzy $\mathcal{B}^*$-open in $\mathcal{X}$. Hence by Lemma 4.2., we have $(h\backslash\psi)^{-1}(\mathcal{V}) = h^{-1}(\mathcal{V}) \wedge \psi \in h\mathcal{B}^*O(\psi)$. Thus, it follows that $h\backslash\psi$ is r-fy $\mathcal{B}^*$-cont..

Lemma 4.3. A fuzzy topological space $(\mathcal{X},\ \sigma)$, let ψ & ω be two fuzzy subsets. If $\psi \in h\delta O(X)$ and $\omega \in h\mathcal{B}^*O(\psi)$, $\Rightarrow$ $\psi \wedge \omega \in h\mathcal{B}^*O(X)$.

Theorem 4.2. Let $(\mathcal{X},\ \sigma) \to (\mathcal{Y},\ \sigma)$ be a function & $\{\mathcal{G}_i : i \in \mathcal{I}\}$ be a cover of $\mathcal{X}$ by r-fy δ-Op sets of $(\mathcal{X},\ \sigma)$. If $h\backslash\mathcal{G}_i : (\mathcal{G}_i, \sigma_{\mathcal{G}\rangle}) \to (\mathcal{Y}, s)$ is r-fy $\mathcal{B}^*$-cont. for all $i \in \mathcal{I}$, $\Rightarrow$ h is r-fy $\mathcal{B}^*$-cont..

Proof. Let ρ be an r-fuzzy open set of $(\mathcal{Y},\ \sigma)$. That implies by hypothesis $h^{-1}(\rho) = \mathcal{X} \wedge h^{-1}(\rho) = \bigvee\{\mathcal{G}_i \wedge h^{-1}(\rho) : i \in \mathcal{I}\} = \bigvee\{(h\backslash\mathcal{G}_i)^{-1}(\rho) : i \in \mathcal{I}\}$. Since $h\backslash G_i$ is r-fy $\mathcal{B}^*$-cont. for all $i \in \mathcal{I}$, $\Rightarrow$ $(h\backslash\mathcal{G}_i)^{-1}(\rho) \in (\mathcal{G}_i)$ for all $i \in \mathcal{I}$. By Lemma 4.3, we have $(h\backslash\mathcal{G}_i)^{-1}(\rho)$ is r-fy $\mathcal{B}^*$-cont. in $\mathcal{X}$. Therefore, h is r-fy $\mathcal{B}^*$-cont. in $(\mathcal{X},\ \sigma)$.

Definition 4.1. The r-fuzzy $\mathcal{B}^*$-frontier of a fuzzy subset ψ of $\mathcal{X}$, denoted by r-fuzzy $\mathcal{B}^*$-$\mathcal{F}r(\psi, r)$, is defined by $\mathcal{B}^*$-$Fr(\psi) = \mathcal{B}^*$-$\mathcal{C}_\sigma(\psi, r) \wedge \mathcal{B}^*$-$\mathcal{C}_\sigma(X\backslash\psi)$ equivalently $\mathcal{B}^*$-$Fr(\psi) = \mathcal{B}^*$ -$\mathcal{C}_\sigma(\psi, r)\backslash\mathcal{B}^*$-$\mathcal{I}_\sigma(\psi, r)$.

Theorem 4.3. The set of all fuzzy points x of $\mathcal{X}$ at which a function $h : (\mathcal{X},\ \sigma) \to (\mathcal{Y},\ \sigma)$ is not r-fy $\mathcal{B}^*$- cont. is identical with the union of the r-fuzzy $\mathcal{B}^* - \mathcal{F}r(\psi, r)$, of the inverse images of r-fuzzy open sets containing $h(x)$.

Proof. Necessity.
Let x be a fuzzy point of $\mathcal{X}$ at which h is not r-fy $\mathcal{B}^*$-cont.. Then, there is an r-fuzzy open set ρ of $\mathcal{Y}$ containing $h(x)$ such that $\varphi \wedge (\mathcal{X}\backslash h^{-1}(\rho)) \neq \phi$, for all $\varphi \in h\mathcal{B}^*O(X)$ containing x. Thus, we have $x \in \mathcal{B}^*$-$\mathcal{C}_\sigma(\mathcal{X}\backslash h^{-1}(\rho)) = X\backslash\mathcal{B}^* - \mathcal{I}_\sigma(h^{-1}(\rho))$ and $x \in h^{-1}(\rho)$. Therefore, we have $x \in \mathcal{B}^*$-$\mathcal{F}r(h^{-1}(\rho))$.

Sufficiency.
Suppose that $x \in \mathcal{B}^*$-$\mathcal{F}r(h^{-1}(\rho))$, for some ρ is r-fuzzy open set containing $h(x)$. Now, we assume that h is r-fy $\mathcal{B}^*$-cont. at $x \in \mathcal{X}$. Then $\exists$ $\varphi \in h\mathcal{B}^*O(\mathcal{X})$ containing x $\ni$ $h(\varphi) \leq \rho$. Therefore, we have $x \in \varphi \leq h^{-1}(\rho)$ and hence $x \in \mathcal{B}^*$-$\mathcal{I}_\sigma(h^{-1}(\rho)) \leq \mathcal{X}\backslash\mathcal{B}^*$-$\mathcal{F}r(h^{-1}(\rho))$. This is a contradiction. This means that h is not r-fy $\mathcal{B}^*$- cont. at x.

Definition 4.2. A r-fuzzy topological space $(\mathcal{X}, \sigma)$

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(1) for each and every pair of distinct fuzzy points η and ν in $\mathcal{X}$, $\exists$ disjoint r-fy $\mathcal{B}^*$-Op sets φ and ρ in $\mathcal{X}$ $\ni$ $\eta \in \varphi$ & $\nu \in \rho$. It is called r-fuzzy $\mathcal{B}^*$-T_2.

(2) for each pair of distinct fuzzy points η and μ of $\mathcal{X}$, $\exists$ r-fy $\mathcal{B}^*$-Op sets φ and ρ containing η and ν respectively $\ni$ $\nu \notin \varphi$ and $\eta \notin \rho$. It is called r-fuzzy $\mathcal{B}^*$-T_1.

(3) for each pair of distinct fuzzy points η and ν of $\mathcal{X}$, $\exists$ a r- fy $\mathcal{B}^*$- Op sets φ $\ni$ either $\eta \notin \varphi$, $\nu \notin \varphi$ or $\nu \notin \varphi$, $\eta \notin \varphi$. It is called r-fuzzy $\mathcal{B}^*$-T_0.

Remark 4.1. The following implications are hold for a topological space X:

$$\begin{tikzcd}
T_2\text{ -space} \arrow[r] \arrow[d] & T_1\text{ -space} \arrow[r] \arrow[d] & T_0\text{ -space} \arrow[d] \\
\beta*\!-\!T_2\text{ -space} \arrow[r] & \beta*\!-\!T_1\text{ -space} \arrow[r] & \beta*\!-\!T_0\text{ -space}
\end{tikzcd}$$

Now, by [[6] and [17]] and The converse of these implications are not true in general and the following examples.

Example 4.1 Let $\mathcal{X} = \mathcal{Y} = \{a\}$ and define the fuzzy sets $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \in \mathcal{I}^X$ as follows:
$\omega_1(a_0) = 0.4$,
$\omega_2(a_0) = 0.7$,
$\omega_3(a_0) = 0.8$,
$\omega_4(a_0) = 0.6$,
$\omega_5(a_0) = 0.2$.
Define fuzzy topology $\sigma, \sigma : \mathcal{I}^X \to \mathcal{I}$ as follows:

$$\sigma(\psi) = \begin{cases} 1 & \textit{if } \psi \in \{\underline{0},\ \underline{1}\}, \\ \frac{1}{4} & \textit{if } \psi \in \{\omega_1\}, \\ \frac{1}{3} & \textit{if } \psi = \{\omega_2\}, \\ 0 & \textit{if otherwise.} \end{cases}$$

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$$\sigma(\psi)=\begin{cases}1 & if\ \psi\in\{\underline{0},\ \underline{1}\},\\ \frac{1}{4} & if\ \psi\in\{\omega_3\},\\ \frac{1}{2} & if\ \psi=\{\omega_3\},\\ \frac{1}{5} & if\ \psi=\{\omega_4\},\\ \frac{1}{6} & if\ \psi=\{\omega_5\},\\ 0 & if\ otherwise.\end{cases}$$

(1) $(\mathcal{X},\ \sigma)$ is r-fuzzy $\mathcal{B}^*$-T_1 but not r-fuzzy $\mathcal{B}^*-T_2$-space

(2) $(X,\ \sigma)$ is r-fuzzy $\mathcal{B}^*$-T_0 but not r-fuzzy T_0-space.

Example 4.2 In Example 2.5 , $(\mathcal{X},\ \sigma)$ is r-fuzzy $\mathcal{B}^*-T_2$ (resp. r-fuzzy $\mathcal{B}^*-T_2$) but not r-fuzzy T_2 (resp. r-fuzzy T_2)-space.

Theorem 4.4. If $h:(\mathcal{X},\ \sigma)\to(\mathcal{Y},\ \sigma)$ is a r-fuzzy $\mathcal{B}^*$-continuous injection and $(\mathcal{Y},\ \sigma)$ is $T_i,\ \Rightarrow\ (\mathcal{X},\ \sigma)$ is r-fuzzy $\mathcal{B}^*$-T_i, where $i=0,1,2$.

Proof. We prove that the theorem for $i=1$.

Let $\mathcal{Y}$ be T_1 and η,ν be distinct points in $\mathcal{X}$. There exist open subsets φ,ρ in ν such that $h(\eta)\in\varphi,\ f(\nu)\notin\varphi,\ h(\eta)\notin\rho$ and $h(\nu)\in\rho$. Since h is r-fy $\mathcal{B}^*$-cont., $h^{-1}(\varphi)$ and $h^{-1}(\rho)$ are r-$\mathcal{B}^*$-fuzzy open subsets of $\mathcal{X}$ such that $\eta\in h^{-1}(\varphi),\nu\notin h^{-1}(\varphi),\eta\notin h^{-1}(\rho)$ and $\nu\in h^{-1}(\rho)$. Hence, $\mathcal{X}$ is r-fuzzy $\mathcal{B}^*$-T_1.

Theorem 4.5. If $h:(\mathcal{X},\ \sigma)\to(\mathcal{Y},\ \sigma)$ is r-fy $\mathcal{B}^*$-cont., $g:(\mathcal{X},\ \sigma)\to(\mathcal{Y},\ \sigma)$ is r-fuzzy super-continuous and $\mathcal{Y}$ is r-fuzzy Hausdorff, then the set $\{\eta\in\mathcal{X}: h(\eta)=g(\eta)\}$ is r- fuzzy $\mathcal{B}^*$-closed in $\mathcal{X}$. h is r-fy $\mathcal{B}^*$- cont. in $(\mathcal{X},\ \sigma)$.

Proof. Let $\psi=\{\eta\in\mathcal{X}:h(\eta)=g(\eta)\}$ and $\eta\notin\psi$. Then $f(\eta)\notin g(\eta)$. Since $\mathcal{Y}$ is r-fuzzy Hausdorff, $\exists$ r-fuzzy open sets φ and ρ of $\mathcal{Y}\ \ni\ h(\eta)\in\varphi, g(\eta)\in\rho$ and $\varphi\wedge\rho=\phi$. Since h is r-fuzzy $\mathcal{B}^*$-continuous, $\exists$ a r-fuzzy $\mathcal{B}^*$-open set G containing η such that $h(\mathcal{G})\leq\varphi$. Since g is r-fuzzy super continuous, there exist an r-fuzzy δ-open set H of X containing η such that $g(\mathcal{H})\leq\rho$. Now, put $\mathcal{W}=\mathcal{G}\wedge\mathcal{H}$, then by Lemma 4.1., we have $\mathcal{W}$ is a r-$\mathcal{B}^*$-fuzzy open set containing η and $h(\mathcal{W})\wedge g(\mathcal{W})\leq\varphi\wedge\rho=\phi$. Therefore, we obtain $\mathcal{W}\wedge\mathcal{A}=\phi$ and hence $\eta\notin\mathcal{B}^*-\mathcal{C}_\sigma(\mathcal{A},r)$. This shows that $\mathcal{A}$ is $\mathcal{B}^*$-closed in $\mathcal{X}$.

Definition 4.3. Let $h:(\mathcal{X},\ \sigma)\to(Y,\ \omega)$ be a mapping.The fuzzy subset $\{(\eta,h(\eta)):\eta\in\mathcal{X}\}$ of the fuzzy product space $\mathcal{X}\times\mathcal{Y}$ is called the fuzzy graph of h and is denoted by $\mathcal{G}(h)$.

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Definition 4.4. A mapping $h : (\mathcal{X}, \sigma) \to (\mathcal{Y}, \sigma)$, has a r-fuzzy $(\mathcal{B}^*, \sigma)$-graph if for each $(\eta, \nu) \in (\mathcal{X} \times \mathcal{Y}) \backslash \mathcal{G}(h)$, $\exists$ a r-fy $\mathcal{B}^*$-Op. φ of $\mathcal{X}$ containing η and an r-fuzzy open set ρ of $\mathcal{Y}$ containing ν such that $(\varphi \times \rho) \wedge \mathcal{G}(h) = h$.

Lemma 4.4. A mapping $h : (\mathcal{X}, \sigma) \to (\mathcal{Y}, \sigma)$, has a has a r-fuzzy $(\mathcal{B}^*, \sigma)$- graph iff for all $(\eta, \nu) \in \mathcal{X} \times \mathcal{Y}$ such that $\nu \notin h(\eta)$, $\exists$ a r- fy $\mathcal{B}^*$- Op set φ and an r-fuzzy open set ρ containing η and ν, respectively, such that $h(\varphi) \wedge \rho = \phi$.

Proof. It easy to prove usuing from definition 4.4..

Theorem 4.6. If $f : (X, \sigma) \to (Y, \sigma)$, is a r-fuzzy $\mathcal{B}^*$-continuous function and Y is r-fuzzy Hausdorff, then f has a r-fuzzy $(\mathcal{B}^*, \sigma)$-graph.

Proof. Let $(\eta, \nu) \in X \times Y$ such that $\nu \neq f(\eta)$. Then there exist r-fuzzy open sets φ and ρ such that $y \in \varphi, f(\eta) \in \rho$ and $\rho \wedge \varphi = \phi$. Since f is r-fuzzy $\mathcal{B}^*$-continuous, there exists r- fuzzy $\mathcal{B}^*$-open W containing η such that $f(W) \leq \rho$. This implies that $f(W) \wedge \varphi \leq \rho \wedge \varphi = \phi$. Therefore, f has a r-fuzzy $(\mathcal{B}^*, \sigma)$-graph.

Definition 4.5. A space X is said to be r-fuzzy $\mathcal{B}^*$-compact if every r- fuzzy $\mathcal{B}^*$-open cover of X has a finite subcover.

Theorem 4.7. If $f : (X,\ \sigma) \to (Y,\ \sigma)$ has a r-fuzzy $(\mathcal{B}^*, \sigma)$-graph, then $f(K)$ is r-fuzzy closed in (Y, s) for each fuzzy subset K which is r-fuzzy $\mathcal{B}^*$-compact relative to (X, σ).

Proof. Suppose that $\nu \notin f(K)$. Then $(\eta, y) \notin G(f)$ for each $\eta \in K$. Since $G(f)$ is r-fuzzy $(\mathcal{B}^*, \sigma)$-graph, there exist a r- fuzzy $\mathcal{B}^*$-open set U_η containing η and an r-fuzzy open set V_η of Y containing ν such that $f(U_\eta) \wedge V_\eta = \phi$. The family $\{U_\eta : \eta \in K\}$ is a cover of K by r-fuzzy $\mathcal{B}^*$- open sets. Since K is r-fuzzy $\mathcal{B}^*$-compact relative to (X, σ), there exists a finite fuzzy subset K_0 of K such that $K \leq \{U_\eta : \eta \in K_0\}$. Let $\rho = \bigwedge\{V_\eta : \eta \in K_0\}$. Then ρ is an r-fuzzy open set in Y containing ν. Therefore, we have $f(K) \wedge \rho \leq \bigvee_{\eta \in K_0} f(U_\eta) \wedge \rho \leq \bigvee_{\eta \in K_0} (f(u_\eta)) \wedge \rho = \phi$. It follows that, $y \notin \mathcal{C}_\sigma(f(K))$. Therefore, $f(K)$ is r-fuzzy closed in (Y, σ).

Corollary 4.1. If $f : (X,\ \sigma) \to (Y,\ \sigma)$ is r-fuzzy $\mathcal{B}^*$-continuous function and Y is r-fuzzy Hausdorff, then $f(K)$ is r-fuzzy closed in $(Y,\ \sigma)$ for each fuzzy subset K which is r-fuzzy $\mathcal{B}^*$-compact relative to (X, σ).

Lemma 4.5. The product of two r-fuzzy $\mathcal{B}^*$-open sets is r- fuzzy $\mathcal{B}^*$-open.

Received: October 30, 2025

International Journal of Applied Mathematics
Volume 38 No. 11s 2025,
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 4.8. If $f:(X,\sigma)\to(Y,\sigma)$, is a r-fuzzy $\mathcal{B}^*$-continuous function and Y is a r-fuzzy Hausdorff space, then f has a r-fuzzy $(\mathcal{B}^*,\sigma)$-graph.

Proof. Let $(\eta,y)\in X\times Y$ such that $y\neq f(\eta)$ and Y be a r-fuzzy Hausdorff space. Then there exist two r-fuzzy open sets φ and ρ such that $y\in\varphi, f(\eta)\in\rho$ and $\rho\wedge\varphi=f$. Since f is r- fuzzy $\mathcal{B}^*$-continuous, there exists a r- fuzzy $\mathcal{B}^*$-open set W containing η such that $f(W)\leq\rho$. This implies that $f(W)\wedge\varphi\in\rho\wedge\varphi=\phi$. Therefore, f has a r-fuzzy $(\mathcal{B}^*,\sigma)$-graph.

Corollary 4.2. If $f:(X,\sigma)\to(Y,\sigma)$, is r- fuzzy $\mathcal{B}^*$-continuous and Y is r-fuzzy Hausdorff, then $G(f)$ is r- fuzzy $\mathcal{B}^*$-closed in $X\times Y$.

Theorem 4.9. If $f:(X,\sigma)\to(Y,\sigma)$, has a $(\mathcal{B}^*,\sigma)$-graph and $g:Y\to Z$ is a r-$\mathcal{B}^*$-fuzzy continuous function, then the set $\{(x,y):f(x)=g(y)\}$ is r- fuzzy $\mathcal{B}^*$-closed in $X\times Y$.

Proof. Let $\psi=\{(x,y):f(x)=g(y)\}$ and $(x,y)\notin\psi$. We have $f(x)\notin g(y)$ and then $(x,g(y))\in(X\times Z)\backslash G(f)$. Since f has a r-fuzzy $(\mathcal{B}^*,\sigma)$-graph, then there exist a r-fuzzy $\mathcal{B}^*$-open set φ and an r-fuzzy open set ρ containing x and $g(y)$, respectively such that $f(\varphi)\wedge\rho=\phi$. Since g is a r-fuzzy $\mathcal{B}^*$-continuous function, then there exist an r-fuzzy $\mathcal{B}^*$-open set G containing y such that $g(G)\leq\rho$. We have $f(\varphi)\wedge g(G)=\phi$. This implies that $(\varphi\times G)\wedge\psi=\phi$. Since $\varphi\times G$ is r- fuzzy $\mathcal{B}^*$-open, then $(x,y)\notin\mathcal{B}^*$-$\mathcal{C}_\sigma(\psi)$. Therefore, ψ is $\mathcal{B}^*$-closed in $X\times Y$.

Corollary 4.3. . If $f:(X,\sigma)\to(Y,\sigma)$, $f:(Y,\sigma)\to(Z,\omega)$, are r-fuzzy $\mathcal{B}^*$-continuous functions and Z is r-fuzzy Hausdorff, then the set $\{(x,y):f(x)=g(y)\}$ is r-fuzzy $\mathcal{B}^*$- closed in $X\times Y$.

Proof. It follows from Corollary 4.2., and Theorem 4.9..

Theorem 4.10. If $f:(X,\sigma)\to(Y,\sigma)$ is a r- fuzzy $\mathcal{B}^*$-continuous function and Y is r-fuzzy Hausdorff, then the set $\{(x,y)\in X\times X:f(x)=f(y)\}$ is r- fuzzy $\mathcal{B}^*$-closed in $X\times X$.

Proof. Let $\psi=\{(x,y):f(x)=f(y)\}$ and let $(x,y)\in(X\times X)\backslash\psi$. Then $f(x)\neq f(y)$. Since Y is r-fuzzy Hausdorff, then there exist r-fuzzy open sets φ and ρ containing $f(x)$ and $f(y)$, respectively, such that $\varphi\wedge\rho=\phi$. But, f is r- fuzzy $\mathcal{B}^*$-continuous, then there exist r- fuzzy $\mathcal{B}^*$-open sets H and G in X containing x and y, respectively, such that $f(H)\leq\varphi$ and $f(G)\leq\rho$. This implies $(H\times G)\wedge\psi=\phi$. By Lemma 4.5., we have $H\times G$ is a r- fuzzy $\mathcal{B}^*$-open set in $X\times X$ containing (x,y). Hence, ψ is r- fuzzy $\mathcal{B}^*$-closed in $X\times X$.

Theorem 4.11. If $f:(X,\sigma)\to(Y,\sigma)$ is r- fuzzy $\mathcal{B}^*$-continuous and S is r-fuzzy closed in $X\times Y$, then $\omega_x(S\wedge G(f))$ is r- fuzzy $\mathcal{B}^*$- closed in X, where ω_x represents the projection of $X\times Y$ onto X.

Proof. Let S be a r-fuzzy closed subset of $X\times Y$ and $x\in\mathcal{B}^*$-$\mathcal{C}_\sigma(\omega_x(S\wedge G(f)))$. Let $\varphi\in\sigma$ containing x and $\rho\in\sigma$ containing $f(x)$. Since f is r- fuzzy $\mathcal{B}^*$-continuous, by Theorem 3.2., $x\in f^{-1}(\rho)\leq\mathcal{B}^*$-$\mathcal{I}_\sigma(f^{-1}(\rho))$. Then $\varphi\wedge\mathcal{B}^*$-$\mathcal{I}_\sigma(f^{-1}(\rho))\wedge\omega_x(S\wedge G(f))$ contains some fuzzy point z of X. This implies that $(z,f(z))\in S$ and $f(z)\in\rho$. Thus we have $(\varphi\times\rho)\wedge S\neq\phi$ and hence $(x,f(x))\in\mathcal{C}_\sigma(S)$. Since ψ is r-fuzzy closed, then $(x,f(x))\in S\wedge G(f)$ and $x\in\omega_x(S\wedge G(f))$. Therefore $\omega_x(S\wedge G(f))$ is r- fuzzy $\mathcal{B}^*$-closed in (X,σ).

Theorem 4.12. A fuzzy topological space (X,σ) is said to be r- fuzzy $\mathcal{B}^*$-connected if it is not the union of two nonempty disjoint r- fuzzy $\mathcal{B}^*$-open sets.

Theorem 4.13. If (X,σ) is a r- fuzzy $\mathcal{B}^*$-connected space and $f:(X,\sigma)\to(Y,\sigma)$ has a r-fuzzy $(\mathcal{B}^*,\sigma)$-graph and r- fuzzy $\mathcal{B}^*$-continuous function, then f is r-fuzzy constant.

Proof. Suppose that f is not constant. There exist disjoint fuzzy points $x,y\in X$ such that $f(x)=f(y)$. Since $(x,f(x))\notin G(f)$, then $y\neq f(x)$, hence by Lemma 4.4., there exist r-fuzzy open sets φ and ρ containing x and $f(x)$ respectively such that $f(\varphi)\wedge\rho=\phi$. Since f is r- fuzzy $\mathcal{B}^*$-continuous, there exist a r- fuzzy $\mathcal{B}^*$-open sets G containing y such that $f(G)\leq\rho$. Since φ and ρ are disjoint r-fuzzy $\mathcal{B}^*$-open sets of (X,σ), it follows that (X,σ) is not r- fuzzy $\mathcal{B}^*$-connected. Therefore, f is r-fuzzy constant.

Theorem 4.14. Let $(X_1,\sigma_1),(X_2,\sigma_2)$ and (X,σ) be fuzzy topological spaces. Define a function $f:(X,\sigma)\to(X_1\times X_2,\sigma_1\times\sigma_2)$ by $f(x)=(f(x_1),f(x_2))$. Then $f_i:X\to(X_i,\sigma_i)$, where $(i=1,2)$ is r- fuzzy $\mathcal{B}^*$-continuous if f is r- fuzzy $\mathcal{B}^*$-continuous.

Proof. It suffices to prove that $f_1:X\to(X_i,\sigma_i)$, where $i=1,2$ is r- fuzzy $\mathcal{B}^*$-continuous. Let U_1 be an r-fuzzy open set of X_1. Then $U_1\times U_2$ is open in $X_1\times X_2$. Then $f^{-1}(U_1)=f^{-1}(U_1\times U_2)$. Since f_1 is r- fuzzy $\mathcal{B}^*$-continuous, Then by Theorem 3.1., $f^{-1}(U_1)=f^{-1}(U_1\times X_2)\leq\mathcal{B}^*$- $\mathcal{I}_\sigma(f^{-1}(U_1\times X_2))=\mathcal{B}^*$-$\mathcal{I}_\sigma(f^{-1}(\varphi_1))$. This impliesthat $f_1:X\to(X_i,\sigma_i)$ is r- fuzzy $\mathcal{B}^*$-continuous.

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Received: October 30, 2025