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COEFFICIENT ESTIMATES ALONG WITH SECOND, THIRD AND FOURTH HANKEL DETERMINANT INEQUALITY FOR THE CLASS OF STARLIKE FUNCTIONS ASSOCIATED WITH THE NEPHRIOD DOMAIN

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Abstract

In this paper we introduce a new subclass $\mathcal{S}_{\mathcal{N}}^{*}$ of starlike functions which is subordinate to a Nephriod function $1 + z\cos z$, our paper establishes some coefficient bounds for this class, along with a derivation of a sharp bound for the well-known Fekete - Szegö functional, which plays a crucial role in the geometric analysis of analytic functions, Further we have derived a sharp bound for the second Hankel determinant, also we worked on Third Hankel determinant of this new subclass

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Introduction

et $\mathcal{A}$ is class of all the complex analytic functions in unit disk : $= \{z \in \mathbb{C}: |z| < 1\}$, normalized under conditions $h(0) = 0$ and $h'(0) = 1$, whose Taylor series expansions of the form

$$h(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots, \ \forall z \in \mathbb{D}, \tag{1.1}$$

Let a subclass of $\mathcal{A}$ consisting of all univalent (one-to-one) analytic functions in the unit disk $\mathbb{D}$denoted by $\mathcal{S}$. All the analytic-univalent functions with Taylor series expansions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

such that $\Re e\big(p(z)\big) > 0$, form a class denoted by $\mathcal{P}$. A conjecture regarding the coefficients of analytic functions of the form (1.1) in $\mathcal{S}$ was proposed a German mathematician Ludwig Bieberbach **[3]** in 1916 ,

$$|a_n| \leq n, \quad n \in \mathbb{N}, \quad \forall \quad h(z) \in \mathcal{S}.$$

The equality occurs for the Koebe function or one of its rotation function i.e

$$h(z) = \frac{z}{(1-z)^2} \quad or \quad \frac{z}{(1-e^{i\theta}z)^2}.$$

This conjecture, was named after him *"Bieberbach conjecture"*, was a significant challenge in the geometric function theory of complex analysis until it was proven by the French-American mathematician Louis de Branges **[4]** in 1985. Before *de Branges's* proof, numerous mathematicians attempted to prove or disprove this conjecture, leading to the discovery of various subfamilies of the class $\mathcal{S}$.

There are some very common subfamilies of $\mathcal{S}$, some of them are class of convex, starlike, and close-to-convex functions, denoted by $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{R}$.

Two functions h and g in $\mathcal{A}$ are said to be subordinate if h is subordinated to g, written as $h(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic over $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$h(z) = g\big(w(z)\big), \quad \forall z \in \mathbb{D}.$$

If the function $g(z)$ is univalent in $\mathbb{D}$, then

$$h(z) \prec g(z) \quad \text{if and only if} \quad h(0) = g(0) \quad \text{and} \quad h(\mathbb{D}) \subset g(\mathbb{D}).$$

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Ma and Minda **[10]** introduced two classes of analytic functions:

$$\mathcal{S}^*(\phi) := \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec \phi(z), \quad \forall z \in \mathbb{D} \right\},$$

$$\mathcal{C}(\phi) := \left\{ h \in \mathcal{A} : 1 + \frac{zh''(z)}{h'(z)} \prec \phi(z), \quad \forall z \in \mathbb{D} \right\}.$$

Here, $\phi(z)$ is a univalent analytic function with positive real part in the unit disk $\mathbb{D}$, such that $\phi(0) = 1$ and $\phi'(0) > 0$, where ϕ maps the open unit disk onto a region starlike with respect to 1 and symmetric with respect to the real axis. Various other classes can be formed by varying the function ϕ. *Sun*, *et al.* **[25]** have worked on some classes associated with activation functions.

The *Hankel determinant* is a fundamental mathematical construct that plays a significant role across various areas of mathematics, including algebra, real analysis, and complex analysis. Its applications are diverse: it is central in the study of orthogonal polynomials, has important uses in approximation theory, and also serves as a powerful tool in applied fields such as signal processing and time-series analysis.

The idea of the Hankel determinant traces back to the 19th-century German mathematician Hermann Hankel, whose work laid the foundation for its development. Later, *Pommerenke* **[14], [15], [16]** extended its scope within the framework of Geometric Function Theory (GFT), highlighting its deeper relevance in the study of analytic and univalent functions. Formally, the p^{th} Hankel determinant, for integers $p \geq 1$ and $n \geq 1$, where $p, n \in \mathbb{N}$, is defined for analytic functions h of the form (1.1). It is expressed as

$$H_p(n)(h) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+p-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+p-1} & a_{n+p} & \cdots & a_{n+2p-2} \end{vmatrix},$$

where a_k are the coefficients of the Taylor expansion of h.

Within the framework of Geometric Function Theory (GFT), one of the central problems is to determine sharp upper bounds for the Hankel determinants corresponding to various subclasses of the analytic function class $\mathcal{A}$. A notable example is the functional

$$\mathcal{H}_{2,1}(h) = a_2^2 - a_3,$$

which is widely recognized as the classical Fekete–Szegő functional.

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The study of higher-order Hankel determinants has attracted considerable attention. *Zaprawa* contributed to this direction by analyzing the third Hankel determinant and obtaining certain non-sharp estimates for the subclasses $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{R}$. However, deriving exact sharp bounds for these classes still remains an open problem in the literature.

Earlier investigations by *Noonan et al.* **[12]** and *Noor* **[13]** focused on the growth behavior of the general Hankel determinant $\mathcal{H}_{p,n}$, particularly for fixed integers p and n as $n \to \infty$, in relation to different subfamilies of the univalent class $\mathcal{S}$. Establishing precise estimates for the third and higher-order Hankel determinants continues to be a challenging and active area of research. Recent studies have further explored the third Hankel determinant for new subclasses of univalent functions, contributing to this ongoing line of inquiry.

In this paper we shall establish a sharp bound fro second Hankel determinant, represented as

$$\mathcal{H}_{(2,2)}(h) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

along with this in this paper we have established a nonsharp result for third Hankel determiant.

$$\mathcal{H}_3(1)(h) = 2a_2 a_3 a_4 - a_3^3 - a_4^2 + a_3 a_5 - a_2^2 a_5.$$

In recent years, considerable research has focused on establishing upper bounds for different orders of the Hankel determinant. For notable studies on the second, third, and fourth-order Hankel determinants can be referred to **[7, 9, 16, 19, 20, 27]** . Recently, Cho *et al.* **[5]** introduced a new function class, $\mathcal{S}^*_{\sin}$.

$$\mathcal{S}^*_{\sin} := \left\{ h \in \mathcal{A} : 1 + \frac{zh'(z)}{h(z)} \prec 1 + \sin(z), \quad \forall z \in \mathbb{D} \right\}.$$

Various researchers **[21], [22], [23], [24], [32], [33]** established Fekete-Szegö inequalities for different class. In this context, we define a new subclass $\mathcal{S}^*_{\mathcal{N}}$ of $\mathcal{S}^*$. This subclass includes all analytic univalent functions in $\mathcal{A}$ such that

$$\mathcal{S}^*_{\mathcal{N}} := \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec 1 + z\cos z \right\}.$$

Some of work on Nephriod domain is done **[26, 34, 35, 36, 37]** introduced the class of starlike functions associated with the nephriod domain and conducted extensive research on this class. This subclass offers new insights and extends the theory of starlike functions, adding to the rich tapestry of geometric function theory in complex analysis.

2 Preliminary Lemmas

Let $\mathcal{P}$ denotes the class of analytic functions of the form

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$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ z \in \mathbb{D}, \tag{2.1}$$

where $\Re\mathrm{e}(p(z)) > 0$ in $\mathbb{D}$.

Let $p(z)$ belongs to $\mathcal{P}$, such that

$$p(z) = \frac{1 + w(z)}{1 - w(z)}.$$

Solving above for $w(z)$, we have

$$w(z) = \frac{c_1 z}{2} + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{1}{4}(-c_1)c_2 + \frac{1}{8}c_1(c_1^2 - 2c_2) + \frac{c_3}{2}\right) z^3 + \cdots.$$

Lemma 2.1 (see [15]). *If $p(z) \in \mathcal{P}$, and c_n be the n^{th} coefficients of $P(z)$, then*

$$|c_n| \leq 2 \text{ for all } n \in \mathbb{N} \tag{2.2}$$

$$|c_2 - \gamma c_1^2| \leq 2\max\{1, |2\gamma - 1|\}, \text{ where } \gamma \in \mathbb{C}, \tag{2.3}$$

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}, \tag{2.4}$$

$$|c_{n+m} - \gamma c_n c_m| \leq 2, \gamma \in [0,1], \text{ Where } n, m \in \mathbb{N}, \tag{2.5}$$

$$|c_2 - \mu c_1^2| \leq \begin{cases} -4\mu + 2, & \text{if } \mu \leq 0, \\ 2, & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 2, & \text{if } 1 \leq \mu. \end{cases} \tag{2.6}$$

Lemma 2.2 ([9]). *If $P(z) \in \mathcal{P}$ then there exists $x, \zeta \in \overline{\mathbb{D}}$ with $|x| \leq 1, |\zeta| \leq 1$, such that*

$$2c_2 = {c_1}^2 + x(4 - {c_1}^2), \tag{2.7}$$

$$4c_3 = {c_1}^3 + 2(4 - {c_1}^2)c_1 x - (4 - {c_1}^2)c_1 x^2 + 2(4 - {c_1}^2)(1 - |x|^2)\zeta, \tag{2.8}$$

$$8c_4 = \left(-4(4 - c_1^2)(1 - |x|^2)\left(\zeta^2 \bar{x} + c_1(x - 1)\zeta - r(1 - |\zeta|^2)\right) + x(4 - c_1^2)(c_1^2(x^2 - 3x + 3) + 4x) + c_1^4\right). \tag{2.9}$$

Lemma 2.3([6]). *Let*

$$R(A, B, C) = \max_{z \in \mathbb{D}} \{|A + Bz + Cz^2| + 1 - |z|^2\}.$$

If $AC \geq 0$, then

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$$R(A,B,C)=\begin{cases}|A|+|B|+|C|, & \text{if } |B|\geq 2(1-|C|),\\ 1+|A|+\dfrac{B^2}{4(1-|C|)}, & \text{if } |B|<2(1-|C|).\end{cases}$$

Important Theorem

Theorem 3.1. *If* $h\in\mathcal{S}_N^*$, *be given by* (1.1). *Then*

$$|a_2|\leq 1,$$
$$|a_3|\leq \frac{1}{2},$$
$$|a_4|\leq \frac{1}{2},$$
$$|a_5|\leq \frac{2}{3}.$$

Proof. As we have defined above

$$\mathcal{S}_{\mathcal{N}}^*:=\left\{h\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec 1+z\cos z\right\}.$$

after simplifying $\frac{zf\prime(z)}{f(z)}$, we have

$$\begin{aligned}=\;& 1+a_2z+(2a_3-a_2^2)z^2+(a_2^3-3a_3a_2+3a_4)z^3+(-a_2^4+4a_3a_2^2-4a_4a_2-2a_3^2+4a_5)z^4\\ &+\left(a_2^5-5a_3a_2^3+5a_4a_2^2+5(a_3^2-a_5)a_2-5a_3a_4+5a_6\right)z^5\\ &+(-a_2^6+6a_3a_2^4-6a_4a_2^3+(6a_5-9a_3^2)a_2^2+6(2a_3a_4-a_6)a_2+2a_3^3-3a_4^2-6a_3a_5+6a_7)z^6\\ &+(a_2^7-7a_3a_2^5+7a_4a_2^4+7(2a_3^2-a_5)a_2^3+7(a_6-3a_3a_4)a_2^2-7(a_3^3-2a_5a_3-a_4^2+a_7)a_2\\ &+7(a_4a_3^2-a_6a_3-a_4a_5+a_8))z^7+\cdots.\end{aligned}$$

(3.1)

Using the definition of subordination we have

$$\frac{zh'(z)}{h(z)}=1+w(z)\cos\left(w(z)\right)$$

where expression of $1+w(z)\cos\left(w(z)\right)$ is

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$$1+\frac{c_1 z}{2}+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+\frac{1}{16}(c_1^3-8c_2c_1+8c_3)z^3+\frac{1}{32}(c_1^4+6c_2c_1^2-16c_3c_1-8c_2^2+16c_4)z^4$$
$$+\frac{1}{768}[-47c_1^5+96c_2c_1^3+144c_3c_1^2+48(3c_2^2-8c_4)c_1+384(c_5-c_2c_3)]z^5$$
$$+\frac{1}{1536}[91c_1^6-470c_2c_1^4+192c_3c_1^3+288(c_2^2+c_4)c_1^2+192(3c_2c_3-4c_5)c_1+96(c_2^3-8c_4c_2-4$$
$$+[-\frac{4231c_1^7}{92160}+\frac{91}{256}c_2c_1^5-\frac{235}{768}c_3c_1^4+\left(\frac{c_4}{8}-\frac{235c_2^2}{384}\right)c_1^3+\frac{3}{16}(2c_2c_3+c_5)c_1^2+\frac{1}{16}(2c_2^3+6c_4c_2+3$$
$$+\frac{1}{16}(3c_3c_2^2-8c_5c_2-8c_3c_4+8c_7)]z^7+\cdots.$$

(3.2)

Comparing (3.2) and (3.3), we derive

$$a_2=\frac{c_1}{2}, \qquad (3.3)$$

$$a_3=\frac{c_2}{4}, \qquad (3.4)$$

$$a_4=\frac{1}{48}(-c_1^3-2c_2c_1+8c_3), \qquad (3.5)$$

$$a_5=\frac{5c_1^4}{384}-\frac{7}{192}c_2c_1^2-\frac{c_3c_1}{24}+\frac{c_4}{8}-\frac{c_2^2}{32}, \qquad (3.6)$$

$$a_6=-\frac{13c_1^5}{1920}+\frac{83c_2c_1^3}{1920}-\frac{1}{40}c_3c_1^2-\frac{19}{960}c_2^2c_1-\frac{3c_4c_1}{80}+\frac{c_5}{10}-\frac{7c_2c_3}{120}, \qquad (3.7)$$

$$a_7=\frac{53c_1^6}{15360}-\frac{61c_2c_1^4}{1920}+\frac{13}{360}c_3c_1^3+\frac{631c_2^2c_1^2}{11520}-\frac{3}{160}c_4c_1^2-\frac{c_5c_1}{30}-\frac{37c_2c_3c_1}{1440}+\frac{c_6}{12}-\frac{c_3^2}{36}-\frac{5c_2c_4}{96}-\frac{c_2^3}{384}.$$

(3.8)

For (3.3) we have,

$$a_2=\frac{c_1}{2},$$

using lemma (2.1), it yields,

$$|a_2|\leq 1.$$

For (3.4) we have,

$$a_3=\frac{c_2}{4},$$

using lemma (2.1), it yields,

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$$|a_3| \leq \frac{1}{2}.$$

For (3.5) we have,

$$a_4 = \frac{1}{48}(-c_1^3 - 2c_2c_1 + 8c_3),$$

re-arranging the terms we have

$$|a_4| \leq \frac{1}{48}\left(|c_1^3| + 8\left|c_3 - \frac{1}{4}c_2c_1\right|\right),$$

using lemma (2.1), it yields,

$$|a_4| \leq \frac{1}{2}.$$

similarly for (3.6) we have,

$$|a_5| \leq \frac{7c_1^2}{192}\left|c_2 - \frac{960c_1^2}{2688}\right| + \frac{1}{8}\left|c_4 - \frac{1}{3}c_3c_1\right| + \frac{c_2^2}{32},$$

Using Lemma (2.1), it yields,

$$|a_5| \leq \frac{2}{3}.$$

Theorem 3.2. *If function* $h \in \mathcal{S}_{\mathcal{N}}^*$ *and* μ *is real, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\mu + \frac{1}{2}, & \text{if } \mu \leq 0, \\ \frac{1}{2}, & \text{if } 0 \leq \mu \leq 1, \\ \mu - \frac{1}{2}, & \text{if } 1 \leq \mu. \end{cases}$$

Proof. From (3.3) and (3.6) . we have,

$a_3 - \mu a_2^2 = \frac{c_2}{4} - \mu\frac{c_1^2}{4}$,

$$|a_3 - \mu a_2^2| = \frac{1}{4}|c_2 - \mu c_1^2|. \tag{3.9}$$

Using Lemma (2.1) in (3.9), we have

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$$|a_3 - \mu a_2^2| \leq \begin{cases} -\mu + \frac{1}{2}, & \text{if } \mu \leq 0, \\ \frac{1}{2}, & \text{if } 0 \leq \mu \leq 1, \\ \mu - \frac{1}{2}, & \text{if } 1 \leq \mu. \end{cases}$$

Theorem 3.3. *Let $h \in \mathcal{S}_{\mathcal{N}}^*$ then*

$$|\mathcal{H}_2(2)(h)| \leq \frac{1}{4}.$$

where the inequality is sharp for

$$h_1(z) := ze^{\sin z}$$

Proof. From (3.3) - (3.5), we have

$$(a_2a_4 - a_3^2) = -\tfrac{c_1^4}{96} - \tfrac{1}{48}c_2c_1^2 + \tfrac{c_3c_1}{12} - \tfrac{c_2^2}{16} \quad . \tag{3.10}$$

Using Lemma (2.2) in above and replacing $c_1 = c$, we have,

$$a_2a_4 - a_3^2 = \tfrac{c(4-c^2)}{24}\left[\tfrac{1}{c(4-c^2)}\left(\tfrac{c^4}{8} + c^2 - 6\right)\delta^2 - \tfrac{3c^3}{8(4-c^2)} + (1 - |\delta|^2)\zeta\right], \tag{3.11}$$

for $c = 0$ and $\delta, \eta \in \bar{\mathbb{D}}$ gives

$$|a_2a_4 - a_3^2| \leq \tfrac{1}{4}. \tag{3.12}$$

When $c = 2$ and $\delta, \eta \in \bar{\mathbb{D}}$ then (3.12) yields

$$|a_2a_4 - a_3^2| \leq \frac{1}{4},$$

When $c \in (0,2)$, and on simplifying (3.11), and choosing $|\eta| = 1$, it will yields,

$$|a_2a_4 - a_3^2| \leq \tfrac{c(4-c^2)}{24}[|-\tfrac{3c^3}{8(4-c^2)} + \tfrac{1}{c(4-c^2)}\left(-6 + c^2 + \tfrac{c^4}{8}\right)\delta^2| + (1 - |\delta|^2)]. \tag{3.13}$$

$$\mathcal{H}(2,2) = \tfrac{c(4-c^2)}{24}[P(A,B,C)], \tag{3.14}$$

where

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$$P(A,B,C) = max_{z\in\mathbb{D}}|-\frac{3c^3}{8(4-c^2)}+\frac{1}{c(4-c^2)}\left(-6+c^2+\frac{c^4}{8}\right)\delta^2|+1-|\delta|^2,$$

using lemma (2.3), we have,

$$A=\frac{-3c^3}{8(4-c^2)}, B=0 \ and \ C=\frac{1}{c(4-c^2)}(-6+c^2+\frac{c^4}{8})$$

All the terms in above equation are positive so we can conclude that $AC>0$. Here $|B|-2(1-|C|)\geq 0$ for all $c\in(0,2)$, hence again using Lemma (2.3), it yields

$$P(A,B,C)=\frac{3c^3}{8(4-c^2)}+\frac{1}{c(4-c^2)}\left(6-c^2-\frac{c^4}{8}\right),$$

which can be written as

$$|\mathcal{H}_2(2)(h)|\leq\frac{c^4}{64}+\frac{1}{24}\left(6-c^2-\frac{c^4}{8}\right).$$

above equation attains maximum value at $c=0$ and $c=2$, which is $\frac{1}{4}$, hence

$$|\mathcal{H}_2(2)(h)|\leq\frac{1}{4}.$$

Theorem 3.4. *Let $h\in\mathcal{S}_{\mathcal{N}}^*$ then $|\mathcal{H}_3(1)(h)|<\frac{1}{6}$.*

Proof. As we know that

$$\mathcal{H}_3(1)(h)=-a_3^3+2a_2a_4a_3+a_5a_3-a_4^2-a_2^2a_5$$

using (3.3) – (3.6) we have

$$\frac{1}{4608}[-17c_1^6+25c_2c_1^4+80c_3c_1^3-2(31c_2^2+72c_4)c_1^2+208c_2c_3c_1 \\ -4(27c_2^3-36c_4c_2+32c_3^2)],$$

using lemmas and rearranging the terms we obtain

$$\frac{1}{4608}[u_1(c,\delta)+u_2(c,\delta)\eta+u_3(c,\delta)\eta^2+\phi(c,\delta,\eta)\rho],$$

where $\rho,\eta,\delta\in\mathbb{D}$ and

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$$\begin{aligned}
u_1(c,\delta) :=\ & \frac{-9}{2}c^6 - \frac{1}{2}(4-c^2)[(4-c^2)(36\delta^3 - 2c^2\delta^4) - 18\delta c^4 \\
& +3c^4\delta^3 + 60c^2\delta^3 - 12c^4\delta^2 + 144c^2\delta^2], \\
u_2(c,\delta) :=\ & 4c(4-c^2)(1-|\delta|^2)(6(c^2+6\delta) + (4-c^2)(-\delta-3)\delta), \\
u_3(c,\delta) :=\ & -4(4-c^2)(1-|\delta|^2)\left((4-c^2)(|\delta|^2+8) - 9c^2\bar{\delta}\right), \\
\phi(c,\delta,\eta) :=\ & 36(4-c^2)(1-|\delta|^2)(1-|\eta|^2)\big((4-c^2)\delta - c^2\big).
\end{aligned}$$

using $|\delta| = d$, $|\eta| = u$ and $|\rho| \leq 1$

$$|\mathcal{H}_3(1)(h)| \leq \frac{1}{4608}[|u_1(c,\delta)| + |u_2(c,\delta)| + |u_3(c,\delta)| + |\psi(c,\delta,\eta)|] \leq M(c,d,u).$$

where

$$M(c,d,u) = \frac{1}{4608}[k_1(c,d) + k_2(c,d)u + k_3(c,d)u^2 + k_4(c,d)(1-u^2)|].$$

where

$$\begin{aligned}
k_1(c,d) :=\ & \frac{9c^6}{2} + \frac{1}{2}(4-c^2)((4-c^2)(2c^2d^4 + 36d^3) \\
& 3c^4d^3 + 12c^4d^2 + 18c^4d + 60c^2d^3 + 144c^2d^2), \\
k_2(c,d) :=\ & 4c(4-c^2)(1-d^2)\big((4-c^2)d(d+3) + 6(c^2+6d)\big), \\
k_3(c,d) :=\ & 4(4-c^2)(1-d^2)\big((4-c^2)(d^2+8) + 9c^2d\big), \\
k_4(c,d) :=\ & 36(4-c^2)(1-d^2)\big((4-c^2)d + c^2\big).
\end{aligned}$$

here we need to maximize the $M(c,d,u)$ on a closed cuboid $\mathcal{D} = [0,2] \times [0,1] \times [0,1]$. For this we need to find the maximum values in the interior of its six faces, twelve edges, and in the interior of D. First we will solve Hankel for the interior of $\mathcal{D}$, i.e $(0,2) \times (0,1) \times (0,1)$.

Differentiating the $M(c,d,u)$ w.r.t u, we have,

$$\begin{aligned}
\frac{\partial M}{\partial u} =\ & \frac{1}{1152}(4-c^2)(d^2-1)[-6c^3 + (c^2-4)d^2(c+2u) \\
& +3d(c(c^2-16) - 12(c^2-2)u) + (34c^2-64)u]
\end{aligned} \tag{3.15}$$

solving for $\frac{\partial M}{\partial u} = 0$, we have

$$u = \frac{6c^3 + (4-c^2)cd^2 + 3(16-c^2)cd}{2(d-1)((c^2-4)d - 17c^2 + 32)} \tag{3.16}$$

if u_0 is the critical point inside the cuboid $\mathcal{D}$, then $u_0 \in (0,1)$, here we can see this is only possible if

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$$c^2 > \frac{4(8-d)}{17-d}. \qquad (3.17)$$

If the point exists then it will satisfies both (3.11) and (3.12). Let $p(d) = \frac{4(8-d)}{17-d}$, as $p'(d) < 0$ for (0,1), p is a decreasing function in (0,1). After calculation we have that above equations are satisfied for $c > 1.6244416638125102$ and $d < \frac{1}{6}$. After this we have to find an inequality for $M(c,d,u)$ inside the cuboid $(1.6244416638,2) \times \left(0,\frac{1}{6}\right) \times (0,1)$. For our convenience we will assume c and d in the larger range, let $d < \frac{1}{4}$ and $c > \frac{3}{2}$.

$$\begin{aligned}
k_1(c,d) \leq\ & \frac{9c^6}{2} + \frac{1}{2}\left(\frac{677c^4}{128} + \frac{301c^2}{32} + \frac{9}{4}\right)(4-c^2) =: m_1(c),\\
k_2(c,d) \leq\ & c(4-c^2)\left(\frac{1245c^2}{64} + \frac{735}{16}\right) =: m_2(c),\\
k_3(c,d) \leq\ & (4-c^2)\left(\frac{1935}{16} - \frac{1395c^2}{64}\right) =: m_3(c),\\
k_4(c,d) \leq\ & (4-c^2)\frac{135}{16}(3c^2+4) =: m_4(c).
\end{aligned}$$

$$M(c,d,u) \leq \frac{1}{4608}[m_1(c) + m_4(c) + m_2(c)u + [m_3(c) - m_4(c)]u^2] := \Psi(c,u).$$

$$\begin{aligned}
\Psi(c,u) =:\ & \frac{1}{1179648}[475c^6 - 4980c^5u + 4c^4(3015u^2 - 1244) + 8160c^3u + c^2(21808 - 70560u^2)\\
& + 47040cu + 17856(5u^2+2)].
\end{aligned}$$

by taking derivative of above w.r.t u we have.

$$\frac{\partial\Psi}{\partial u} = \frac{1}{4608}\left[m_2(c) + 2\left(m_3(c) - m_4(c)\right)u\right] \quad \text{also} \quad \frac{\partial^2\Psi}{\partial u^2} = \frac{2}{4608}[m_3(c) - m_4(c)]$$

from above it is clear that, $m_3(c) - m_4(c) \leq 0$ for $c \in (3/2,2)$, it yields $\frac{\partial^2\Psi}{\partial u^2} \leq 0$ for $u \in (0,1)$. Therefore, it can concluded that,

$$\frac{\partial\Psi}{\partial u} \geq \frac{\partial\Psi}{\partial u}|_{u=1} = \frac{-4980c^5 + 24120c^4 + 8160c^3 - 141120c^2 + 47040c + 178560}{1179648}.$$

as a result

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$$\Psi(c,u) \le \Psi(c,1)$$
$$\le \frac{475c^6 - 4980c^5 + 7084c^4 + 8160c^3 - 48752c^2 + 47040c + 124992}{1179648} := j(c).$$

On evaluating for j, we observe that largest value is attained at $c = 3/2$ which is $\frac{830979}{8388608} \approx 0.099060$

Hence

$$M(c,d,u) < \frac{1}{6}, \quad \text{for} \quad (c,d,u) \in (1.6244416637,2) \times \left(0,\frac{1}{6}\right) \times (0,1).$$

From all above it is concluded that $M(c,d,u) < \frac{1}{6}$, inside the cuboid $\mathcal{D}$.

II). Now we will consider inside the six faces of cuboid. On the side $c = 0$, $M(c,d,u)$ will be

$$p_1(d,u) := M(0,d,u) = \frac{288d^3 + 576(1-d^2)d(1-u^2) + \left(64(1-d^2)(d^2+8)\right)u^2}{4608}.$$

above function $p_1(d,u)$ has no critical point in $(0,1) \times (0,1)$, because

$$\frac{\partial p_1}{\partial u} = \frac{4(1-d^2)(8-d)(1+d)u}{144} \neq 0, \quad \text{for} \quad d,u \in (0,1)$$

On $c = 2$, $M(c,d,u)$, yields

$$M(2,d,u) = \frac{1}{16}, \quad \text{where} \quad d,u \in (0,1),$$

On $d = 0$, $M(c,d,u)$ yields

$$p_2(c,u) = \frac{8(4-c^2)(32-17c^2)u^2 + 9(c^4 - 8c^2 + 32)c^2 + 48(4-c^2)c^3u}{9216}.$$

as $c \in (0,2)$ and $u \in (0,1)$. We now solve the system $\frac{\partial p_2}{\partial u} = 0$ and $\frac{\partial p_2}{\partial c} = 0$ to obtain the critical points, hence, we obtain

$$u = \frac{3c^3}{17c^2 - 32} =: u_1 \tag{3.18}$$

for the given range of u, u_1 should belong to $(0,1)$, which is possible only if $c > c_0 \approx 1.6244416638$. And upon calculation for $\frac{\partial p_2}{\partial c} = 0$ we obtain

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$$c(288c - 120c^3)u + c(272c^2 - 800)u^2 + c(27c^4 - 144c^2 + 288) = 0 \quad (3.19)$$

substituting (3.18) in (3.19), it yields

$$c^8 - 5776c^6 + 26656c^4 - 51200c^2 + 32768 = 0 \quad (3.20)$$

A simple calculation of (3.20) implies, $c \approx 1.6959796194820452$, so the p_2 have a critical point $(1.69597961948, 0.86606500)$ in $(0,2) \times (0,1)$. At this point p_2 have maximum value 0.0608522 On $d = 1, M(c,d,u)$ reduces to

$$M(c,1,u) := p_3(c,u) := \frac{-11c^6 - 26c^4 + 280c^2 + 288}{4608}.$$

solving $\frac{\partial p_3}{\partial c} = 0$ we have the critical points $c_0 = 0$ and $c_1 = \sqrt{\frac{2}{33}\left(\sqrt{2479} - 13\right)} \approx 1.4932079994350664$ at $c = c_1$. thus p_3 attains its maximum value 0.143472, or

$$M(c,1,u) < \frac{1}{6}.$$

On $u = 0$ we have

$$\begin{aligned} M(c,d,0) = & : p_4(c,d) \\ = & \frac{1}{4608}\left[\frac{9c^6}{2} + 36(4-c^2)(1-d^2)\left((4-c^2)d + c^2\right) + \frac{1}{2}(4-c^2)(3c^4d^3 + 12c^4d^2 + 18c^4\right. \\ & \left. +144c^2d^2 + (4-c^2)(2c^2d^4 + 36d^3))\right] \end{aligned}$$

on solving the system of equations i.e $\frac{\partial p_4}{\partial d} = 0$, $\frac{\partial p_4}{\partial c} = 0$, this system have no solutions in $(0,2) \times (0,1)$. On $u = 1$ we have

$$\begin{aligned} M(c,d,1) = & : p_5(c,d) \\ = & \frac{1}{9216}[9c^6 + 8(4-c^2)c(1-d^2)\left(4d(d+12) - c^2(d^2+3d-6)\right) + 8(4-c^2)(1-d^2)(\\ & -(c^2-4)(d^2+8)) + (4-c^2)d(c^4(-2d^3+3d^2+12d+18) + 8c^2d(d^2+3d+18) + \end{aligned}$$

after calculations we get that critical point lies at $(1.3522835, 0.783107476)$ and the value of $M(c,d,1)$ at this point is 0.151995. i.e

$$M(c,d,1) < \frac{1}{6}.$$

III). Now we will find the values of $M(c,d,u)$ on all the vertices of cuboid $\mathcal{D}$, below is the table for $M(c,d,u)$ on all the vertices,

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M(0,0,0) = 0	M(0,0,1) $=\frac{1}{9}$	M(0,1,0) $=\frac{1}{16}$	M(0,1,1) $=\frac{1}{16}$
M(2,0,0) $=\frac{1}{16}$	M(2,0,1) $=\frac{1}{16}$	M(2,1,0) $=\frac{1}{16}$	M(2,1,1) $=\frac{1}{16}$

IV). At last, we will find the greatest value of $M(c,d,u)$ on the 12 edges of cuboid $\mathcal{D}$.

$$M(c,0,0)=\frac{c^2(c^4-8c^2+32)}{1024}\leq M(2,0,0)\leq\frac{1}{16},$$

$$M(c,0,1)=\frac{(3c^3-8c^2+32)^2}{9216}\leq\frac{32041}{531441}<M(0,0,1)=\frac{1}{9},$$

$$M(c,1,0)=\frac{-11c^6-26c^4+280c^2+288}{4608}\leq M(\beta,1,0)\approx 0.143472,\quad c\in(0,2),$$

$$M(0,d,0)=-\frac{1}{16}d(d^2-2)\leq M\left(0,\sqrt{\frac{2}{3}},0\right)=0.06804,$$

$$M(0,d,1)=\frac{1}{144}(-2d^4+9d^3-14d^2+16)\leq M(0,0,1)=\frac{1}{9},\qquad d\in(0,1),$$

$$M(2,d,0)=\frac{1}{16},\quad M(2,d,1)=\frac{1}{16},\quad d\in(0,1),$$

$$M(0,0,u)=\frac{u^2}{9}\leq\frac{1}{9},M(0,1,u)=\frac{1}{16},\quad u\in(0,1),$$

$$M(2,0,u)=\frac{1}{16},\quad M(2,1,u)=\frac{1}{16},\quad u\in(0,1).$$

From all above we conclude that

$$|\mathcal{H}_3(1)(h)|:=M(c,d,u)<\frac{1}{6}$$

Theorem 3.5. *Let* $f\in\mathcal{S}_{\mathcal{N}}^{*}$, *then*

$$|a_2a_3-a_4|\leq\frac{1}{2}.$$

The inequality is sharp for the function

$$f_2:=ze^{\sin(z)}.$$

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Proof : From (3.3) - (3.5), we have,

$$(a_2a_3 - a_4) = \frac{1}{48}(c_1^3 + 8c_2c_1 - 8c_3) \quad . \tag{3.21}$$

The functional $(a_2a_3 - a_4)$ and the class $\mathcal{S}_{\mathcal{N}}^*$ both are rotationally invariant, so we can assume that $c_1 = c$. Using Lemma (2.2) and after simplifying, we have

$$a_2a_3 - a_4 = \frac{1}{48}[c^3(3 - 2\delta^2) + 8c\delta^2 - 4c^2\eta(|\delta|^2 - 1) + 16\eta(|\delta|^2 - 1)] := \kappa. \tag{3.22}$$

where $\delta,\ \eta \in \bar{\mathbb{D}}$. When $c = 0$,

$$|a_2a_3 - a_4| \leq \frac{1}{3}. \tag{3.23}$$

When $c = 2$ and $\delta \in \bar{\mathbb{D}}$ then (3.22) yields

$$|a_2a_3 - a_4| \leq \frac{1}{2},$$

Re-arranging the (3.22), we have

$$a_2a_3 - a_4 := \frac{c^3}{16} + \frac{1}{24}c(4 - c^2)\delta^2 - \frac{1}{12}(4 - c^2)\eta(1 - |\delta|^2) \quad . \tag{3.24}$$

$$|a_2a_3 - a_4| \leq \frac{(4-c^2)}{12}\{|-\frac{3c^3}{4(4-c^2)} - \frac{c}{2}\delta^2| + (1 - |\delta|^2)\}. \tag{3.25}$$

When $c \in (0,2)$, and on simplifying (3.24), it yields,

$$\frac{(4 - c^2)}{12}[K(A, B, C)],$$

where $K(A, B, C) = |-\frac{3c^3}{4(4-c^2)} - \frac{c}{2}\delta^2| + 1 - |\delta|^2$. comparing above with $|Cz^2 + Bz + A| + 1 - |z|^2$, we have

$$A = \frac{-3c^3}{8(4 - c^2)}, B = 0 \ and \ C = -\frac{c}{2}$$

All the terms in above equation are negative so we can conclude that $AC > 0$. Also $|B| - 2(1 - |C|) \leq 0$ for all $c \in (0,2)$, hence again using Lemma (2.3), it yields

$$K(A, B, C) \leq 1 + \frac{3c^3}{4(4 - c^2)},$$

from equation (3.25), we have

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$$|a_2a_3 - a_4| \leq \frac{(4-c^2)}{12}\left(1 + \frac{3c^3}{4(4-c^2)}\right) = \frac{(4-c^2)}{12} + \frac{c^3}{16}.$$

above inequality attains maximum value at $c = 2$, which is $\frac{1}{2}$, hence

$$|a_2a_3 - a_4| \leq \frac{1}{2}.$$

the above inequality is sharp for the function, $f_2(z) = ze^{\sin(z)}$

$$f_2(z) := z + z^2 + \frac{z^3}{2} - \frac{z^5}{8} + O(z^6).$$

here $a_1 = 1, a_2 = 1, a_3 = \frac{1}{2}$ and $a_4 = 0$, which gives

$$|a_2a_3 - a_4| = \frac{1}{2}.$$

Conclusions

In our this research paper, we have examined the subclass $\mathcal{S}_{\mathcal{N}}^*$ of analytic univalent functions and investigated several associated coefficient problems. We have obtained the sharp bounds for both the Fekete-Szegö functional and the second-order Hankel determinant, thereby contributing new sharp results to the existing literature. For the third-order Hankel determinant, we derived an explicit estimate; however, the bound is not optimal, leaving scope for refinement in future studies. In addition, we established a sharp bound for the Zalcman functional $|a_2a_3 - a_4|$ for the considered class, providing further insight into its coefficient structure. The work of this research paper may serve as a foundation for further research in order to improving the third-order Hankel determinant estimate of this class.

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