International Journal of Applied Mathematics

Volume 38 No. 10s, 2025

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

ON NANO FUZZY CONTINUOUS FUNCTION VIA NOVEL CLASSES OF OPEN SETS

Jackson S[1] Gnanaselvam Jude I [2]

[1]Assistant Professor, [2]Research Scholar (Reg.no 23112232091006)

P.G. and Research Department of Mathematics,

V. O. Chidambaram College, Thoothukudi - 628 008,

(Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India)

Abstract.

This research study focuses on the notions of nano fuzzy continuous, which play an important role in understanding the topological and algebraic aspects of fuzzy spaces. The Nano fuzzy J-open set is the new type of open set which was utilizing Nano fuzzy semi-interior and closure operators. In this study, we define and examine the Nano fuzzy J continuous of the Nano fuzzy J open set, as well as the essential features and interactions between them.

Key words: Nano fuzzy topology, Nano fuzzy J open set, Nano fuzzy J closed set, Nano fuzzy J continuous.

1. Introduction:

Zadeh (1965) created fuzzy set theory[1], which gives a paradigm for dealing with imprecise and uncertain information. In 1968 Chang introduced Fuzzy topology[2] as an extension of classical topology that incorporates the concept of fuzzy sets. Later, Lellis Thivagar and Richard (2013) proposed Nano topology[3], which expanded classical topology by introducing granular computing concepts, allowing for more detailed investigation of micro-level topological systems. The study of Nano fuzzy open sets is a new subject in mathematics that combines the ideas of fuzzy set theory with Nano topology to deal with ambiguity and vagueness in a variety of applications like decision-making, medical diagnostics, and artificial intelligence.

An important paradigm for applying fuzzy and classical topological ideas to situations including uncertainty, ambiguity, and granular structures is Nano fuzzy topology[4]. This approach allows for a more comprehensive understanding of continuity, openness, and closure in situations where information is ambiguous or poorly defined by fusing the concepts of Nano topology with fuzzy set theory. Weaker notions of open sets in Nano fuzzy topological spaces, including Nano fuzzy Z-open, semi-open, pre-open, b-open and β-open sets, have been investigated[5-8]. Deeper understanding of the structural behaviour of mappings and their related topological features has resulted from the introduction and study of different generalised forms of continuity in a variety of fuzzy and Nano topological environments in recent years.[9,10]

Received: Aug 09, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

. Nano fuzzy J open sets [11, 12] and their characterizations have already been studied. Inspired by these developments, the present work introduces and examines the concept of Nano fuzzy J continuous functions. After defining Nano fuzzy J continuity, we investigate its fundamental properties and related characteristics. In order to show the parallels, differences, and hierarchical connections within the larger continuity framework, the interconnections between Nano fuzzy J continuity and other well-known kinds of continuity are investigated. Additionally, illustrative examples are given to show how the suggested notions behave and are applicable in both theoretical and practical contexts.

2. Preliminaries:

Definition 2.1.[4] Let S be a non-empty set and R be an equivalence relation on . Let K be a $F_Y s$ in S with the membership function μ_F . The fuzzy nano lower (upper) approximations and fuzzy nano boundary of K in the approximation (S, R) denoted by $\underline{F_Z N}(K)$, $\overline{F_Z N}(K)$ and $BF_Z N(K)$ are respectively defined as follows:

(i) $\underline{F_Z N}(K) = \{l, \mu_{\underline{R}(A)}(l) \,/\, y \in [l]_R, l \in S\}$,

(ii) $\overline{F_Z N}(K) = \{l, \mu_{\overline{R}(A)}(l) \,/\, y \in [l]_R, l \in S\}$,

(iii) $BF_Z N(K) = \overline{F_Z N}(K) - \underline{F_Z N}(K)$,

where $\mu_{\underline{R}(A)}(l) = \wedge_{y \in [l]_R} \mu_A(y)$ and $\mu_{\overline{R}(A)}(l) = \vee_{y \in [l]_R} \mu_A(y)$.

The collection $\tau_R(K) = \{0_F, 1_F, \underline{F_Z N}(K), \overline{F_Z N}(K), BF_Z N(K)\}$ forms a topology called as Nano fuzzy topology and $(S, \tau_R(K))$ as **Nano Fuzzy Topological space** (simply, $NF_Y Ts$). The elements of $\tau_R(K)$ are called **Nano fuzzy open** (simply, $NF_Y os$) sets. Elements of $\tau_{R^c}(K)$ are called **Nano fuzzy closed** (simply, $NF_Y cs$) sets.

Definition 2.2.[4] Let $(S, \tau_R(K))$be a $NF_Y Ts$. Let γ be a F_Y subs of S. Then

i) **Nano fuzzy interior** of γ (simply $NF_Y int(\gamma)$) is described as $NF_Y int(\gamma) = \vee\{\beta : \beta \leq \gamma \,\&\, \beta \textit{ is a } NF_Y \; os\}$

ii) **Nano fuzzy closure** of γ (simply $NF_Y cl(\gamma)$) is described as $NF_Y cl(\gamma) = \wedge\{\beta : \gamma \leq \beta \,\&\, \beta \textit{ is a } NF_Y \; cs\}$

Definition 2.3.[10] Let S_1, S_2 be a non-empty set and R be an equivalence relation on S_1, S_2. Let K_1, K_2 be a $F_Y s$'s in S_1, S_2 respectively with the membership function μ_F. Then the mapping $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ is said to be

i) **Nano fuzzy continuous** (simply, $NF_Y Cts$) if for each $NF_Y os\ A$ of S_2, the set $\sigma^{-1}(A)$ is $NF_Y os$ of S_1.

ii) **Nano fuzzy β continuous** (simply, $NF_Y \beta - Cts$) if for each $NF_Y os\ A$ of S_2, the set $\sigma^{-1}(A)$ is $NF_Y \beta - os$ of S_1.

iii) **Nano fuzzy semi continuous** (simply, $NF_Y SCts$) if for each $NF_Y os\ A$ of S_2, the set $\sigma^{-1}(A)$ is $NF_Y Sos$ of S_1.

iv) **Nano fuzzy α continuous** (simply, $NF_Y \alpha - Cts$) if for each $NF_Y os\ A$ of S_2, the set $\sigma^{-1}(A)$ is $NF_Y \alpha - os$ of S_1.

Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

v) **Nano fuzzy pre continuous** (simply, NF_YPCts) if for each $NF_Yos\ A$ of S_2, the set $\sigma^{-1}(A)$ is NF_YPos of S_1.

vi) **Nano fuzzy regular continuous** (simply, NF_YRCts) if for each $NF_Yos\ A$ of S_2, the set $\sigma^{-1}(A)$ is NF_YRos of S_1.

Definition 2.4.[12] Let $(S, \tau_R(K))$ be a NF_YTs. Let γ be a $F_Y$subs of S. Then γ is defined as

i) $\boldsymbol{NF_Y\ J}$ **open** set (simply, $\boldsymbol{NF_YJos}$) if $\gamma \leq NF_Yscl(NF_Ysint(\gamma))$.

ii) $\boldsymbol{NF_Y\ J}$ **closed** set (simply, $\boldsymbol{NF_Y\ Jcs}$) if $NF_Ysint(NF_Yscl(\gamma)) \leq \gamma$.

The $\boldsymbol{NF_Y\ J}$ **interior** of γ (simply, $\boldsymbol{F_YJint(\gamma)}$), is described as a union of all $NF_Y\ Jos$'s contained in γ. The $\boldsymbol{NF_Y\ J}$ **closure** of γ (simply, $\boldsymbol{NF_YJcl(\gamma)}$), is described as a intersection of all $NF_Y\ Jcs$'s contain γ.

3. NANO FUZZY J CONTINUOUS

In this section, we define the nano fuzzy J open set and nano fuzzy J continuous functions, establish their relationships with other types of continuity, and present illustrative examples.

Definition 3.1. A mapping $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ is said to be Nano fuzzy J −continuous (simply, NF_YJCts) if for each $NF_Y\ os\ A$ of S_2, the set $\sigma^{-1}(A)$ is $NF_Y\ Jos$ of S_1.

Lemma 3.2. Let $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ be a mapping. Then the following statements are hold.

i) If T_1 and T_2 are F_Ysubs's of S_1 such that $T_1 \leq T_2$, then $\sigma(T_1) \leq \sigma(T_2)$.

ii) If T_1 and T_2 are F_Ysubs's of S_1 such that $T_1 \leq T_2$, then $\sigma^{-1}(T_1) \leq \sigma^{-1}(T_2)$.

Lemma 3.3. Let $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ be a mapping. If T_1 is a F_Ysubs's of S_1 and T_2 is a F_Ysubs's of S_2. Then the following statements are hold.

i) $\sigma(\sigma^{-1}(T_2)) \leq T_2$

ii) $\sigma(\sigma^{-1}(T_2)) = T_2$ iff σ is surjective.

iii) $\sigma^{-1}(\sigma(T_1)) \geq T_1$

iv) $\sigma^{-1}(\sigma(T_1)) = T_1$ whenever σ is injective.

Theorem 3.4. Let $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ be a mapping. Then, every

i) NF_YCts is NF_YJCts.

ii) NF_YJCts is NF_YSCts.

iii) NF_YJCts is $NF_Y\beta - Cts$.

iv) $NF_Y\alpha - Cts$ is NF_YJCts

v) NF_YPCts is NF_YJCts.

vi) NF_YRCts is NF_YJCts.

Proof:

Received: Aug 09, 2025

International Journal of Applied Mathematics
Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

i) Let σ be a NF_YCts and T is a $NF_Y\ os$ in S_2. Then $\sigma^{-1}(T)$ is $NF_Y\ os$ in S_1. Since every $NF_Y\ os$ is $NF_Y\ Jos$, $\sigma^{-1}(T)$ is $NF_Y\ Jos$ in S_1. Therefore σ is NF_YJCts.
ii) Let σ be a NF_YJCts and T is a $NF_Y\ os$ in S_2. Then $\sigma^{-1}(T)$ is $NF_Y\ Jos$ in S_1. Since every $NF_Y\ Jos$ is $NF_Y\ Sos$, $\sigma^{-1}(T)$ is $NF_Y\ Sos$ in S_1. Therefore σ is NF_YSCts.
iii) Let σ be a NF_YJCts and T is a $NF_Y\ os$ in S_2. Then $\sigma^{-1}(T)$ is $NF_Y\ Jos$ in S_1. Since every $NF_Y\ Jos$ is $NF_Y\ \beta - os$, $\sigma^{-1}(T)$ is $NF_Y\ \beta - os$ in S_1. Therefore σ is $NF_Y\beta - Cts$.
iv) Let σ be a $NF_Y\alpha - Cts$ and T is a $NF_Y\ os$ in S_2. Then $\sigma^{-1}(T)$ is $NF_Y\alpha - os$ in S_1. Since every $NF_Y\ \alpha - os$ is $NF_Y\ Jos$, $\sigma^{-1}(T)$ is $NF_Y\ Jos$ in S_1. Therefore σ is NF_YJCts.
v) Let σ be a NF_YPCts and T is a $NF_Y\ os$ in S_2. Then $\sigma^{-1}(T)$ is $NF_Y\ Pos$ in S_1. Since every $NF_Y\ Pos$ is $NF_Y\ Jos$, $\sigma^{-1}(T)$ is $NF_Y\ Jos$ in S_1. Therefore σ is NF_YJCts.
vi) Let σ be a NF_YRCts and T is a $NF_Y\ os$ in S_2. Then $\sigma^{-1}(T)$ is $NF_Y\ Ros$ in S_1. Since every $NF_Y\ Ros$ is $NF_Y\ Jos$, $\sigma^{-1}(T)$ is $NF_Y\ Jos$ in S_1. Therefore σ is NF_YJCts.

Remark 3.5. The converse of above theorem need not be true as shown in the following Examples.

Example 3.6. Let $S_1 = \{t, u, v, w\}$ and $S_2 = \{p, q, r, s\}$ be the universe set and $S_1/R = \{\{t, v\}, u, w\}$ & $S_2/R = \{\{p, q\}, \{r, s\}\}$ be the equivalence relations.

Let $K_1 = \{t_{0.45}, u_{0.41}, v_{0.25}, w_{0.43}\}$ be a $F_y$subs of S_1. Then

$\underline{NF_z}(K_1) = \{t_{0.25}, u_{0.41}, v_{0.25}, w_{0.43}\}$, $\overline{NF_z}(K_1) = \{t_{0.45}, u_{0.41}, v_{0.45}, w_{0.43}\}$,

$NF_ZB(K_1) = \{t_{0.2}, u_0, v_{0.2}, w_0\}$.

Thus $\tau_R(K_1) = \{0_F, 1_F, \underline{NF_Z}(K_1), \overline{NF_Z}(K_1), NF_ZB(K_1)\}$ and

$NF_Y\ Jos = \{0_F, 1_F, \overline{NF_Z}(K_1), \overline{NF_Z}^{c}(K_1), \{t_{0.25-0.45}, u_{0.41}, v_{0.25-0.45}, w_{0.43}\},$
$\{t_{0.2-0.45}, u_{0-0.41}, v_{0.2-0.25}, w_{0-0.43}\}\}$.

Let $K_2 = \{p_{0.2}, q_{0.4}, r_{0.2}, s_{0.4}\}$ be a $F_y$subs of S_2. Then $\underline{NF_Z}(K_2) = \{p_{0.2}, q_{0.2}, r_{0.2}, s_{0.2}\}$, $\overline{NF_z}(K_2) = \{p_{0.4}, q_{0.4}, r_{0.4}, s_{0.4}\}$, $NF_ZB(K_2) = \{p_{0.2}, q_{0.2}, r_{0.2}, s_{0.2}\}$.

Thus $\tau_R(K_2) = \{0_F, 1_F, \underline{NF_z}(K_2), \overline{NF_Z}(K_2), NF_ZB(K_2)\}$.

Now we define the mapping $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ be an identity mapping. Thus the inverse image of every NF_Yos in S_2 is $NF_Y\ Jos$ in S_1. Therefore σ is NF_YJCts.

Here $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ is NF_YJCts but not NF_YCts and NF_YRCts. Since $T = \{p_{0.4}, q_{0.4}, r_{0.4,} s_{0.4}\}$ is NF_Yos in S_2 but $\sigma^{-1}(T)$ is not $NF_Y\ os$ and $NF_Y\ Ros$ in S_1.

Example 3.7. Let $S_1 = \{t, u, v, w\}$ and $S_2 = \{p, q, r, s\}$ be the universe set and $S_1/R = \{\{t, v\}, \{u, w\}\}$ & $S_2/R = \{\{p, r\}, \{q, s\}\}$ be the equivalence relations.

Let $K_1 = \{t_{0.41}, u_{0.32}, v_{0.63}, w_{0.54}\}$ be a $F_y$subs of S_1. Then

$\underline{NF_z}(K_1) = \{t_{0.41}, u_{0.32}, v_{0.41}, w_{0.32}\}$, $\overline{NF_z}(K_1) = \{t_{0.63}, u_{0.54}, v_{0.63}, w_{0.54}\}$,

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Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$NF_ZB(K_1) = \{t_{0.22}, u_{0.22}, v_{0.22}, w_{0.22}\}$.

Thus $\tau_R(K_1) = \{0_F, 1_F, \underline{NF_Z}(K_1), \overline{NF_Z}(K_1), NF_ZB(K_1)\}$ and

$NF_Y\,Jos = \{0_F, 1_F, \underline{NF_Z}(K_1), \overline{NF_Z}(K_1), NF_ZB(K_1), \underline{NF_Z}^c(K_1), \overline{NF_Z}^c(K_1), NF_ZB^c(K_1)\}$.

Let $K_2 = \{p_{0.37}, q_{0.46}, r_{0.59}, s_{0.68}\}$ be a $F_y$subs of S_2. Then $\underline{NF_Z}(K_2) = \{p_{0.37}, q_{0.46}, r_{0.37}, s_{0.46}\}$, $\overline{NF_Z}(K_2) = \{p_{0.59}, q_{0.68}, r_{0.59}, s_{0.68}\}$, $NF_ZB(K_2) = \{p_{0.22}, q_{0.22}, r_{0.22}, s_{0.22}\}$.

Thus $\tau_R(K_2) = \{0_F, 1_F, \underline{NF_Z}(K_2), \overline{NF_Z}(K_2), NF_ZB(K_2)\}$.

Now we define the mapping $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ be an identity mapping. Thus the inverse image of every $NF_Y\,os$ in S_2 is $NF_Y\,Jos$ in S_1. Therefore σ is NF_YJCts.

Here $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ is NF_YJCts but not NF_YPCts and $NF_Y\alpha - Cts$. Since $T = \{p_{0.59}, q_{0.68}, r_{0.59}, s_{0.68}\}$ is $NF_Y\,os$ in S_2 but $\sigma^{-1}(T)$ is not $NF_Y\,Pos$ and $NF_Y\,\alpha - os$ in S_1.

Example 3.8. Let $S_1 = \{t, u, v\}$ and $S_2 = \{q, r, s\}$ be the universe set and $S_1/R = \{\{t, u\}, v\}$ & $S_2/R = \{\{q, r\}, s\}$ be the equivalence relations.

Let $K_1 = \{t_{0.61}, u_{0.36}, v_{0.7}\}$ be a $F_y$subs of S_1. Then

$\underline{NF_Z}(K_1) = \{t_{0.36}, u_{0.36}, v_{0.7}\}$, $\overline{NF_Z}(K_1) = \{t_{0.61}, u_{0.61}, v_{0.7}\}$, $NF_ZB(K_1) = \{t_{0.25}, u_{0.25}, v_0\}$.

Thus $\tau_R(K_1) = \{0_F, 1_F, \underline{NF_Z}(K_1), \overline{NF_Z}(K_1), NF_ZB(K_1)\}$ and

$NF_Y\,Jos =$
$\{0_F, 1_F, \underline{NF_Z}(K_1), \overline{NF_Z}(K_1), NF_ZB(K_1), \overline{NF_Z}^c(K_1), NF_ZB^c(K_1), \{t_{0.36-0.61}, u_{0.36-0.61}, v_{0.7}\}\}$.

$NF_YSos = NF_Y\beta - os =$
$\{0_F, 1_F, \overline{NF_Z}^c(K_1), NF_ZB^c(K_1), \{t_{0.36-0.75}, u_{0.36-0.75}, v_{0.7-1}\}, \{t_{0.25-0.39}, u_{0.25-0.39}, v_{0-0.3}\}\}$.

Let $K_2 = \{q_{0.4}, r_{0.7}, s_{0.8}\}$ be a $F_y$subs of S_2. Then $\underline{NF_Z}(K_2) = \{q_{0.4}, r_{0.4}, s_{0.8}\}$, $\overline{NF_Z}(K_2) = \{q_{0.7}, r_{0.7}, s_{0.8}\}$, $NF_ZB(K_2) = \{q_{0.3}, r_{0.3}, s_0\}$.

Thus $\tau_R(K_2) = \{0_F, 1_F, \underline{NF_Z}(K_2), \overline{NF_Z}(K_2), NF_ZB(K_2)\}$.

Now we define the mapping $\sigma: (S_1, \tau_R(K_1)) \to (S_2, \tau_R(K_2))$ be an identity mapping. Thus the inverse image of every $NF_Y\,os$ in S_2 $NF_Y\,Sos$ and $NF_Y\,\beta - os$ in S_1. Therefore σ is NF_YSCts and $NF_Y\beta - Cts$ but not NF_YJCts. Since $T = \{q_{0.7}, r_{0.7}, s_{0.8}\}$ is $NF_Y\,os$ in S_2 but $\sigma^{-1}(T)$ is not $NF_Y\,Jos$ in S_1.

4. CHARACTERIZATIONS AND PROPERTIES OF NANO FUZZY J CONTINUOUS

In this section, we present the characterizations and fundamental properties of Nano fuzzy J continuous functions, derive key results based on these characterizations, and discuss their theoretical significance within the framework of nano fuzzy topology.

Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Theorem 4.1. A mapping $\sigma:(S_1,\tau_R(K_1))\to(S_2,\tau_R(K_2))$ is NF_YJCts iff the inverse image of every $NF_Y\ cs$ in S_2 is $NF_Y\ Jcs$ in S_1.

Proof. Let σ be NF_YJCts and J_0 be $NF_Y\ cs$ in S_2. Then S_2-J_0 is $NF_Y\ os$ in S_2. Since σ is NF_YJCts, $\sigma^{-1}(S_2-J_0)$ is $NF_Y\ Jos$ in S_1. i.e. $S_1-\sigma^{-1}(J_0)$ is $NF_Y\ Jos$ in S_1. Therfore $\sigma^{-1}(J_0)$ is $NF_Y\ Jcs$ in S_1.

Conversely, let the inverse image of every $NF_Y\ cs$ be $NF_Y\ Jcs$. Let β be $NF_Y\ os$ in S_2. Then $S_2-\beta$ be $NF_Y\ cs$ in S_2. That implies $\sigma^{-1}(S_2-\beta)$ be $NF_Y\ Jcs$ in S_1. i.e. $S_1-\sigma^{-1}(\beta)$ is $NF_Y\ Jcs$ in S_1. Therefore $\sigma^{-1}(\beta)$ is $NF_Y\ Jos$ in S_1. Thus the inverse image of every $NF_Y\ os$ in S_2 is $NF_Y\ Jos$ in S_1. Hence σ is NF_YJCts on S_1.

Theorem 4.2. A mapping $\sigma:(S_1,\tau_R(K_1))\to(S_2,\tau_R(K_2))$ is NF_YJCts iff $\sigma(NF_YJCl(J_0))\le NF_YCl(\sigma(J_0))$ for every $F_Y\ subs\ J_0$ of S_1.

Proof. Let σ be NF_YJCts and J_0 be $NF_Y\ subs$ in S_1. Then $\sigma(J_0)\le S_2$. Since σ be NF_YJCts and $NF_YCl(\sigma(J_0))$ is $NF_Y\ cs$ in S_2, $\sigma^{-1}\left(NF_YCl(\sigma(J_0))\right)$ is $NF_Y\ Jcs$ in S_1. Since $\sigma(J_0)\le NF_YCl(\sigma(J_0))$, $\sigma^{-1}(\sigma(J_0))\le\sigma^{-1}\left(NF_YCl(\sigma(J_0))\right)$, then $J_0\le\sigma^{-1}\left(NF_YCl(\sigma(J_0))\right)\Rightarrow NF_YJCl(J_0)\le NF_YJCl\left(\sigma^{-1}\left(NF_YCl(\sigma(J_0))\right)\right)=\sigma^{-1}\left(NF_YCl(\sigma(J_0))\right)$. Thus $NF_YJCl(J_0)\le\sigma^{-1}\left(NF_YCl(\sigma(J_0))\right)$. Therefore $\sigma(NF_YJCl(J_0))\le NF_YCl(\sigma(J_0))$ for every $F_Y\ subs\ J_0$ of S_1.

Conversely, let $\sigma(NF_YJCl(J_0))\le NF_YCl(\sigma(J_0))$ for every $F_Y\ subs\ J_0$ of S_1. If M_0 is $NF_Y\ cs$ in S_2 and since $\sigma^{-1}(M_0)\le S_1$, $\sigma\left(NF_YJCl(\sigma^{-1}(M_0))\right)\le NF_Ycl\left(\sigma(\sigma^{-1}(M_0))\right)=NF_Ycl(M_0)=M_0$. That is $\sigma\left(NF_YJCl(\sigma^{-1}(M_0))\right)\le M_0$. Thus $NF_YJCl(\sigma^{-1}(M_0))\le\sigma^{-1}(M_0)$. But $\sigma^{-1}(M_0)\le NF_YJCl(\sigma^{-1}(M_0))$. Therefore $\sigma^{-1}(M_0)$ is $NF_Y\ Jcs$ in S_1, for every $NF_Y\ cs\ M_0$ in S_2. Thus σ is NF_YJCts.

Remark 4.3. A mapping $\sigma:(S_1,\tau_R(K_1))\to(S_2,\tau_R(K_2))$ is NF_YJCts then $\sigma(NF_YJCl(J))$ is not necessarily equal to $NF_YCl(\sigma(J))$ where $J\in S_1$.

Example 4.4. In example 3.1.6 , $\sigma:(S_1,\tau_R(K_1))\to(S_2,\tau_R(K_2))$ is NF_YJCts. Let $J=\{t_{0.45},u_{0.41},v_{0.45}\ ,w_{0.43}\}\in S_1$. Then $\sigma(NF_YJCl(J))=\sigma(J)=J$. But $NF_YCl(\sigma(J))=NF_YCl(J)=\{t_{0.55},u_{0.59},v_{0.55}\ ,w_{0.57}\}$. Thus $\sigma(NF_YJCl(J))\ne NF_YCl(\sigma(J))$.

Theorem 4.5. A mapping $\sigma:(S_1,\tau_R(K_1))\to(S_2,\tau_R(K_2))$ is NF_YJCts iff $NF_YJCl(\sigma^{-1}(J_0))\le\sigma^{-1}(NF_YCl(J_0))$ for every $F_Y\ subs\ J_0$ of S_2.

Proof. Let σ be NF_YJCts and J_0 be $NF_Y\ subs$ in S_2. $NF_Ycl(J_0)$ is $NF_Y\ cs$ in S_2. Thus, $\sigma^{-1}(NF_Ycl(J_0))$ is $NF_Y\ Jcs$ in S_1. Therefore $NF_YJcl\left(\sigma^{-1}(NF_Ycl(J_0))\right)=\sigma^{-1}(NF_Ycl(J_0))$. Since $J_0\le NF_Ycl(J_0)$, $\sigma^{-1}(J_0)\le\sigma^{-1}(NF_Ycl(J_0))$. Therefore $NF_YJCl(\sigma^{-1}(J_0))\le$

Received: Aug 09, 2025

Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$NF_YJcl\left(\sigma^{-1}\left(NF_Ycl(J_0)\right)\right) = \sigma^{-1}\left(NF_Ycl(J_0)\right)$. That is $NF_YJCl\left(\sigma^{-1}(J_0)\right) \leq \sigma^{-1}\left(NF_YCl(J_0)\right)$.

Conversely, let $NF_YJCl\left(\sigma^{-1}(J_0)\right) \leq \sigma^{-1}\left(NF_YCl(J_0)\right)$ for every $F_Y\ subs\ J_0$ of S_2. If J_0 is $NF_Y\ cs$ in S_2, then $NF_YCl(J_0) = J_0$. By assumption, $NF_YJCl\left(\sigma^{-1}(J_0)\right) \leq \sigma^{-1}\left(NF_YCl(J_0)\right) = \sigma^{-1}(J_0)$. Thus $NF_YJCl\left(\sigma^{-1}(J_0)\right) \leq \sigma^{-1}(J_0)$. But $\sigma^{-1}(J_0) \leq NF_YJCl\left(\sigma^{-1}(J_0)\right)$. Therfore $NF_YJCl\left(\sigma^{-1}(J_0)\right) = \sigma^{-1}(J_0)$. Hence $\sigma^{-1}(J_0)$ is a $NF_Y\ Jcs$ in S_1, for every $NF_Y\ cs\ J_0$ in S_2. Therfore σ is NF_YJCts on S_1.

Remark 4.6. A mapping $\sigma: \left(S_1, \tau_R(K_1)\right) \to \left(S_2, \tau_R(K_2)\right)$ is NF_YJCts then $NF_YJCl\left(\sigma^{-1}(J)\right)$ is not necessarily equal to $\sigma^{-1}\left(NF_YCl(J)\right)$ where $J \in S_2$.

Example 4.7. In example 3.6., $\sigma: \left(S_1, \tau_R(K_1)\right) \to \left(S_2, \tau_R(K_2)\right)$ is NF_YJCts. Let $J = \{t_{0.45}, u_{0.41}, v_{0.45}, w_{0.43}\} \in S_1$. Then $NF_YJCl\left(\sigma^{-1}(J)\right) = NF_YJCl(J) = J$. But $\sigma^{-1}\left(NF_YCl(J)\right) = \sigma^{-1}(\{t_{0.55}, u_{0.59}, v_{0.55}, w_{0.57}\}) = \{t_{0.55}, u_{0.59}, v_{0.55}, w_{0.57}\}$. Thus $NF_YJCl\left(\sigma^{-1}(J)\right) \neq \sigma^{-1}\left(NF_YCl(J)\right)$.

Theorem 4.8. A mapping $\sigma: \left(S_1, \tau_R(K_1)\right) \to \left(S_2, \tau_R(K_2)\right)$ is NF_YJCts iff $\sigma^{-1}\left(NF_Yint(J_0)\right) \leq NF_YJint\left(\sigma^{-1}(J_0)\right)$ for every $F_Y\ subs\ J_0$ of S_2.

Proof. Let σ be NF_YJCts and J_0 be $NF_Y\ subs$ in S_2. $NF_Yint(J_0)$ is $NF_Y\ os$ in S_2 and hence $\sigma^{-1}\left(NF_Yint(J_0)\right)$ is $NF_Y\ Jos$ in S_1. Therefore $NF_YJint\left(\sigma^{-1}\left(NF_Yint(J_0)\right)\right) = \sigma^{-1}\left(NF_Yint(J_0)\right)$. So, $NF_Yint(J_0) \leq J_0 \Longrightarrow \sigma^{-1}\left(NF_Yint(J_0)\right) \leq \sigma^{-1}(J_0)$. Then, $NF_YJint\left(\sigma^{-1}\left(NF_Yint(J_0)\right)\right) \leq NF_YJint\left(\sigma^{-1}(J_0)\right)$. That is $\sigma^{-1}\left(NF_Yint(J_0)\right) \leq NF_YJint\left(\sigma^{-1}(J_0)\right)$.

Conversely, let $\sigma^{-1}\left(NF_Yint(J_0)\right) \leq NF_YJint\left(\sigma^{-1}(J_0)\right)$ for every $F_Y\ subs\ J_0$ of S_2. If J_0 is $NF_Y\ os$ in S_2, then $NF_Yint(J_0) = J_0$. Based on our assumption, $\sigma^{-1}\left(NF_Yint(J_0)\right) \leq NF_YJint\left(\sigma^{-1}(J_0)\right)$. Thus $\sigma^{-1}(J_0) \leq NF_YJint\left(\sigma^{-1}(J_0)\right)$. But $NF_YJint\left(\sigma^{-1}(J_0)\right) \leq \sigma^{-1}(J_0)$. Therefore $NF_YJint\left(\sigma^{-1}(J_0)\right) = \sigma^{-1}(J_0)$. That is $\sigma^{-1}(J_0)$ is $NF_Y\ Jos$ in S_1, for every $NF_Y\ os$ J_0 in S_2. Therfore σ is NF_YJCts on S_1.

Remark 4.9. A mapping $\sigma: \left(S_1, \tau_R(K_1)\right) \to \left(S_2, \tau_R(K_2)\right)$ is NF_YJCts then $\sigma^{-1}\left(NF_Yint(J_0)\right)$ is not necessarily equal to $NF_YJint\left(\sigma^{-1}(J_0)\right)$ where $J \in S_2$.

Example 4.10. In example 3.6., $\sigma: \left(S_1, \tau_R(K_1)\right) \to \left(S_2, \tau_R(K_2)\right)$ is NF_YJCts. Let $J = \{p_{0.6}, q_{0.6}, r_{0.6}, s_{0.6}\} \in S_2$. Then $\sigma^{-1}\left(NF_Yint(J_0)\right) = \sigma^{-1}(\{p_{0.4}, q_{0.4}, r_{0.4}, s_{0.4}\}) = \{p_{0.4}, q_{0.4}, r_{0.4}, s_{0.4}\}$. But $NF_YJint\left(\sigma^{-1}(J_0)\right) = NF_YJint(J_0) = J_0$. Thus $\sigma^{-1}\left(NF_Yint(J_0)\right) \neq NF_YJint\left(\sigma^{-1}(J_0)\right)$.

Received: Aug 09, 2025

Definition 4.11. A $F_y\, subs\ \ J$ in a $NF_YTs\ (S, \tau_R(K))$ is called a **Nano fuzzy $\boldsymbol{J}$ q-neighbourhood** (simply, $NF_YJq - nbhd$) of a fuzzy point l_r if there exists a $NF_Y\, Jos\, T$ with $l_r \sim T \leq J$.

Preposition 4.12. Let $(S_1, \tau_R(K_1)), (S_2, \tau_R(K_2))$ be a NF_YTs's and $\sigma: (S_1, \tau_R(K_1)) \rightarrow (S_2, \tau_R(K_2))$ be a mapping. Then the following assertions are equivalent.

(i) σ is NF_YJCts.

(ii) For each fuzzy point $l_r \in S_1$ and every $NF_YJq - nbhd\ J_1$ of $\sigma(l_r)$, there exists a $NF_Y\, Jos\, J_2$ in $S_1 \ni l_r \in J_2 \leq \sigma^{-1}(J_1)$.

(iii) For each fuzzy point $l_r \in S_1$ and every $NF_YJq - nbhd\ J_1$ of $\sigma(l_r)$, there exists a $NF_Y\, Jos\, J_2$ in $S_1 \ni l_r \in \sigma(J_2) \leq J_1$.

Proof.

(i)⇒(ii) let l_r be a fuzzy point in S_1 and let J_1 be a $NF_YJq - nbhd$ of $\sigma(l_r)$. Then there exists a $NF_Y\, Jos\, J_2$ in $S_2 \ni \sigma(l_r) \in J_2 \leq J_1$. Since σ is NF_YJCts, we know that $\sigma^{-1}(J_2)$ is a $NF_Y\, Jos$ in S_1 and $l_r \in \sigma^{-1}(\sigma(l_r)) \leq \sigma^{-1}(J_2) \leq \sigma^{-1}(J_1)$. Consequently (ii) is valid.

(ii)⇒(iii) let l_r be a fuzzy point in S_1 and let J_1 be a $NF_YJq - nbhd$ of $\sigma(l_r)$. The condition (ii) implies that there exists a $NF_Y\, Jos\, J_2$ in $S_1 \ni l_r \in J_2 \leq \sigma^{-1}(J_1)$ so that $l_r \in J_2$ and $\sigma(J_2) \leq \sigma(\sigma^{-1}(J_1)) \leq J_1$. Hence (iii) is true.

(iii)⇒(i) let J_2 be a $NF_Y\, os$ in S_2 and let $l_r \in \sigma^{-1}(J_2)$. Since J_2 is a $NF_Y\, os$, $\sigma(l_r) \in J_2$ and so J_2 is a $NF_YJq - nbhd$ of $\sigma(l_r)$. It follows from (iii) that there exists a $NF_Y\, Jos\, J_1$ in $S_1 \ni l_r \in J_1$ and $\sigma(J_1) \leq J_2$ so that $l_r \in J_1 \leq \sigma^{-1}(\sigma(J_1)) \leq \sigma^{-1}(J_2)$. Applying definition 20 induces that $\sigma^{-1}(J_2)$ is $NF_Y\, Jos$ in S_1. Therefore, σ is a NF_YJCts mapping.

Theorem 4.13. Let $\sigma_1: (S_1, \tau_R(K_1)) \rightarrow (S_2, \tau_R(K_2)), \sigma_2: (S_2, \tau_R(K_2)) \rightarrow (S_3, \tau_R(K_3))$ be any two mappings. If σ_1 is a NF_YJCts and σ_2 is a NF_YCts mappings, then $\sigma_2 \circ \sigma_1$ is NF_YJCts.

Proof let J be any $NF_Y\, cs$ in S_3. As σ_2 is a NF_YCts, $\sigma_2^{-1}(J)$ is $NF_Y\, cs$ in S_2. Since σ_1 is NF_YJCts, implies $\sigma_1^{-1}(\sigma_2^{-1}(J)) = (\sigma_2 \circ \sigma_1)^{-1}(J)$ is a $NF_Y\, Jcs$ in S_1. Therefore $\sigma_2 \circ \sigma_1$ is NF_YJCts.

Remark 4.14. The composition of NF_YJCts need not be NF_YJCts as seen from the below example.

Example 4.15. In example 3.6., $\sigma: (S_1, \tau_R(K_1)) \rightarrow (S_2, \tau_R(K_2))$ is NF_YJCts.

Now let $S_3 = \{l, m, n, x\}$ be the universe set and $S_3/R = \{\{l, n\}, \{m, x\}\}$ be the equivalence relations.

Let $K_3 = \{l_{0.6}, m_{0.4}, n_{0.4}, x_{0.6}\}$ be a $F_y$subs of S_3. Then $\tau_R(K_3) = \{0_F, 1_F, \{l_{0.4}, m_{0.4}, n_{0.4}, x_{0.4}\}, \{l_{0.6}, m_{0.6}, n_{0.6}, x_{0.6}\}, \{l_{0.2}, m_{0.2}, n_{0.2}, x_{0.2}\}\}$.

Now we define the mapping $\sigma': (S_2, \tau_R(K_2)) \rightarrow (S_3, \tau_R(K_3))$ be an identity mapping. Thus the inverse image of every $NF_Y\, os$ in S_2 is $NF_Y\, Jos$ in S_1. Therefore σ' is NF_YJCts.

Volume 38 No. 10s, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Here σ and σ' are $NF_Y JCts$ but $\sigma' \circ \sigma$ is not $NF_Y JCts$. since, $J = \{l_{0.6}, m_{0.6}, n_{0.6}, x_{0.6}\}$ is $NF_Y\ os$ in S_3 but $(\sigma' \circ \sigma)^{-1}(J)$ is not $NF_Y\ Jos$ in S_1.

5. CONCLUSION

The notion of Nano fuzzy J continuous functions was presented in this research along with an analysis of their fundamental characteristics. Illustrative examples were used to establish the links between Nano fuzzy J continuity and other types of continuity. The findings support the theoretical advancement of Nano fuzzy topology and serve as a foundation for additional expansions and applications in decision-making and generalised topological contexts.

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Received: Aug 09, 2025