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DYNAMICAL BEHAVIOR OF SEMI-ANALYTICAL SOLUTIONS IN THE TIME-FRACTIONAL BENJAMIN–ONO EQUATION AND STABILITY ANALYSIS

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ABSTRACT

This research presents the study of the time-fractional Benjamin-Ono equation (BOEs) with the support of Caputo-Fabrizio fractional derivative(CFFD). In the present work, the fractional homotopy perturbation transform (FHPTM) method has been taken to obtain some new semi-analytical solution of the fractional BOE. The stability study of the model with Banach's fixed-point theory is used in an iterative manner. After that, we provide some numerical examples to demonstrate the theoretical results that were attained. Lastly, a graphical representation of the results is given.

Keywords: FHPTM; The fractional Benjamin-Ono equation; Fixed point; Caputo Fabrizio derivative(CFD); Stability Analysis.

AMSC: 35G50, 35Q35, 35B06, 35C08

1 Introduction

Fractional non-linear partial differential equations (FNPDEs) are a class of equations that generalize classical partial differential equations (PDEs) by incorporating fractional derivatives. These equations are used to model complex phenomena in fields such as Finance,

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Biology, Physics and Engineering, where memory effects and long-range interactions are significant[1−2]. Benjamin-Ono equation is a non-linear PDE without dissipation and arises as a model equation for certain waves in stratified fluid; that is, a fluid in which the density can depend only on the vertical coordinate. In addition, there is a well-defined interface between fluid layers of different densities. BOEs exhibit some interesting analytical features when subjected to the so-called time-fractional derivative[3−5]. By incorporating a time-fractional order into the Benjamin-Ono equation, we are essentially broadening its application to fluids that have yet to fully stratify and establish an internal wave pattern. This is a novel and fascinating new direction for the analysis of such wave systems and has many potential applications in fields such as coastal Engineering and renewable marine energy[6−7].

1.1 Historical developments of Benjamin-Ono Equation

The BOEs are special type of equations known as a partial differential equation. Partial differential equations are used to find unknown functions of several variables, for example $S(x, y, z), P(x, y, z)$ *or* $Q(t, x, y, z)$ given the relationship between the function and its partial derivatives. The BOEs are of particular interest, as it is one of the few equations which can be solved exactly under certain conditions. This is important for mathematicians, as it allows them to make definitive progress, that is, to make theorems that apply for all time by starting at one time. Solutions of the form given in the graph at the beginning of the article are particularly simple and are called solitons. These are a special type of wave. What makes solitons so interesting is that they retain their shape and speed after they have collided with each other. Solitons have been widely studied in applied mathematics and have been found to have use in physics, for example, to model the behavior of particles, Biology, for example, to model the movement of nerve pulses, and also in oceanography where solitons can form large waves in sea currents. In the 1980s, two Japanese mathematicians K. Mizuhara and M. Wadati made a major breakthrough - they discovered that the Benjamin-Ono equation arises in a physical context when one models large-amplitude internal waves in a two-layer fluid[8−10]. This has increased the interest of the physical community in the Benjamin-Ono equation. Now, the equation is studied by both mathematicians and physicists. However, many questions remain about the equation. For example, we know that in 2 space dimensions it is possible to produce chaotic solutions, that is, solutions which seem to have no pattern and jump around erratically. But no one knows whether this can happen in three or more dimensions. This is the kind of open problem that motivates the work of mathematicians. These straightforward methods are especially helpful for solving nonlinear equations, which are frequently challenging to resolve using conventional methods because of their complexity and variety of forms[11−15].

1.2 Understanding of the Time-Fractional Benjamin-Ono Equation

BOE is a nonlinear partial differential equation (PDE) that designs one-dimensional wave dynamics in certain physical contexts. It was introduced in the context of fluid mechanics to

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elaborate the propagation of long internal waves in deep stratified fluids. It is necessary to be familiar with both the classical BOEs and the ideas of fractional calculus to comprehend the time-fractional BOEs. Here is a summary of the essential elements.

Classical Benjamin-Ono Equation: The evolution of one-dimensional waves in a medium with weak dispersion and weak non-linearity is described by the BOEs. It is given by[16]

$$\Phi_\nu + \Phi\Phi_\mu + H\Phi_{\mu\mu} = 0. \tag{1.1}$$

were,$\Phi(\mu,\nu)$ denotes wave amplitude, Φ_ν and Φ_μ partial derivatives with respect to space μ and time ν, respectively, H is the Hilbert transform, and $\Phi.\Phi_\mu$ represents the non-linearity term.

Fractional Derivatives: The concept of differentiation and integration is expanded to non- integer orders via fractional calculus. Notion of differentiation and integration to non-integer orders extends the Fractional calculus. In the context of the time-fractional BOE, the fractional derivative in time is typically expressed using the Caputo Fabrizio derivative, denoted as D_ν^α where, α the fractional order[17].

Time-Fractional Benjamin-Ono Equation: The time-fractional BOEs can be obtained by combining fractional calculus with the traditional BOEs[18−19].

$${}_{0}^{CF}D_\nu^\alpha + \Phi\Phi_\mu + H\Phi_{\mu\mu} = 0, \quad 0 < \alpha \leq 1. \tag{1.2}$$

Here, ${}_{0}^{CF}D_\nu^\alpha\Phi$ represents the fractional derivative of Caputo Fabrizio in time.
Analyzing the impact of fractional derivatives on the dynamics of wave propagation is necessary to comprehend the time-fractional Benjamin-Ono equation. Fractional derivatives give rise to memory effects in the system, whereby the wave's evolution at a given location is contingent upon both its past and present states. Studying the characteristics of fractional derivatives, examining stability and bifurcation behavior, determining existence and uniqueness of solutions, also creating numerical techniques for solving the problem which are all parts of the analysis process. For more details regarding the FHPTM authors can refer to [20−21], conformable derivative[22], the sine-Gordon expansion method[23], rational sine–cosine and rational sinh–cosh methods[24], few traveling wave solutions[25], one may refer[26−28].
The Time-Fractional Benjamin–Ono Equation is used in many other areas.

- **Surface Water Waves:** The analysis of long waves on the superficies of water bodies can be immediately applied to the BOEs. Understanding the propagation, interaction, and evolution of these waves is aided by it, especially in situations involving shallow water where linear wave theories might not be appropriate.

- **Wave Breaking:** In fluid dynamics, wave breaking is a key mechanism that affects things such as mixing, sediment movement, and energy dissipation. The subsequent evolution of the wave field and the circumstances under which waves break can both be studied with the aid of the BOEs. Designing offshore structures, environmental impact assessments, and coastal management all depend on an understanding of wave breaking.

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- **Wave-Structure Interaction:** Wave-structure interactions are complicated when structures like breakwaters, offshore platforms, and coastal defenses interact with incident waves. The BOEs can be used to study wave-structure interactions and improve the performance and design of marine constructions when combined with suitable boundary conditions and structural models.

This article devided in some parts. In section 2 we present for a basic idea of CFD and Laplace transforms (LT), together FC . Section 3, depicts the techniques for the solution of nonlinear PDEs with the support of FHPM method. The stability analysis of result in Section 4, to the FBEs. Section 5, based on approximate solution of the mentioned nonlinear BOEs. In last Section 6, include the conclusions and remarks.

2 Preliminaries

In the present section, the notions of fractional order Caputo-Fabrizio derivatives and integrals are discussed.

Definition 2.1 [26]The fractional integration of α order is defined as

$$ {}_{a}D_{\nu}^{\alpha}\,\Phi(\nu) = \frac{1}{\Gamma(\alpha)}\int_{a}^{\nu}(\nu - W)^{\alpha-1}\Phi(W)dW. \tag{2.1}$$

Definition 2.2 [26]For each α, the Riemnn-Liouville derivative of α order is defined as fractional derivative of order α

$$ {}_{a}D_{\nu}^{\alpha}\,\Phi(\nu) = \frac{d}{dt}({}_{a}D_{\nu}^{\alpha}\,\Phi)(\nu) = \frac{\frac{d}{d\nu}}{\Gamma(1-\alpha)}\int_{a}^{\nu}(\nu - W)^{-\alpha}\Phi(W)dW. \tag{2.2}$$

Definition 2.3 [26] The Caputo fractional derivative of order $\alpha \geq 0$ of a functions $\Phi(\nu)$: $\alpha \in (0,1) \rightarrow R$

$$ {}_{a}^{C}D_{\nu}^{\alpha}\,\Phi(\nu) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{\nu}(\nu - W)\frac{d^{n}}{dW^{n}}\Phi(W)dW. \tag{2.3}$$

where ${}_{a}^{C}D_{\nu}^{\alpha}\,\Phi(\nu)$ is a Caputo derivative.
Definition 2.4 [26] for $\Phi \in H^{1}(a,b)$ with the condition that $b > a$ then the Caputo-Fabrizio fractional derivative.

$$ {}_{0}^{CF}D_{\nu}^{\alpha}\,\Phi(\nu) = \frac{\lambda(\alpha)}{1-\alpha}\int_{0}^{\nu} exp\left[-\frac{\alpha(1-W)}{1-\alpha}\right]\Phi'(W)dW. \tag{2.4}$$

where, $\lambda(\alpha)$ is the normalization function with condition $\lambda(0) = \lambda(1) = 1$.
Definition 2.5 [26] Let $\Phi \in H^{1}(a_1,b_1), a_1 < b_1$, and order $0 < \alpha < 1$. the fractional integral of a function Φ is defined as

$$ {}_{0}^{CF}D_{\nu}^{\alpha}\,\Phi(\nu) = \frac{2(1-\alpha)}{\lambda(\alpha)(2-\alpha)}\Phi(\nu) + \frac{2\alpha}{\lambda(\alpha)(2-\alpha)}\int_{0}^{\nu}\Phi(W)dW,\ \nu \geq 0. \tag{2.5}$$

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where $\lambda(\alpha)$ is an order normalization function.
Definition 2.6 [26] Laplace transform of CFFD of is gives by

$$\begin{aligned} L\left[{}_0^{CF}\ D_\nu^{(m+\alpha)}\ \Phi(\nu)\right](p) =& \frac{1}{1-\alpha} L[\Phi^{m+1}(\nu)] L\left[exp\left(\frac{-\alpha}{(1-\alpha)}\nu\right)\right] \\ =& \frac{p^{m+1}L[\Phi(\nu)] - p^m\Phi(0) - p^{m-1}\Phi'(0)\ldots - \Phi^m(0)}{p+\alpha(1-p)}. \end{aligned} \tag{2.6}$$

when $m = 0, 1, 2, 3...,$

$$L\left[{}_0^{CF}\ D_\nu^{(m+\alpha)}\ \Phi(\nu)\right](p) = \frac{pL(\Phi(\nu))}{p+\alpha(1-p)}, \quad m = 0,$$

$$L\left[{}_0^{CF}\ D_\nu^{(m+\alpha)}\ \Phi(\nu)\right](p) = \frac{p^2L(\Phi(\nu)) - p\Phi(o) - \Phi'(0)}{p+\alpha(1-p)}, \quad m = 1.$$

3 Well-Developed Technique in FHPTM

In this section, in sequence to achieve semi-analytical solution that converges to the analytic solution of first order fractional partial differential equation, we first introduce the FHPT approach and build an algorithm.

$${}_0^{CF}\ D_\nu^\alpha \Phi(\mu,\nu) + \beta\Phi(\mu,\nu) + \varphi\Phi(\mu,\nu) = k(\mu,\nu), \tag{3.1}$$

with the initial conditions

$$\frac{\partial^l \Phi(\mu,0)}{\partial \nu^l} = f_l(\mu), \quad l = 0, 1, 2, \ldots n-1. \tag{3.2}$$

where, ${}_0^{CF}\ D_\nu^\alpha \Phi(\mu,\nu)$ is indicating Caputo Fabrizio fractional derivative β is representing a linear differential operator, φ is a nonlinear differential operator and $k(\mu,\nu)$ is standing for the source term.

Apply Laplace transform on both side of Eqs. (3.1),

$$L[\Phi(\mu,\nu)] = \Theta(\mu,p) - \left(\frac{p+\alpha(1-p)}{p^{n+1}}\right) L[\beta\Phi(\mu,\nu) + \varphi\Phi(\mu,\nu)]. \tag{3.3}$$

Where

$$\Theta(\mu,p) = \frac{1}{p^{m+1}}[p^m f_0(x) + p^{m-1}h_r(\mu) + \ldots + f_m(x)] + \frac{p+\alpha(1-p)}{p^{n+1}}\tilde{k}(\mu,p). \tag{3.4}$$

Taking inverse Laplace transform on Eq. (3.3),

$$\Phi(\mu,\nu) = \Theta(\mu,p) - L^{-1}\left[\left(\frac{p+\alpha(1-p)}{p^{n+1}}\right) L[\beta\Phi(\mu,\nu) + \varphi\Phi(\mu,\nu)]\right]. \tag{3.5}$$

where, $\Theta(\mu,p)$ originate from the starting point and the source phrase

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Now, applying HPTM

$$\Phi(\mu,\nu)=\sum_{n=0}^{\infty}p^n\Phi_n(\mu,\nu). \tag{3.6}$$

the nonlinear term is

$$\varphi\Phi(\mu,\nu)=\sum_{n=0}^{\infty}p^nH_n(\mu,\nu). \tag{3.7}$$

He's polynomials are a mathematical tool introduced by Dr. Ji-Huan He[11], often used in conjunction with the Homotopy Perturbation Method $H_n(\mu,\nu)$ which are.

$$H_m(\Phi_0,\Phi_1,\Phi_2,\ldots\Phi_n)=\frac{1}{n!}\frac{\partial^m}{\partial r^m}\left[\left(\sum_{m=0}^{\infty}r^i\Phi_i\right)\right]_{r=0},\qquad m=0,1,2,\ldots; \tag{3.8}$$

From Eqs.(3.5), (3.6) and Eq.(3.7), we obtain

$$\sum_{m=0}^{\infty}\Phi_m(\mu,\nu)=\Theta(\mu,p)-rL^{-1}\left[\left(\frac{p+\alpha(1-p)}{p^{m+1}}\right)L\left[\beta\sum_{m=0}^{\infty}r^m\Phi_m(\mu,\nu)+\sum_{m=0}^{\infty}r^mH_m\right]\right]. \tag{3.9}$$

This is a combination of the HPTM using He polynomials and the Laplace transform. Comparing comparable powers of p coefficients

$$\begin{aligned}
r^0:\Phi_0(\mu,\nu) &= \Theta(\mu,p),\\
r^1:\Phi_1(\mu,\nu) &= -L^{-1}\left[\left(\frac{p+\alpha(1-p)}{p^{m+1}}\right)L[H_0(\Phi)+\beta\Phi_0(\mu,\nu)]\right],\\
r^2:\Phi_2(\mu,\nu) &= -L^{-1}\left[\left(\frac{p+\alpha(1-p)}{p^{m+1}}\right)L[\beta\Phi_1(\mu,\nu)+H_1(\Phi)]\right],\\
&\vdots\\
p^m:\Phi_m(\mu,\nu) &= -L^{-1}\left[\left(\frac{p+\alpha(1-p)}{p^{m+1}}\right)L[\beta\Phi_{m-1}(\mu,\nu)+H_{m-1}(\Phi)]\right].
\end{aligned}$$

Lastly, FHPTM series solution is given as

$$\Phi(\mu,\nu)=\sum_{m=0}^{\infty}\Phi_m(\mu,\nu). \tag{3.10}$$

4 Stability Analysis

Present section devoted to the stability analysis of time-fractional BOEs. Let $(\prod,\|.\|)$ as a space of Banach. Moreover, clarify W as a self-map $\prod$, and $\Phi_{m+1}=g(W,\Phi_m)$ displays a precise repeatable procedure. The fixed-point set on W is signified by $\mathbb{N}(W)$. Moreover, W involved minimum single component that $k\in\mathbb{N}(W)$ converge by χ_m . Let $\{\rho_m\}\subseteq\prod$ $x_m=\|\rho_{m+1}-f(W,\rho_m)\|$. If $\lim_{m\to\infty}x^m$ approaches to zero implies $\lim_{m\to\infty}\rho^m$ approaches to k. Iteration $\Phi_{m+1}=g(W,\Phi_m)$ is called as W-stable. Comparatively, we

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believe that the upper bound for this sequence $\{\rho_m\}$. If all of these conditions are met for $\Phi_{m+1} = W\Phi_m$, this iteration is known as Picard's iteration and it is $W-$ stable. In the following, we shall explain the result described by Theorem 1.
Let $(\prod, \|.\|)$ is a Banach space, W be as a self-map on $\prod$ that satisfies the below inequality.

$$\|W_y - W_z\| \leq \Re \|y - W_y\| + \wp\|y - z\|. \tag{4.1}$$

for all the values of $y, z \in \prod$ where $0 \leq \wp < 1, 0 \leq \Re,$. Suppose that W is Picard W-stable.
Now, consider the time-fractional BOEs expressed in the form

$$\Phi_{y+1}(\mu,\nu) = \Phi_y(\mu,\nu) - L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left(\Phi_y\frac{\partial \Phi_y}{\partial \mu} + \frac{\partial^2\Phi_y}{\partial \mu^2}\right)\right]. \tag{4.2}$$

In (20) the term $\frac{s+\alpha(1-s)}{s}$ is the Lagrange's Multiplier
Now, we want to show the subsequent result presented in form of the theorem.

Theorem 1. Assume that W is a self-map which is given defined by

$$W\left(\Phi_y(\mu,\nu)\right) = \Phi_{y+1}(\mu,\nu) = \Phi_y(\mu,\nu) - L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right) L\left(\Phi_y\frac{\partial \Phi_y}{\partial \mu} + \frac{\partial^2\Phi_y}{\partial \mu^2}\right)\right]. \tag{4.3}$$

is W-stable in $L^2(y,z)$ if

$$\left\{1 + T_1(W)\delta_2 + \frac{T_2(W)\delta_1}{2}(\kappa_1+\kappa_2)\right\} < 1. \tag{4.4}$$

where, δ_1, δ_2 are functions arising from $L^{-1}\left[\frac{p+\beta(1-p)}{p}L(.)\right]$

Proof. We will now demonstrate that W consists of a fixed point. Therefore, for all $(y,z) \in \mathbb{Q}\times\mathbb{Q}$, let us suppose

$$\begin{aligned} W\left(\Phi_y(\mu,\nu)\right) - W\left(\Phi_z(\mu,\nu)\right) &= \Phi_y(\mu,\nu) - \Phi_z(\mu,\nu) - L^{-1}\left[\left(\frac{p+\alpha(1-p)}{p}\right) L\left((\Phi_y\frac{\partial \Phi_y}{\partial \mu} + \frac{\partial^2\Phi_y}{\partial \mu^2}\right)\right] \\ &- L^{-1}\left[\left(\frac{p+\beta(1-p)}{p}\right) L\left((\Phi_z\frac{\partial \Phi_z}{\partial \mu} + \frac{\partial^2\Phi_z}{\partial \mu^2}\right)\right]. \end{aligned} \tag{4.5}$$

Without losing generality and by applying the standard to both sides of (4.5) we obtain

$$\begin{aligned} \|W\left(\Phi_y(\mu,\nu)\right) - W\left(\Phi_z(\mu,\nu)\right)\| &= \|\Phi_y(\mu,\nu) - \Phi_z(\mu,\nu) - L^{-1}\left[\left(\frac{p+\alpha(1-p)}{p}\right) L\left((\Phi_y\frac{\partial \Phi_y}{\partial \mu} + \frac{\partial^2\Phi_y}{\partial \mu^2}\right)\right] \\ &- L^{-1}\left[\left(\frac{p+\beta(1-p)}{p}\right) L\left((\Phi_z\frac{\partial \Phi_z}{\partial \mu} + \frac{\partial^2\Phi_z}{\partial \mu^2}\right)\right]\|. \end{aligned} \tag{4.6}$$

Next, by using the triangle inequality and further simplification (4.6), we obtain,

$$\begin{aligned} \|W\left(\Phi_y(\mu,\nu)\right) - W\left(\Phi_z(\mu,\nu)\right)\| &\leq \|\Phi_y(\mu,\nu) - \Phi_z(\mu,\nu)\| - L^{-1}\{\left(\frac{p+\beta(1-p)}{p}\right) L\left[\left\|\frac{\partial^2\Phi_y}{\partial \mu^2} - \frac{\partial^2\Phi_z}{\partial \mu^2}\right\|\right] \\ &+ L\left[\left\|\Phi_y\frac{\partial \Phi_y}{\partial \mu} - \Phi_z\frac{\partial \Phi_z}{\partial \mu}\right\|\right]\right\}. \end{aligned} \tag{4.7}$$

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$$\begin{aligned}
&\|W\left(\Phi_y(\mu,\nu)\right)-W\left(\Phi_z(\mu,\nu)\right)\| \leq \|\Phi_y(\mu,\nu)-\Phi_z(\mu,\nu)\|+L^{-1}\left\{\left(\frac{p+\beta(1-p)}{p}\right)\right.\\
&L\left\|\left(\frac{\partial^2\Phi_y}{\partial\mu^2}-\frac{\partial^2\phi_z}{\partial\mu^2}\right)\right\|\Big]+L\left[\left\|\Phi_y\frac{\partial\Phi_y}{\partial\mu}-\Phi_z\frac{\partial\Phi_z}{\partial\mu}\right\|\right]\}\\
&\leq \|\Phi_y(\mu,\nu)-\Phi_z(\mu,\nu)\|+L^{-1}\left\{\left(\frac{p+\beta(1-p)}{p}\right)\right.\\
&L\left[\left\|\frac{\partial^2\Phi_y}{\partial\mu^2}-\frac{\partial^2\Phi_z}{\partial\mu^z}\right\|\right]+L\left[\frac{1}{2}\left\|\frac{\partial\Phi_m^2}{\partial\mu}-\frac{\partial\Phi_\nu^2}{\partial\mu}\right\|\right]\Bigg\}.
\end{aligned} \tag{4.8}$$

Now substituting differential operators $\frac{\partial}{\partial\mu}=\delta_1, and \frac{\partial^2}{\partial\mu^2}=\delta_2$, we get

$$\begin{aligned}
&\|W\left(\Phi_y(\mu,\nu)\right)-W\left(\Phi_z(\mu,\nu)\right)\| \leq \|\Phi_y(\mu,\nu)-\Phi_z(\mu,\nu)\|+L^{-1}\left\{\left(\frac{p+\beta(1-p)}{p}\right)\right.\\
&L\left[\sigma_2\|\Phi_y(\mu,\nu)-\Phi_z(\mu,\nu)\|\right]+L\left[\frac{\sigma_1}{2}\left\|\left(\Phi_y(\mu,\nu)+\Phi_z(\mu,\nu)\right)\left(\Phi_y(\mu,\nu)-\Phi_z(\mu,\nu)\right)\right\|\right]\Bigg\}.
\end{aligned} \tag{4.9}$$

Since, Φ_y and Φ_z are bounded, $\|\Phi_y\|\leq\kappa_1$ and $\|\Phi_z\|\leq\kappa_2$. Thus simplifying (4.9), we obtain

$$\|W\left(\Phi_y(\mu,\nu)\right)-W\left(\Phi_z(\mu,\nu)\right)\| \leq \left\{1+T_1(W)+\frac{T_2(W)\delta_1}{2}\left(\kappa_1+\kappa_2\right)\right\}\|\Phi_y(\mu,\nu)+\Phi_z(\mu,\nu)\|. \tag{4.10}$$

where T_1, and T_2 are functions of $L^{-1}\left\{\left(\frac{p+\alpha(1-p)}{p}\right)L\right\}$.
Thus, the nonlinear W-self mapping attains a fixed point. Now, we shall prove that W satisfies the criteria with the inequality (4.1). Precisely, let Eq. (4.10) be held, then we have

$$\wp=0,\quad \Re=\left\{1+T_1(W)\delta_2+\frac{T_2(W)\delta_1}{2}\left(\kappa_1+\kappa_2\right)\right\}. \tag{4.11}$$

Therefore Eq. (4.11) reveals that the nonlinear mapping W together the inequality (4.1) holds. For the defined nonlinear mapping W together all the conditions with inequality (4.1) fulfil, then W Picard's W -Stable. Which closes the proof and satisfies the Theorem 1.

5 Examples

To illustrate our main findings two examples are provided in this section.

Example 1. Consider the time-fractional BOEs

$${}_{0}^{CF}D_{\nu}^{\alpha}\Phi_\nu(\mu,\nu)+\Phi\Phi_\mu+\lambda\Phi_{\mu\mu}=0, \tag{5.1}$$

$$\Phi(\mu,0)= \ -2\eta\lambda+2\eta\lambda\tanh(\eta\mu). \tag{5.2}$$

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Taking Laplace transform's on both side's of Eq. (5.1)

$$L[\Phi(\mu,\nu)] = \frac{1}{p}\ \left(-2\eta\lambda + 2\eta\lambda\tanh(\eta\mu)\right) - \left(\frac{p+\alpha(1-p)}{p}\right) L\left[\Phi\Phi_{\mu} + \lambda\Phi_{\mu\mu}\right]. \tag{5.3}$$

Taking inverse Laplace transform's of Eq.(5.3)

$$\Phi(\mu,\nu) = -2\eta\lambda + 2\eta\lambda\tanh(\eta\mu) - L^{-1}\left[\left(\frac{p+\alpha(1-p)}{p}\right) L\left[\Phi\Phi_{\mu} + \lambda\Phi_{\mu\mu}\right]\right]. \tag{5.4}$$

Now, we apply the FHPTM, The first few components of He's polynomial H_m are written as

$$\begin{aligned}\sum_{m=0}^{\infty}\Phi(\mu,\nu) =&\ -2\eta\lambda + 2\eta\lambda\tanh(\eta\mu)\\ &+ rL^{-1}\left[\left(\frac{p+\alpha(1-p)}{p}\right) L\left[\left(\sum_{m=0}^{\infty}p^m\Phi_m(\mu,\nu)\right)_{\mu\mu} + \sum_{m=0}^{\infty}p^m H_m(\Phi)\right]\right].\end{aligned} \tag{5.5}$$

where $H_m(\Phi)$ are the nonlinear terms represented by He's polynomials[11]. For instance, the initial few elements of He's polynomials given by

$$\begin{aligned}H_0(\mu) =&\Phi_0(\Phi_0)_{\mu},\\ H_1(\mu) =&\Phi_1(\Phi_0)_{\mu} + \Phi_0(\Phi_1)_{\mu}.\\ &\vdots\end{aligned} \tag{5.6}$$

When we compare the like powers of r's coefficient, we get

$$\begin{aligned}
&r^0 : \Phi_0(\mu,\nu)\ = -2\eta\lambda + 2\eta\lambda\tanh(\eta\mu),\\
&r^1 : \Phi_1(\mu,\nu)\ = (1-\alpha+\alpha\nu)\left(-4\eta^3\lambda^2\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) + 2\eta^2\lambda\mathrm{sech}(\eta\mu)^2(-2\eta\lambda + 2\eta\lambda\tanh(\eta\mu))\right),\\
&r^2 : \Phi_2(\mu,\nu) = 4\alpha^2\nu^2\left(\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2 + \eta^5\lambda^3\mathrm{sech}(\eta\mu)^4 - \eta^4\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) - 2\eta^5\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu)\right)\\
&-16\nu\ \left(-\alpha\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2 + \alpha^2\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2 - \alpha\eta^5\lambda^3\mathrm{sech}(\eta\mu)^4 + \alpha^2\eta^5\lambda^3\mathrm{sech}(\eta\mu)^4 + \alpha\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) - \alpha^2\eta^4\lambda^3\mathrm{se}\right.\\
&\left.+2\alpha\eta^5\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) - 2\alpha^2\eta^5\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu)\right) +\\
&8\left(\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2 - 2\alpha\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2 + \alpha^2\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2 + \eta^5\lambda^3\mathrm{sech}(\eta\mu)^4 - 2\alpha\eta^5\lambda^3\mathrm{sech}(\eta\mu)^4 + \alpha^2\eta^5\lambda^3\mathrm{sech}(\eta\mu)^4 - \eta^4\lambda^3\mathrm{se}\right.\\
&+2\alpha\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) - \alpha^2\eta^4\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) - 2\eta^5\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) + 4\alpha\eta^5\lambda^3\mathrm{sech}(\eta\mu)^2\tanh(\eta\mu) - 2\alpha^2\eta^5\\
&\vdots
\end{aligned}$$

Example 2. Take the time-fractional Benjamin-Ono equation

$${}_{0}^{CF}D_{\nu}^{\alpha}\Phi_{\nu}(\mu,\nu) + \Phi\Phi_{\mu} + \lambda\Phi_{\mu\mu} = 0, \tag{5.7}$$

$$\Phi(\mu,0) = \ -2\eta\lambda\frac{1}{1+e^{\eta\mu}}. \tag{5.8}$$

Taking Laplace transform's on both side's of Eq(5.7)

$$L[\Phi(\mu,\nu)] = \frac{1}{s}\left(-2\eta\lambda\frac{1}{1+e^{\eta\mu}}\right) - \left(\frac{p+\alpha(1-p)}{p}\right) L\left[\Phi\Phi_{\mu} + \lambda\Phi_{\mu\mu}\right]. \tag{5.9}$$

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Taking inverse Laplace transform's of Eq.(5.9)

$$\Phi(\mu,\nu) = -2\eta\lambda\frac{1}{1+e^{\eta\mu}} - L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)L\left[\Phi\Phi_{\mu}+\lambda\Phi_{\mu\mu}\right]\right]. \tag{5.10}$$

Now, we apply the FHPTM. The first few components of He's polynomials H_m are written as

$$\begin{aligned}\sum_{m=0}^{\infty}\Phi(\mu,\nu) = &-2\eta\lambda\frac{1}{1+e^{\eta\mu}}\\ &+rL^{-1}\left[\left(\frac{p+\alpha(1-p)}{s}\right)L\left[\left(\sum_{m=0}^{\infty}p^m\Phi_m(\mu,\nu)\right)_{\mu\mu}+\sum_{m=0}^{\infty}p^mH_m(\Phi)\right]\right].\end{aligned} \tag{5.11}$$

where $H_m(\Phi)$ are the nonlinear terms represented by He's polynomials [11]. For instance, the initial some member of He's polynomials are

$$\begin{aligned}H_0(\mu) =&\Phi_0(\Phi_0)_{\mu},\\ H_1(\mu) =&\Phi_1(\Phi_0)_{\mu}+\Phi_0(\Phi_1)_{\mu}.\\ &\vdots\end{aligned} \tag{5.12}$$

When we compare the like powers of r's coefficient, we get

$$\begin{aligned}r^0:\Phi_0(\mu,\nu) &= -2\eta\lambda\left(\tfrac{1}{1+e^{\eta\mu}}\right),\\ r^1:\Phi_1(\mu,\nu) &= \left(-\tfrac{4e^{\eta\mu}\eta^3\lambda^2}{(1+e^{\eta\mu})^3}+\lambda\left(-\tfrac{4e^{2\eta\mu}\eta^3\lambda}{(1+e^{\eta\mu})^3}+\tfrac{2e^{\eta\mu}\eta^3\lambda}{(1+e^{\eta\mu})^2}\right)\right)(1-\alpha+\alpha\nu),\\ r^2:\Phi_2(\mu,\nu) &= -\tfrac{e^{\eta\mu}(-1+e^{\eta\mu})\eta^5\lambda^3\left(2-4\alpha+2\alpha^2+4\alpha\nu-4\alpha^2\nu+\alpha^2\nu^2\right)}{(1+e^{\eta\mu})^3};\\ &\vdots\end{aligned}$$

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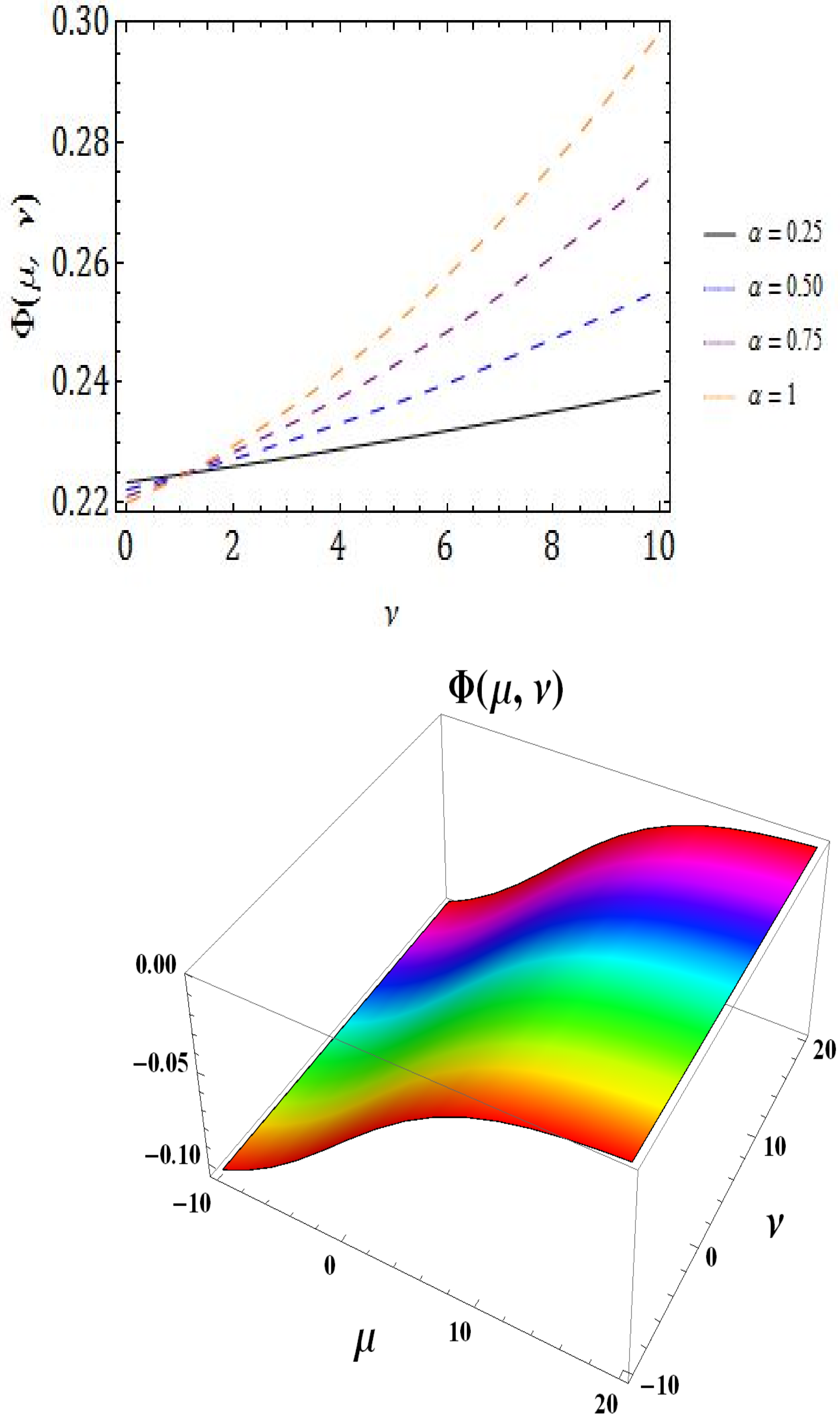

Figure 1: graphs showing the FHPTM approximate and exact solutions $\Phi(\mu, \nu)$ for various values of $\alpha = 0.25,\ \alpha = 0.5, \alpha = 0.75$ and $\alpha = 1$.

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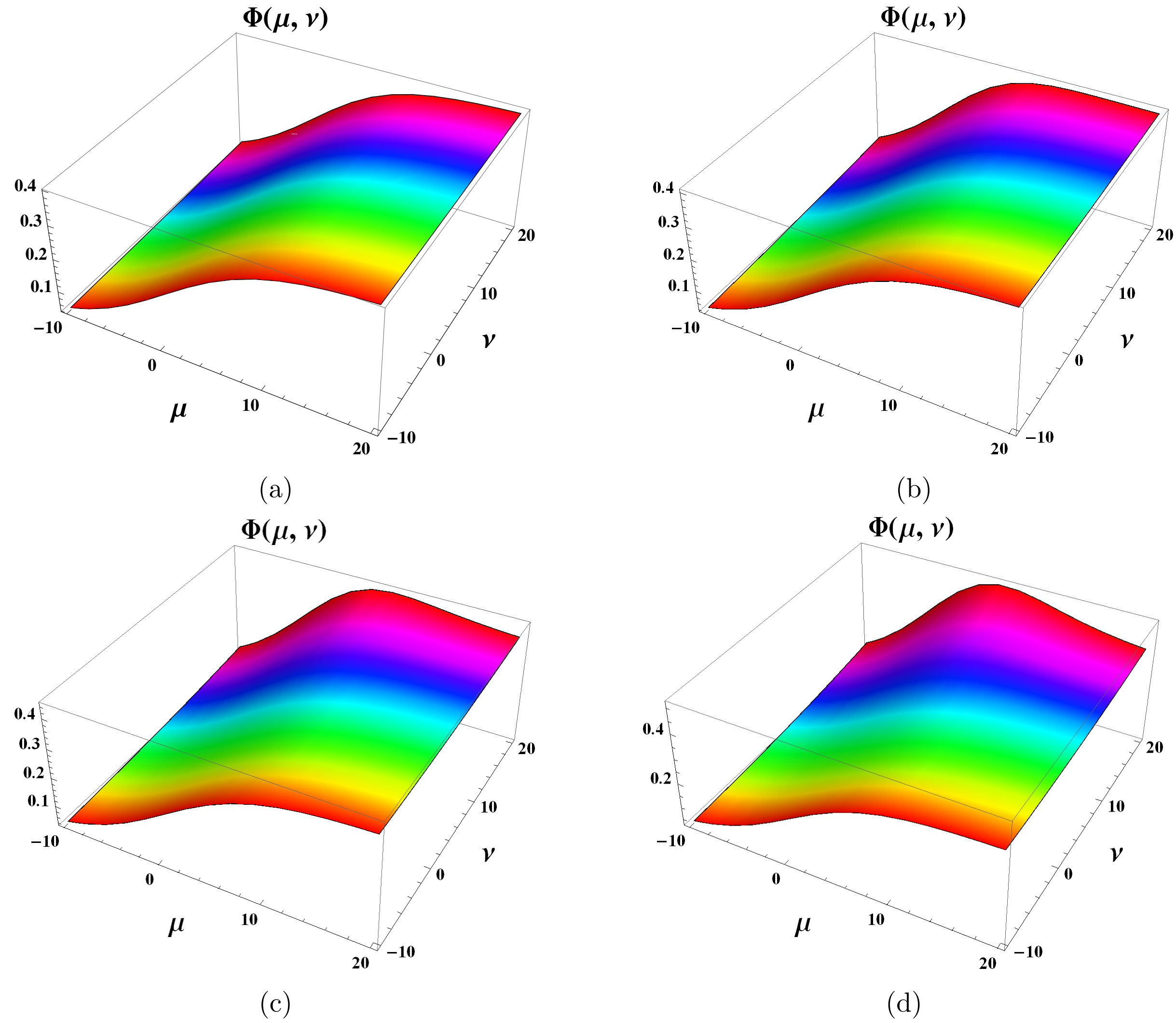

Figure 2: The proposed approaches' resolution is illustrated graphically at (*a*) $\alpha = 0.25$, (*b*) $\alpha = 0.5$, (*c*) $\alpha = 0.75$ and (*d*) $\alpha = 1$ from example 1.

Figure 2: The proposed approaches' resolution is illustrated graphically at (*a*) $\alpha = 0.25$, (*b*) $\alpha = 0.5$, (*c*) $\alpha = 0.75$ and (*d*) $\alpha = 1$.

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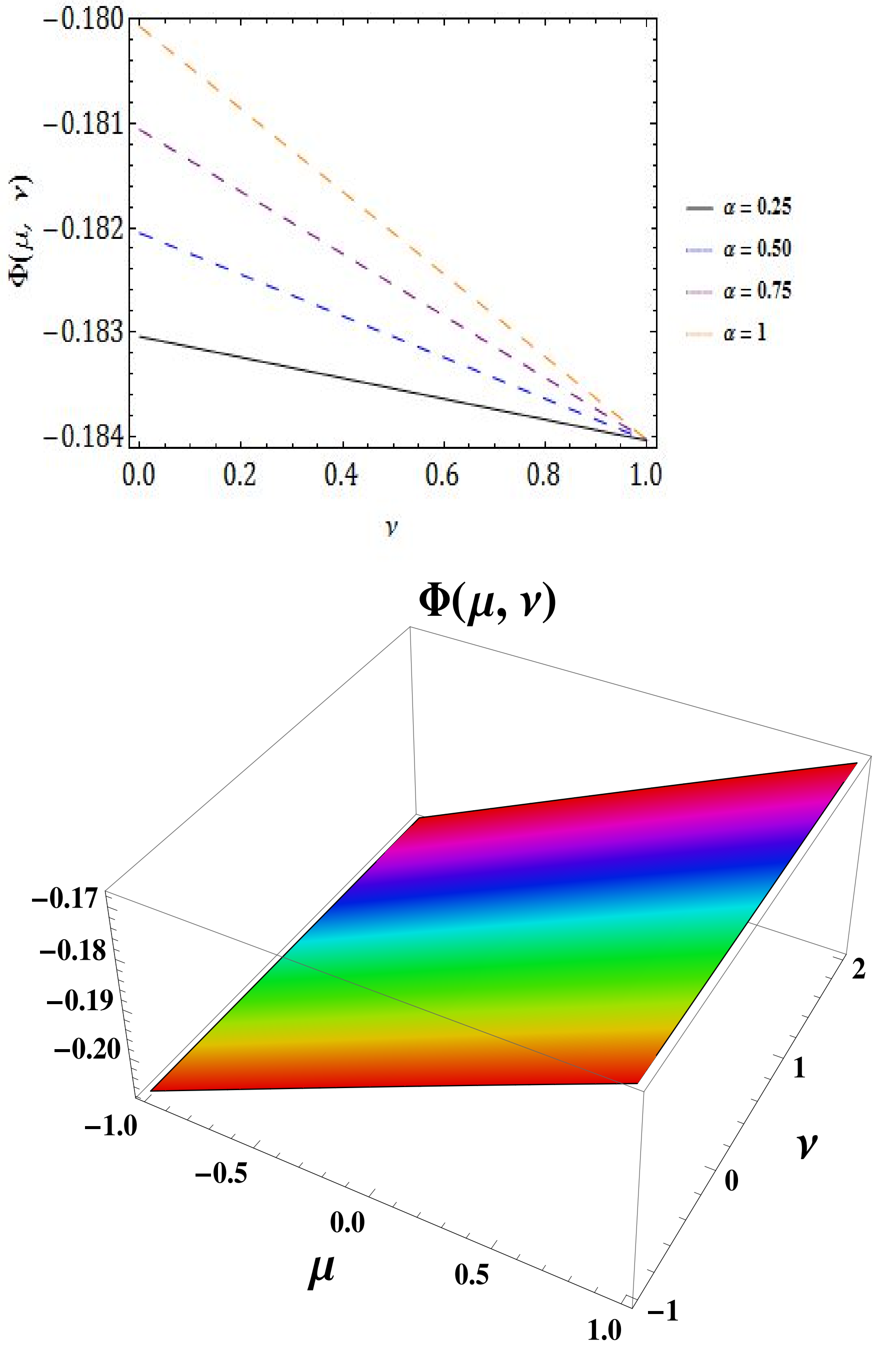

Figure 3: graphs showing the FHPTM approximate and exact solutions $\Phi(\mu,\nu)$ for various values of $\alpha = 0.25$, $\alpha = 0.5, \alpha = 0.75$ and $\alpha = 1$.

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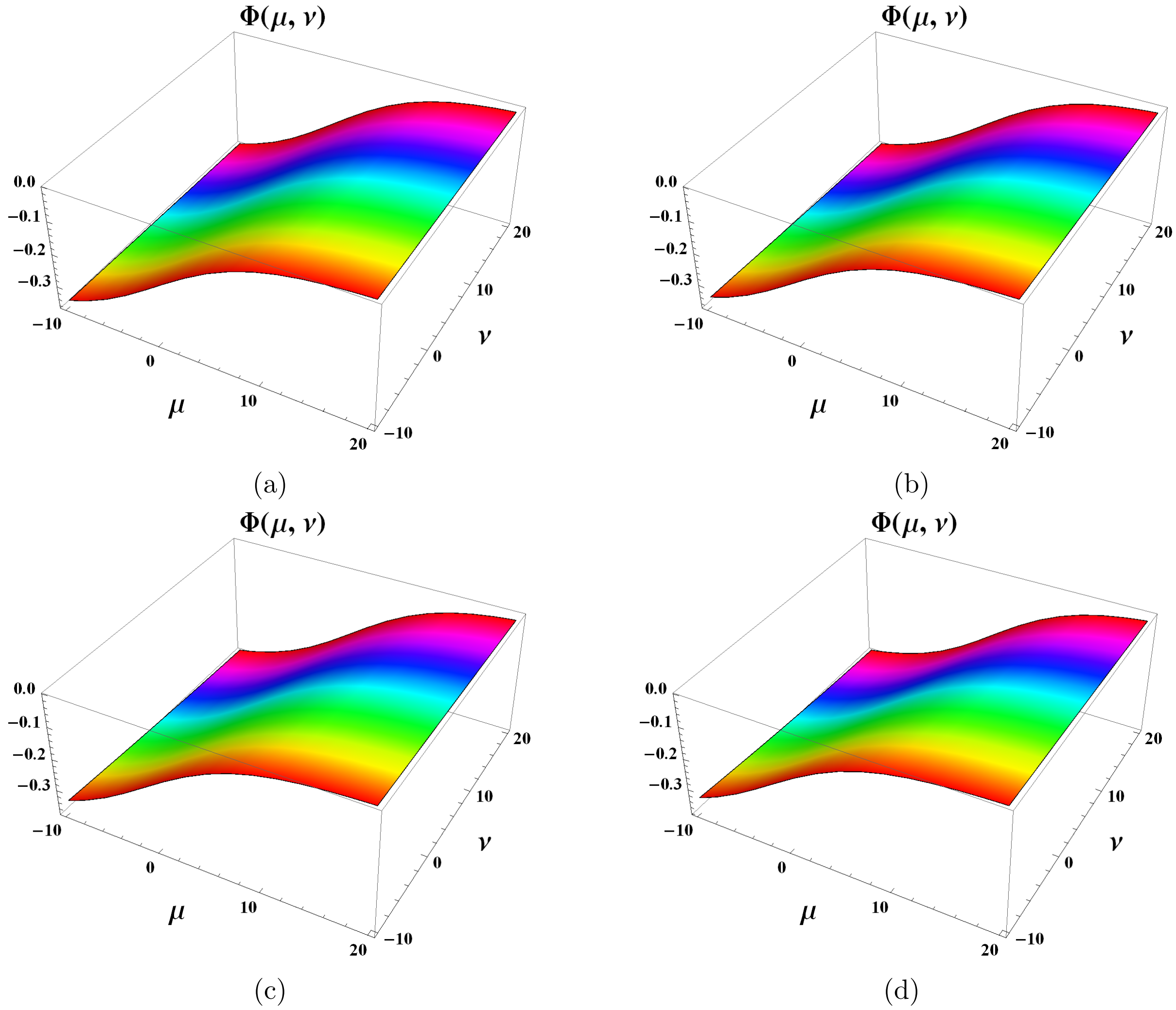

Figure 4: The proposed approaches' resolution is illustrated graphically at (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$ from example 2.

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6 Conclusions and Applications

In this work, the FHPTM has been applied lucratively to solve time fractional nonlinear partial differential equations via a novel Caputo-Fabrizio fractional derivative of order $\alpha \in (0, 1]$, including the Benjamin-Ono equation. This equation intertwines four pivotal aspects: time fractional derivative, nonlinearity, dispersion, internal wave dynamics. The results covers explanations of the model's stability analysis and semi-analytical solution, which are made possible via an iterative process based on the Laplace transform. The equation specifically models the dynamics of internal waves, which are waves that propagate within a fluid medium where density varies with depth. These waves play a crucial role in various geophysical and oceanographic processes. By selecting appropriate parameter values using Mathematica, 2D and 3D curves have been plotted to display the graphical representation. Semi-analytical solutions offer significant advantages in fields where pure analytical solutions are unfeasible, and fully numerical methods are computationally demanding. Their applications span a wide array of disciplines, from Engineering and Physics to Biology and Environmental Sciences, allowing for the efficient modeling of complex systems while maintaining reasonable accuracy and insight.
The time-fractional BOEs has a wide range of applications in various fields. Here are some examples:

- Geophysical Fluid Dynamics: Understanding the dynamics of internal waves in the ocean is crucial for studying processes such as ocean mixing, nutrient transport, and climate variability. The time-fractional BOEs provides a valuable tool for simulating and analyzing these phenomena.

- Nonlinear Optics: In optics, the Benjamin-Ono equation has been used to model the propagation of optical solitons in nonlinear media. By incorporating fractional derivatives in time, the time-fractional BOEs allows for more accurate modeling of dispersion and memory effects in optical systems.

- Plasma Physics: Plasma, the fourth state of matter consisting of charged particles, exhibits complex wave behavior. The time-fractional BOEs can be used to study the propagation of waves in plasma, with applications in fusion research, space physics, and astrophysics.

- Biological Systems: Biological systems often exhibit complex dynamics that can be modeled using fractional differential equations. The time-fractional BOEs has been applied to study phenomena such as nerve impulse propagation, pattern formation in biological tissues, and the dynamics of biochemical reactions.

- Finance and Economics: Fractional calculus has found applications in financial modeling and economics, particularly in the analysis of long-term memory effects and volatility clustering in financial time series. The time-fractional BOEs may provide insights into the dynamics of financial markets and the behavior of complex economic systems.

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