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Product and Commutator of pseudo-differential operators involving the coupled fractional Fourier transform

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Abstract: The symbol class $\Lambda(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ is discussed and it is shown that the product of any two symbols from this class is again in Λ. The Pseudo-differential operators (p.d.o.) $A(x,y,D'_{x,y})$ and $\mathscr{A}(x,y,D'_{x,y})$ involving the coupled fractional Fourier transform $\mathscr{F}_{\alpha_1,\alpha_2}$ associated with symbol classes are defined. Product and Commutator for the p.d.o. and their boundedness results are obtained.

Keywords: Fractional Fourier transform, Coupled fractional Fourier transform, Schwartz space, Sobolev type spaces, pseudo-differential operators.

2020 Mathematics Subject Classification: 35S05, 46E10, 46E35, 46F05, 47G30, 46F12.

1 Introduction

Wiener originally proposed the concept of the fractional Fourier transform (FrFT) in 1929 [1]. In 1980, Namias also explored the fractional Fourier transform [2] as a means of determining the solutions to certain differential equations that sometimes arise in quantum physics. McBride and Kerr [3] further refined his findings by creating an operational calculus for the fractional Fourier transform. Fractional Fourier transform has drawn increased attention in recent years due to its many applications in the fields of image processing, signal analysis, and optics. This transformation is crucial for resolving a number of issues in signal processing,

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optics, and quantum physics [2, 4, 5, 6, 7, 8, 9, 6, 10, 11, 12]. A variety of mathematical analytic fields have examined the fractional Fourier transform, which is a generalisation of the ordinary Fourier transform. These areas include wavelets [13, 14], pseudo-differential operators [15], and generalised functions [16, 11, 17]. The well-known Fourier transform of a function $\phi \in L_1(\mathbb{R})$, represented by $\widehat{\phi}$, is described as

$$\widehat{\phi}(\eta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta\zeta}\phi(\zeta)d\zeta$$

so that its inverse is given by

$$\phi(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\eta\zeta}\widehat{\phi}(\eta)d\eta$$

provided the integrals exist.
We recall the one-dimensional fractional Fourier transform [5, 17, 18, 19, 17, 17, 20, 21, 22, 23, 24, 25] of a function $\phi \in L_1(\mathbb{R})$ with parametre α, denoted by $(\mathscr{F}_\alpha\phi)(\eta) = \widehat{\phi}_\alpha(\eta)$ is given in $L_1(\mathbb{R})$ as follows:

$$(\mathscr{F}_\alpha\phi)(\eta) = \widehat{\phi}_\alpha(\eta) = \int_{\mathbb{R}} K_\alpha(\zeta,\eta)\phi(\zeta)d\zeta \tag{1}$$

where the kernel $K_\alpha(\zeta,\eta)$ is given by

$$K_\alpha(\zeta,\eta) = \begin{cases} C_\alpha e^{\frac{i(\zeta^2+\eta^2)\cot\alpha}{2} - i\zeta\eta\csc\alpha}, & \alpha \neq n\pi, n\in\mathbb{Z} \\ \frac{1}{\sqrt{2\pi}}e^{-i\zeta\eta}, & \alpha = \frac{\pi}{2} \\ \delta(\zeta-\eta), & \alpha = 2n\pi \\ \delta(\zeta+\eta), & \alpha = (2n+1)\pi, \end{cases}$$

$C_\alpha = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and studied some properties of this transform.
The corresponding inversion formula of $(\mathscr{F}_\alpha\phi)(\eta)$ is defined in the following ways

$$\phi(\zeta) = \int_{\mathbb{R}} \overline{K_\alpha(\zeta,\eta)}(\mathscr{F}_\alpha\varphi)(\eta)d\eta \tag{2}$$

$$\overline{K_\alpha(\zeta,\eta)} = \overline{C}_\alpha e^{\frac{-i(\zeta^2+\eta^2)\cot\alpha}{2} + i\zeta\eta\csc\alpha}$$

and $\overline{C}_\alpha = \sqrt{\frac{1+i\cot\alpha}{2\pi}} = C_{-\alpha}$.
Hence, $\overline{K_\alpha(\zeta,\eta)} = K_{-\alpha}(\zeta,\eta)$.
It implies that the inverse of a FrFT with the parameter α is the FrFT with the parameter $-\alpha$.
Exploiting the tensor product of n copies of the one-dimensional fractional Fourier transform each of order α_p, $p = 1, 2, 3, ..., n$ [18], the fractional Fourier transform has been extended to the higher-dimensional transform.

We assume that $\alpha = (\alpha_1, \alpha_2)$, $\mathbf{x} = (x,\eta)$, $\mathbf{y} = (y,\zeta)$, $\mathscr{K}_\alpha(\mathbf{x},\mathbf{y}) = \mathscr{K}_{\alpha_1}(x,\eta).\mathscr{K}_{\alpha_2}(y,\zeta) = \mathscr{K}_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$, where $\mathscr{K}_{\alpha_1}(x,\eta)$ and $\mathscr{K}_{\alpha_2}(y,\zeta)$

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defined as above.

The two-dimensional fractional Fourier transform [26, 27, 28] is defined as follows

$$\begin{aligned}[\mathscr{F}_\alpha\phi](\eta,\zeta)=[\mathscr{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_\alpha(\mathbf{x},\mathbf{y})\phi(x,y)dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1}(x,\eta)\mathscr{K}_{\alpha_2}(y,\zeta)\phi(x,y)dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)\phi(x,y)dxdy. \qquad (3)\end{aligned}$$

The corresponding inversion formula of (3) is defined as follows

$$\phi(x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)}[\mathscr{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta)d\eta d\zeta. \qquad (4)$$

It is easy to observe that for $\alpha_1=\alpha_2=\frac{\pi}{2}$, the two-dimensional fractional Fourier transform $\mathscr{F}_{\alpha_1,\alpha_2}$ becomes a classical two-dimensional Fourier transform.

Definition 1. *A tempered distribution ϕ belongs to the Sobolev type spce $\mathscr{H}^s(\mathbb{R}\times\mathbb{R})$, and $s\in\mathbb{R}$ if its coupled fractional Fourier transform $\mathscr{F}_{\alpha_1,\alpha_2}\phi$ corresponding to a locally integrable function $(\mathscr{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)$ over $\mathbb{R}\times\mathbb{R}$ such that*

$$\|\phi\|_s=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\{(1+|\xi|^2)(1+|\eta|^2)\}^{\frac{s}{2}}(\mathscr{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)\right|^2 d\eta d\xi\right)^{\frac{1}{2}}<\infty. \qquad (5)$$

This space is complete with respect to the norm $\|\phi\|_s$.

Definition 2. *The space $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is the collection of all complex valued infinitely differentiable functions $\phi(\xi,\eta)\in\mathbb{R}\times\mathbb{R}$ for every choice of $l_1,\ l_2,\ m_1,\ m_2\in\mathbb{N}_0$ which for*

$$\Gamma^{l_1,\ l_2}_{m_1,\ m_2}(\phi)=\sup_{(x,y)\in\mathbb{R}\times\mathbb{R}}\left|x^{l_1}y^{l_2}\frac{\partial^{m_1}}{\partial x^{m_1}}\frac{\partial^{m_2}}{\partial y^{m_2}}\phi(x,y)\right|<\infty. \qquad (6)$$

The dual of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is denoted by $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$.
If φ is a locally integrable and polynomial growth function on $\mathbb{R}\times\mathbb{R}$, then φ generates a distribution in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ as follows:

$$\langle\varphi,\ \phi\rangle=\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(\xi,\eta)\phi(\xi,\eta)d\xi d\eta,\quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}). \qquad (7)$$

The elements of $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ are known as tempered distributions.

Theorem 1. *Let $K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$ be the kernel of the two-dimensional fractional Fourier transform. Then, for all $\varphi(x,y)\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$, we have*
(i) $D^r_{x,y}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)=\{i(\eta\csc\alpha_1+\zeta\csc\alpha_2)\}^r K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$,
(ii) $\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x,y)D^r_{x,y}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)dxdy=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)(D'_{x,y})^r\varphi(x,y)dxdy$,
(iii) $\mathscr{F}_{\alpha_1,\alpha_2}\{(D'_{x,y})^r\varphi(x,y)\}(\eta,\zeta)=\{i(\eta\csc\alpha_1+\zeta\csc\alpha_2)\}^r(\mathscr{F}_{\alpha_1,\alpha_2}\varphi(x,y))(\eta,\zeta)$,
for all $r\in\mathbb{N}$, where $D_{x,y}=[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+i(x\cot\alpha_1+y\cot\alpha_2)]$ and $D'_{x,y}=-[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-i(x\cot\alpha_1+y\cot\alpha_2)]$.

Proof. See [26]. □

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2 Symbol Classes

Let $a(x,y,\xi,\zeta)$ be a complex valued function defined for $x,y,\xi\neq 0,\zeta\neq 0\in\mathbb{R}$. The function $a(x,y,\xi,\zeta)\in\mathbb{C}^{\infty}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$ is said to be an element of the class Λ if and only if $a(x,y,t_1\xi,t_2\zeta)=a(x,y,\xi,\zeta)$ for $t_1>0,\ t_2>0$, and assume also that

$$\lim_{(|x|,|y|)\to(\infty,\infty)}=a(\infty,\infty,\xi,\zeta)$$

exists for $\xi\neq 0,\zeta\neq 0\in\mathbb{R}$ and $a(\infty,\infty,\xi,\zeta)$ is a mapping $\mathbb{C}^{\infty}$-function.

Now we define $a'(x,y,\xi,\zeta)=a(x,y,\xi,\zeta)-a(\infty,\infty,\xi,\zeta)$, and assume the estimates

$$(1+x^2+y^2)^p\left|\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial\xi^m}\frac{\partial^n}{\partial\zeta^n}a'(x,y,\xi,\zeta)\right|\leq\mathbb{C}_{p,k,l,m,n},\quad\forall x,\ y,\ \xi\neq 0,\ \zeta\neq 0\in\mathbb{R}\tag{8}$$

here, p=1,2,3,.....,k, l, m, n are natural numbers.

Theorem 2. *(i) We get* $|a(\infty,\infty,\xi,\zeta)-a(\infty,\infty,\delta,\eta)|\leq\mathbb{C}\big((|\xi-\delta|+|\zeta-\eta|)/\big(|\xi|+|\zeta|+|\delta|+|\eta|\big)\big),\quad\forall\xi,\ \zeta,\ \delta,\ \eta$ *arbitray in* $\mathbb{R}-\{0\}$.

(ii) The estimates $(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)|\leq\mathbb{M}_p$, $\forall x,\ y,\ \xi\neq 0,\ \zeta\neq 0\in\mathbb{R},\ p=1,\ 2,\ 3,\ 4,\ 5...;$

(iii) $(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)|\leq\mathbb{M}_p(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1},\quad\forall\xi,\ \zeta,\ \delta,\ \eta\in\mathbb{R}-\{0\},\ \forall x,y\in\mathbb{R},\ p=1,2,...$ *being*

$$\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)a'(t,u,\xi,\zeta)dtdu,\quad\forall x,y,\xi\neq 0,\zeta\neq 0\in\mathbb{R}$$

are verified.

Proof. (i) Similar proof of Theorem 1 (a)[29].
(ii) We get the equality

$$\begin{aligned}&(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)\\=\;&\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)(I-D'_{x,y})^p a'(t,u,\xi,\zeta)dtdu,\\&x,y,\xi\neq 0,\zeta\neq 0\in\mathbb{R},\ D'_{x,y}=-\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-i(xcot\alpha_1+ycot\alpha_2)\right]\end{aligned}\tag{9}$$

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and therefore is verified the estimate

$$
\begin{aligned}
& \left|(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)\right| \\
\leq\ & C_{\alpha_1}C_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q|(I-D'_{x,y})^p a'(t,u,\xi,\zeta)| \\
& \times(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu \\
\leq\ & C_{\alpha_1}C_{\alpha_2}C_1\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q}dtdu=\mathbb{M}_p
\end{aligned}
\tag{10}
$$

for q sufficient large.
(iii) We get

$$
\begin{aligned}
& (1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)| \\
=\ & \int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^p \\
\times\ & \left[a'(t,u,\xi,\zeta)-a'(t,u,\delta,\eta)\right](1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu.
\end{aligned}
$$

Let us put now

$$
b_{p,q}(t,u,\xi,\zeta)=(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^p a'(t,u,\xi,\zeta). \tag{11}
$$

We obtain then the estimate

$$
\begin{aligned}
& (1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\left|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)\right| \\
\leq\ & |C_{\alpha_1}C_{\alpha_2}|\int_{\mathbb{R}}\int_{\mathbb{R}}\Big|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta) \\
& \times\Big|(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu.
\end{aligned}
\tag{12}
$$

Consequently, it will be sufficient to show here that with a constant independent of $x,y\in\mathbb{R}$ we have, for $x,y\in\mathbb{R},\ \xi,\ \zeta,\ \delta,\ \eta\in\mathbb{R}-\{0\}$, the estimate

$$
\left|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta)\right|\leq D_{\alpha_1,\alpha_2}(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1}. \tag{13}
$$

It can be easily proved from (ii), (12)and (13). □

The term "pseudo-differential operators"[30, 31, 32, 33] has a fairly broad definition and covers such topics as harmonic analysis, partial differential equation, geometry, mathematical physics, microlocal analysis, time-frequency analysis, imaging, computations, and quantum mechanics. In mathematics, natural sciences, medicine, scientific computing, and engineering, current trends and novel applications are highlighted. The emphasis is on contemporary developments in different

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branches of engineering, mathematical sciences, the natural sciences, medicine, scientific computers.

In reality, Kohn-Nirenberg and Hörmander [34] were the ones who first introduced the pseudo-differential calculus, and later authors expanded on it, primarily in a local context, to examine local regularity and local solvability of PDEs.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

Many faculties, scientists, Ph.D students and researchers of other field developed the theory of pseudo-differential operators with the help of different types of integral operators like Fourier transforms (see [35, 29]), Hankel transform (see [36, 37, 38]), Fourier Bessel Transform on $\mathbb{R}_+$ (see [20, 21]), Weinstein transform (see [39]), Laguerre hypergroups (see [40]) and Jacobi differential operators (see [41]).

3 Pseudo-Differential Operator $A(x,y,D'_{x,y})$ related to $\mathscr{F}_{\alpha_1,\alpha_2}$

Let $a(x,y,\xi,\zeta) = a'(x,y,\xi,\zeta) + a(\infty,\infty,\xi,\zeta)$ be a symbol, and, as previously,

$$\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)a'(t,u,\xi,\zeta)dtdu, \quad \forall x,y,\xi \neq 0, \zeta \neq 0 \in \mathbb{R}.$$

Let us define, for any $\phi \in \mathscr{S}^2(\mathbb{R})$ and $x,y \in \mathbb{R}$, the pseudo-differential operator $A(x,y,D'_{x,y})$ related to $\mathscr{F}_{\alpha_1,\alpha_2}$ [42] is defined
as follows

$$((A(x,y,D'_{x,y})\phi)(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)G_{\alpha_1,\alpha_2}(t,u)dtdu, \tag{14}$$

where the function $G_{\alpha_1,\alpha_2}(t,u)$ is given by

$$G_{\alpha_1,\alpha_2}(t,u) = a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{a'}_{\alpha_1,\alpha_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)d\xi d\eta. \tag{15}$$

4 The pseudo-differential operator $\mathscr{A}(x,y,D'_{x,y})$

We consider a symbol $a(x,y,\xi,\zeta)$. We introduce an operator $\mathscr{A}(x,y,D'_{x,y})$ of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ [42] by means of the formula

$$[\mathscr{A}(x,y,D'_{x,y})\phi](x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)\mathscr{H}_{\alpha_1,\alpha_2}(t,u)dtdu,$$

where, for $\phi \in \mathscr{S}$, the function $\mathscr{H}_{\alpha_1,\alpha_2}(t,u)$ is defined by the relation

$$\begin{aligned}\mathscr{H}_{\alpha_1,\alpha_2}(t,u) \quad = \quad & a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) + \overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2} \\ & \times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2,\end{aligned}$$

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$\forall \phi \in \mathscr{S}$ and $t \neq 0,\ u \neq 0 \in \mathbb{R}$.

5 Product and Commutators

Theorem 3. *If* $a(x,y,\xi,\eta),\ b(x,y,\xi,\eta) \in \Lambda$, *then* $c(x,y,\xi,\eta) = a(x,y,\xi,\eta)b(x,y,\xi,\eta)$ $\in \Lambda$.

Proof. Since $a(x,y,\xi,\eta),\ b(x,y,\xi,\eta) \in \mathbb{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$, obviously we have $c(x,y,\xi,\eta) \in \mathbb{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$. From the definition of the symbol class Λ, $\forall t_1 > 0,\ t_2 > 0$ we get

$$c(x,y,t_1\xi,t_2\eta) = a(x,y,t_1\xi,t_2\eta)b(x,y,t_1\xi,t_2\eta) = a(x,y,\xi,\eta)b(x,y,\xi,\eta) = c(x,y,\xi,\eta)$$

and that

$$\lim_{(x,y)\to(\infty,\infty)} a(x,y,\xi,\eta) = a(\infty,\infty,\xi,\eta), \qquad \lim_{(x,y)\to(\infty,\infty)} b(x,y,\xi,\eta) = b(\infty,\infty,\xi,\eta),$$

exist for $\xi,\ \eta \in \mathbb{R}-\{0\}$. Also,

$$\lim_{(x,y)\to(\infty,\infty)} c(x,y,\xi,\eta) = c(\infty,\infty,\xi,\eta) = a(\infty,\infty,\xi,\eta)b(\infty,\infty,\xi,\eta)$$

exists for $\xi,\ \eta \in \mathbb{R}-\{0\}$. Thus, if we put

$$c'(x,y,\xi,\eta) = c(x,y,\xi,\eta) - c(\infty,\infty,\xi,\eta), \tag{16}$$

we get

$$\begin{aligned} c(x,y,\xi,\eta) &= \big(a'(x,y,\xi,\eta) + a(\infty,\infty,\xi,\eta)\big)\big(b'(x,y,\xi,\eta) + b(\infty,\infty,\xi,\eta)\big) \\ &= a'(x,y,\xi,\eta)b'(x,y,\xi,\eta) + a(\infty,\infty,\xi,\eta)b'(x,y,\xi,\eta) \\ &\quad + a'(x,y,\xi,\eta)b(\infty,\infty,\xi,\eta) + a(\infty,\infty,\xi,\eta)b(\infty,\infty,\xi,\eta) \\ &= c'(x,y,\xi,\eta) + c(\infty,\infty,\xi,\eta), \end{aligned}$$

where

$$\begin{aligned} c'(x,y,\xi,\eta) &= a'(x,y,\xi,\eta)b'(x,y,\xi,\eta) + a(\infty,\infty,\xi,\eta)b'(x,y,\xi,\eta) \\ &\quad + a'(x,y,\xi,\eta)b(\infty,\infty,\xi,\eta). \end{aligned}$$

Further, we show that $c'(x,y,\xi,\eta)$ satisfies (8).

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Applying Leibniz's theorem, we have

$$
\begin{aligned}
& (1+x^2+y^2)^p\left|\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial \xi^m}\frac{\partial^n}{\partial \eta^n}c'(x,y,\xi,\eta)\right| \\
= \; & (1+x^2+y^2)^p\left|\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial \xi^m}\frac{\partial^n}{\partial \eta^n}a'(x,y,\xi,\eta)b'(x,y,\xi,\eta)\right| \\
& +(1+x^2+y^2)^p\left|\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial \xi^m}\frac{\partial^n}{\partial \eta^n}a(\infty,\infty,\xi,\eta)b'(x,y,\xi,\eta)\right| \\
& +(1+x^2+y^2)^p\left|\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial \xi^m}\frac{\partial^n}{\partial \eta^n}b(\infty,\infty,\xi,\eta)a'(x,y,\xi,\eta)\right| \\
\leq \; & \sum_{s_1=0}^{k}\binom{k}{s_1}\sum_{s_2=0}^{l}\binom{l}{s_2}\sum_{s_3=0}^{m}\binom{m}{s_3}\sum_{s_4=0}^{n}\binom{n}{s_4}(1+x^2+y^2)^p \\
& \times\left|\frac{\partial^{k-s_1}}{\partial x^{k-s_1}}\frac{\partial^{l-s_2}}{\partial y^{l-s_2}}\frac{\partial^{m-s_3}}{\partial \xi^{m-s_3}}\frac{\partial^{n-s_4}}{\partial \eta^{n-s_4}}a'(x,y,\xi,\eta)\right| \\
& \times\left|\frac{\partial^{s_1}}{\partial x^{s_1}}\frac{\partial^{s_2}}{\partial y^{s_2}}\frac{\partial^{s_3}}{\partial \xi^{s_3}}\frac{\partial^{s_4}}{\partial \eta^{s_4}}b'(x,y,\xi,\eta)\right| \\
& +\sum_{s_1=0}^{k}\binom{k}{s_1}\sum_{s_2=0}^{l}\binom{l}{s_2}\sum_{s_3=0}^{m}\binom{m}{s_3}\sum_{s_4=0}^{n}\binom{n}{s_4}(1+x^2+y^2)^p \\
& \times\left|\frac{\partial^{k-s_1}}{\partial x^{k-s_1}}\frac{\partial^{l-s_2}}{\partial y^{l-s_2}}\frac{\partial^{m-s_3}}{\partial \xi^{m-s_3}}\frac{\partial^{n-s_4}}{\partial \eta^{n-s_4}}a(\infty,\infty,\xi,\eta)\right| \\
& \times\left|\frac{\partial^{s_1}}{\partial x^{s_1}}\frac{\partial^{s_2}}{\partial y^{s_2}}\frac{\partial^{s_3}}{\partial \xi^{s_3}}\frac{\partial^{s_4}}{\partial \eta^{s_4}}b'(x,y,\xi,\eta)\right| \\
& +\sum_{s_1=0}^{k}\binom{k}{s_1}\sum_{s_2=0}^{l}\binom{l}{s_2}\sum_{s_3=0}^{m}\binom{m}{s_3}\sum_{s_4=0}^{n}\binom{n}{s_4}(1+x^2+y^2)^p \\
& \times\left|\frac{\partial^{k-s_1}}{\partial x^{k-s_1}}\frac{\partial^{l-s_2}}{\partial y^{l-s_2}}\frac{\partial^{m-s_3}}{\partial \xi^{m-s_3}}\frac{\partial^{n-s_4}}{\partial \eta^{n-s_4}}a'(x,y,\xi,\eta)\right| \\
& \times\left|\frac{\partial^{s_1}}{\partial x^{s_1}}\frac{\partial^{s_2}}{\partial y^{s_2}}\frac{\partial^{s_3}}{\partial \xi^{s_3}}\frac{\partial^{s_4}}{\partial \eta^{s_4}}b(\infty,\infty,\xi,\eta)\right| \\
\leq \; & \sum_{s_1=0}^{k}\binom{k}{s_1}\sum_{s_2=0}^{l}\binom{l}{s_2}\sum_{s_3=0}^{m}\binom{m}{s_3}\sum_{s_4=0}^{n}\binom{n}{s_4} \\
& \times\left[\mathbb{C}_{1,p,k,l,m,n}+\mathbb{C}_{2,p,k,l,m,n}+\mathbb{C}_{3,p,k,l,m,n}\right] \\
\leq \; & \mathbb{M}_{p,k,l,m,n}.
\end{aligned}
$$

Hence from (8) and (16), we obtain $c(x,y,\xi,\eta)\in\Lambda$. □

Let $A(x,y,D'_{x,y})$, $B(x,y,D'_{x,y})$ and $C(x,y,D'_{x,y})$, defined as in (14), be the corresponding to the symbols $a(x,y,\xi,\eta)$, $b(x,y,\xi,\eta)$ and $c(x,y,\xi,\eta)$, respectively, in Λ. Then we get

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$$\begin{aligned}
A(x,y,D'_{x,y}) &= A(\infty,\infty,D'_{x,y}) + A'(x,y,D'_{x,y}),\\
B(x,y,D'_{x,y}) &= B(\infty,\infty,D'_{x,y}) + B'(x,y,D'_{x,y}),\\
C(x,y,D'_{x,y}) &= A(x,y,D'_{x,y})B(x,y,D'_{x,y})\\
&= (A(\infty,\infty,D'_{x,y}) + A'(x,y,D'_{x,y}))(B(\infty,\infty,D'_{x,y}) + B'(x,y,D'_{x,y}))\\
&= A(\infty,\infty,D'_{x,y})B(\infty,\infty,D'_{x,y}) + (A(\infty,\infty,D'_{x,y})B'(x,y,D'_{x,y}))\\
&\quad + A'(x,y,D'_{x,y})B(\infty,\infty,D'_{x,y}) + A'(x,y,D'_{x,y})B'(x,y,D'_{x,y}). \qquad (17)
\end{aligned}$$

Furthermore, we denote $a(\infty,\infty,\xi,\eta)b(\infty,\infty,\xi,\eta) = \gamma(\xi,\eta)$, $a'(x,y,\xi,\eta)b'(x,y,\xi,\eta) = p(x,y,\xi,\eta)$,
$a(\infty,\infty,\xi,\eta)b'(x,y,\xi,\eta) = p_1(x,y,\xi,\eta)$, and $a'(x,y,\xi,\eta)b(\infty,\infty,\xi,\eta) = p_2(x,y,\xi,\eta)$. Then

$$C(x,y,D'_{x,y}) = \Gamma(D'_{x,y}) + P(x,y,D'_{x,y}) + P_1(x,y,D'_{x,y}) + P_2(x,y,D'_{x,y}). \qquad (18)$$

Lemma 1. *We have*

$$\Gamma(D'_{x,y})\phi = A(\infty,\infty,D'_{x,y})B(\infty,\infty,D'_{x,y})\phi,$$

for $\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$.

Proof.

$$\begin{aligned}
\mathscr{F}_{\alpha_1,\alpha_2}[\Gamma(D'_{x,y})\phi](\xi,\eta) &= \gamma(\xi,\eta)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\\
&= a(\infty,\infty,\xi,\eta)b(\infty,\infty,\xi,\eta)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\\
&= a(\infty,\infty,\xi,\eta)\mathscr{F}_{\alpha_1,\alpha_2}[B(\infty,\infty,D'_{x,y})\phi](\xi,\eta)\\
&= \mathscr{F}_{\alpha_1,\alpha_2}[\big(A(\infty,\infty,D'_{x,y})B(\infty,\infty,D'_{x,y})\big)\phi](\xi,\eta);
\end{aligned}$$

hence, by coupled fractional Fourier's inversion formula, holds in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$, we achieve the result of Lemma 1. □

Lemma 2. *We have*

$$P_1(x,y,D'_{x,y})\phi = A(\infty,\infty,D'_{x,y})B'(x,y,D'_{x,y})\phi,$$

for $\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$.

Proof. In fact, we get

$$\begin{aligned}
\mathscr{F}_{\alpha_1,\alpha_2}[P_1(x,y,D'_{x,y})\phi](\xi,\eta) &= p_1(x,y,\xi,\eta)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\\
&= a(\infty,\infty,\xi,\eta)b'(x,y,\xi,\eta)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\\
&= a(\infty,\infty,\xi,\eta)\mathscr{F}_{\alpha_1,\alpha_2}[B(\infty,\infty,D'_{x,y})\phi](\xi,\eta)\\
&= \mathscr{F}_{\alpha_1,\alpha_2}[A(\infty,\infty,D'_{x,y})B(\infty,\infty,D'_{x,y})\phi](\xi,\eta).
\end{aligned}$$

Hence, by inversion formula of coupled fractional Fourier transform, we get the result of Lemma 2. □

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Lemma 3. *We have*

$$P_2(x,y,D'_{x,y})\phi = B(\infty,\infty,D'_{x,y})A'(x,y,D'_{x,y})\phi.$$

for $\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$,

Proof. The proof is similar to the one of Lemma 2. □

Now, using (17) and (18), let us examine here the difference

$$\begin{aligned}
& A(x,y,D'_{x,y})B(x,y,D'_{x,y}) - C(x,y,D'_{x,y}) \\
= \; & A(\infty,\infty,D'_{x,y})B(\infty,\infty,D'_{x,y}) + A'(x,y,D'_{x,y})B(\infty,\infty,D'_{x,y} \\
& + A(\infty,\infty,D'_{x,y})B'(x,y,D'_{x,y}) + A'(x,y,D'_{x,y})B'(x,y,D'_{x,y}) \\
& - A(\infty,\infty,D'_{x,y})B(\infty,\infty,D'_{x,y}) - A(\infty,\infty,D'_{x,y})B'(x,y,D'_{x,y}) \\
& - B(\infty,\infty,D'_{x,y})A'(x,y,D'_{x,y}) - P(x,y,D'_{x,y}) \\
= \; & [A'(x,y,D'_{x,y}),B(\infty,\infty,D'_{x,y})] + A'(x,y,D'_{x,y})B'(x,y,D'_{x,y})] \\
& - P(x,y,D'_{x,y}),
\end{aligned}$$

where

$$[A'(x,y,D'_{x,y}),B(\infty,\infty,D'_{x,y})] = A'(x,y,D'_{x,y})B(\infty,\infty,D'_{x,y}) - B(\infty,\infty,D'_{x,y})A'(x,y,D'_{x,y}).$$

Here $[\ .\ ,\ .\ ]$ denotes the commutators between the two operators $A'(x,y,D'_{x,y})$ and $B(\infty,\infty,D'_{x,y})$. The term $P(x,y,D'_{x,y})$ represents the pseudo-differential operator associated with the symbol $p(x,y,\xi,\eta) = a'(x,y,\xi,\eta)b'(x,y,\xi,\eta)$.

Theorem 4. *We obtain the relation*

$$\|[A'(x,y,D'_{x,y}),B(\infty,\infty,D'_{x,y})]\phi\|_s \le C(\alpha_1,\alpha_2)\|\phi\|_{s-1}, \quad \forall\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}), \quad \forall s \in \mathbb{R}$$

for a certain constant $C(\alpha_1,\alpha_2)$, *where* $[\ .\ ,\ .\ ]$ *denotes the commutators between the two operators* $A'(x,y,D'_{x,y})$ *and* $B(\infty,\infty,D'_{x,y})$.

Proof. In fact, we use the formula

$$\begin{aligned}
& \mathscr{F}_{\alpha_1,\alpha_2}\left[A'(x,y,D'_{x,y})B(\infty,\infty,D'_{x,y})\phi\right](t,u) \\
= \; & \overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda^2)\cot\alpha_1 + i(u\lambda_2-\lambda_2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u) \\
& \times\mathscr{F}_{\alpha_1,\alpha_2}\left[B(\infty,\infty,D'_{x,y})\phi\right](\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\
= \; & \overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda^2)\cot\alpha_1 + i(u\lambda_2-\lambda_2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u) \\
& \times b(\infty,\infty,\lambda_1,\lambda_2)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2.
\end{aligned}$$

Besides,

$$\begin{aligned}
& \mathscr{F}_{\alpha_1,\alpha_2}\left[B(\infty,\infty,D'_{x,y})A'(x,y,D'_{x,y})\phi\right](t,u) \\
= \; & b(\infty,\infty,t,u)\mathscr{F}_{\alpha_1,\alpha_2}\left[A'(x,y,D'_{x,y})\phi\right](t,u) \\
= \; & b(\infty,\infty,t,u)\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda^2)\cot\alpha_1 + i(u\lambda_2-\lambda_2)cot\alpha_2} \\
& \times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2
\end{aligned}$$

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and hence

$$
\begin{aligned}
& \mathscr{F}_{\alpha_1,\alpha_2}\left[A'(x,y,D'_{x,y}),B(\infty,\infty,D'_{x,y})\phi\right](t,u) \\
= \; & \overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda^2)\cot\alpha_1+i(u\lambda_2-\lambda_2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u) \\
& \times[b(\infty,\infty,\lambda_1,\lambda_2)-b(\infty,\infty,t,u)]\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2.
\end{aligned}
$$

From Theorem 2 (i), we get

$$
\begin{aligned}
|b(\infty,\infty,\lambda_1,\lambda_2)-b(\infty,\infty,t,u)| & \leq C\frac{|\lambda_1-t|+|\lambda_2-u|}{|\lambda_1|+|\lambda_2|+|t|+|u|} \\
& \leq C\frac{(1+|\lambda_1-t|^2)^{\frac{1}{2}}(1+|\lambda_2-u|^2)^{\frac{1}{2}}}{(1+|t|^2)^{\frac{1}{2}}(1+|u|^2)^{\frac{1}{2}}}. \quad (19)
\end{aligned}
$$

Hence, we are obliged to estimate the norm $L^2(\mathbb{R}\times\mathbb{R})$ of the expression

$$
\begin{aligned}
\mathscr{U}_s(t,u) = \; & (1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda^2)\cot\alpha_1+i(u\lambda_2-\lambda_2)cot\alpha_2} \\
& \times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)[b(\infty,\infty,\lambda_1,\lambda_2)-b(\infty,\infty,t,u)] \\
& \times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2.
\end{aligned}
$$

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We get

$$
\begin{aligned}
& |\mathscr{U}_s(t,u)| \\
\leq\ & |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}} |\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1, u-\lambda_2, t, u)| \\
& \times |b(\infty,\infty,\lambda_1,\lambda_2) - b(\infty,\infty,t,u)||\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)| d\lambda_1 d\lambda_2. \\
=\ & |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(1+|t|^2)^{\frac{s}{2}}}{(1+|\lambda_1|^2)^{\frac{s}{2}}} \frac{(1+|u|^2)^{\frac{s}{2}}}{(1+|\lambda_2|^2)^{\frac{s}{2}}} |\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1, u-\lambda_2, t, u)| \\
& \times |b(\infty,\infty,\lambda_1,\lambda_2) - b(\infty,\infty,t,u)||\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|(1+|\lambda_1|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{s}{2}} d\lambda_1 d\lambda_2. \\
\leq\ & \mathscr{D}_l |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(1+|t|^2)^{\frac{s}{2}}}{(1+|\lambda_1|^2)^{\frac{s}{2}}} \frac{(1+|u|^2)^{\frac{s}{2}}}{(1+|\lambda_2|^2)^{\frac{s}{2}}} (1+|t-\lambda_1|^2\csc^2\alpha_1)^{-\frac{l}{2}} \\
& \times (1+|u-\lambda_2|^2\csc^2\alpha_2)^{-\frac{l}{2}} C \frac{(1+|\lambda_1-t|^2)^{\frac{1}{2}}(1+|\lambda_2-u|^2)^{\frac{1}{2}}}{(1+|t|^2)^{\frac{1}{2}}(1+|u|^2)^{\frac{1}{2}}} |\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2). \\
& \times |(1+|\lambda_1|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{s}{2}} d\lambda_1 d\lambda_2 \\
=\ & \mathscr{D}_l |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(1+|t|^2)^{\frac{s-1}{2}}}{(1+|\lambda_1|^2)^{\frac{s-1}{2}}} \frac{(1+|u|^2)^{\frac{s-1}{2}}}{(1+|\lambda_2|^2)^{\frac{s-1}{2}}} (1+|t-\lambda_1|^2\csc^2\alpha_1)^{-\frac{l}{2}} \\
& \times (1+|u-\lambda_2|^2\csc^2\alpha_2)^{-\frac{l}{2}} C (1+|\lambda_1-t|^2)^{\frac{1}{2}}(1+|\lambda_2-u|^2)^{\frac{1}{2}} |\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)| \\
& \times (1+|\lambda_1|^2)^{\frac{s-1}{2}}(1+|\lambda_2|^2)^{\frac{s-1}{2}} d\lambda_1 d\lambda_2 \\
\leq\ & \mathscr{D}_l C |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| 2^{|s-1|} \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|t-\lambda_1|^2)^{\frac{|s-1|}{2}} (1+|u-\lambda_2|^2)^{\frac{|s-1|}{2}} \\
& \times (1+|t-\lambda_1|^2\csc^2\alpha_1)^{-\frac{l}{2}} (1+|u-\lambda_2|^2\csc^2\alpha_2)^{-\frac{l}{2}} (1+|\lambda_1-t|^2)^{\frac{1}{2}}(1+|\lambda_2-u|^2)^{\frac{1}{2}} \\
& \times |\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|(1+|\lambda_1|^2)^{\frac{s-1}{2}}(1+|\lambda_2|^2)^{\frac{s-1}{2}} d\lambda_1 d\lambda_2 \\
=\ & \mathscr{D}_l C |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| 2^{|s-1|} \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|t-\lambda_1|^2)^{\frac{|s-1|+1}{2}} (1+|u-\lambda_2|^2)^{\frac{|s-1|+1}{2}} \\
& \times (1+|t-\lambda_1|^2\csc^2\alpha_1)^{-\frac{l}{2}} (1+|u-\lambda_2|^2\csc^2\alpha_2)^{-\frac{l}{2}} |\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|(1+|\lambda_1|^2)^{\frac{s-1}{2}} \\
& \times (1+|\lambda_2|^2)^{\frac{s-1}{2}} d\lambda_1 d\lambda_2 \\
=\ & \mathscr{D}_l C |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| 2^{|s-1|} \int_{\mathbb{R}}\int_{\mathbb{R}} g_{\alpha_1,\alpha_2}(t-\lambda_1, u-\lambda_2) f_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2) d\lambda_1 d\lambda_2 \quad (say) \\
=\ & \mathscr{D}_l C |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| 2^{|s-1|} (g_{\alpha_1,\alpha_2} * f_{\alpha_1,\alpha_2})(t,u).
\end{aligned}
$$

If l is large, $g_{\alpha_1,\alpha_2} \in L_1(\mathbb{R}\times\mathbb{R})$. Also, since $\widehat{\phi}_{\alpha_1,\alpha_2} \in \mathscr{G}(\mathbb{R}\times\mathbb{R}),\ f_{\alpha_1,\alpha_2} \in L_2(\mathbb{R}\times\mathbb{R})$. Then $(g_{\alpha_1,\alpha_2} * f_{\alpha_1,\alpha_2})(t,u)$ belongs to $L_2(\mathbb{R}\times\mathbb{R})$ and the inequality

$$||g_{\alpha_1,\alpha_2} * f_{\alpha_1,\alpha_2}||_{L_2(\mathbb{R}\times\mathbb{R})} \leq ||g_{\alpha_1,\alpha_2}||_{L_1(\mathbb{R}\times\mathbb{R})} ||f_{\alpha_1,\alpha_2}||_{L_2(\mathbb{R}\times\mathbb{R})}.$$

It implies that

$$||\mathscr{U}_s||_{L_2(\mathbb{R}\times\mathbb{R})} \leq C_{\alpha_1,\alpha_2,l,s} ||\phi||_{s-1},$$

where $C_{\alpha_1,\alpha_2,l,s} = \mathscr{D}_l C |\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}| 2^{|s-1|} > 0$ is a constant.

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Therefore

$$\left|\left|[A'(x,y,D'_{x,y}),B(\infty,\infty,D'_{x,y})]\phi\right|\right|_s \leq C_{\alpha_1,\alpha_2,l,s}||\phi||_{s-1}, \quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}), \quad \forall s\in\mathbb{R}.$$

□

Theorem 5. *We have the relation*

$$||[A'(x,y,D'_{x,y})B'(x,y,D'_{x,y})-P(x,y,D'_{x,y})]\phi||_s \leq C_{s,\alpha_1,\alpha_2}||\phi||_s, \ \forall\phi\in\mathscr{G}(\mathbb{R}\times\mathbb{R}), \ \forall \textit{ real } s.$$

Proof. Let us consider the operator $P(x,y,D'_{x,y})$ associated with $p(x,y,\eta,\xi)$.

$$\begin{aligned} &\mathscr{F}_{\alpha_1,\alpha_2}[P(x,y,D'_{x,y})\phi](t,u) \\ = \ &\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)\cot\alpha_1+i(u\lambda_2-\lambda_2)^2\cot\alpha_2}\widehat{p}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u) \\ &\times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2. \end{aligned} \tag{20}$$

But we have for $p(x,y,\xi,\eta)=a'(xy,\xi,\eta)b'(xy,\xi,\eta)$ that

$$\begin{aligned} &(\mathscr{F}_{\alpha_1,\alpha_2}p)(t_1,t_2,t,u) \\ = \ &\overline{C}_{\alpha 1}\overline{C}_{\alpha 2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\mu_1 t-\mu_1^2)\cot\alpha_1+i(\mu_2 t-\mu_2^2)\cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t_1-\mu_1,t_2-\mu_2,t,u) \\ &\times\widehat{b'}_{\alpha_1,\alpha_2}(\mu_1,\mu_2,t,u)d\mu_1 d\mu_2. \end{aligned} \tag{21}$$

Now we compute $(\mathscr{F}_{\alpha_1,\alpha_2}p)(t-\lambda_1,u-\lambda_2,t,u)=\widehat{p}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)$ as follows:

$$\widehat{p}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u) \tag{22}$$

$$\begin{aligned} = \ &\overline{C}_{\alpha 1}\overline{C}_{\alpha 2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\mu_1 t-\mu_1^2)\cot\alpha_1+i(\mu_2 t-\mu_2^2)\cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1-\mu_1,u-\lambda_2-\mu_2,t,u) \\ &\times\widehat{b'}_{\alpha_1,\alpha_2}(\mu_1,\mu_2,t,u)d\mu_1 d\mu_2. \end{aligned} \tag{23}$$

Applying (22) in (20), we get

$$\begin{aligned} &\mathscr{F}_{\alpha_1,\alpha_2}[P(x,y,D'_{x,y})\phi](t,u) \\ = \ &\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)\cot\alpha_1+i(u\lambda_2-\lambda_2)^2\cot\alpha_2}\Bigg(\overline{C}_{\alpha 1}\overline{C}_{\alpha 2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\mu_1 t-\mu_1^2)\cot\alpha_1} \\ &\times e^{i(\mu_2 t-\mu_2^2)\cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1-\mu_1,u-\lambda_2-\mu_2,t,u)\widehat{b'}_{\alpha_1,\alpha_2}(\mu_1,\mu_2,t,u)d\mu_1 d\mu_2\Bigg) \\ &\times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\ = \ &(\overline{C}_{\alpha_1})^2(\overline{C}_{\alpha_2})^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i[(\lambda_1+\mu_1)t-(\mu_1^2+\lambda_1^2)]cot\alpha_1}e^{i[(\lambda_2+\mu_2)u-(\mu_2^2+\lambda_2^2)]cot\alpha_2} \\ &\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1-\mu_1,u-\lambda_2-\mu_2,t,u)\widehat{b'}_{\alpha_1,\alpha_2}(\mu_1,\mu_2,t,u)d\mu_1 d\mu_2 \\ &\times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2. \end{aligned}$$

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In the interior integral, we make; $\lambda_1+\mu_1=\tau_1,\ d\lambda_1=d\tau_1;\ \lambda_2+\mu_2=\tau_2,\ d\lambda_2=d\tau_2;$ it follows

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}[P(x,y,D'_{x,y})\phi](t,u)\\
=\ &(\overline{C}_{\alpha_1})^2(\overline{C}_{\alpha_2})^2\int_{\mathbb{R}}\int_{\mathbb{R}}\Bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i[\tau_1 t-\{(\tau_1-\mu_1)^2+\mu_1^2\}]\cot\alpha_1+i[\tau_1 t-\{(\tau_1-\mu_1)^2+\mu_1^2\}]\cot\alpha_2}\\
&\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\tau_1-\mu_1,\tau_2-\mu_2)d\tau_1d\tau_2\Bigg)\\
&\times\widehat{b'}_{\alpha_1,\alpha_2}(\mu_1,\mu_2,t,u)d\mu_1d\mu_2.
\end{aligned}$$

And once more in the interior, we make; $\tau_1-\mu_1=\vartheta_1,\ \tau_2-\mu_2=\vartheta_2;\ -d\mu_1=d\vartheta_1$ and $-d\mu_2=d\vartheta_2$. We have now

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}[P(x,y,D'_{x,y})\phi](t,u)\\
=\ &(\overline{C}_{\alpha_1})^2(\overline{C}_{\alpha_2})^2\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i[\tau_1 t-\vartheta_1^2-(\tau_1-\vartheta_1)^2]\cot\alpha_1+i[\tau_2 t-\vartheta_2^2-(\tau_2-\vartheta_2)^2]\cot\alpha_2}\\
\times\ &\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\Bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\vartheta_1,\tau_2-\vartheta_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\vartheta_1,\vartheta_2)d\vartheta_1d\vartheta_2\Bigg)\\
\times\ &d\tau_1d\tau_2\\
=\ &(\overline{C}_{\alpha_1})^2(\overline{C}_{\alpha_2})^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i[\tau_1 t-\vartheta_1^2-(\tau_1-\vartheta_1)^2]\cot\alpha_1+i[\tau_2 t-\vartheta_2^2-(\tau_2-\vartheta_2)^2]\cot\alpha_2}\\
&\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\vartheta_1,\tau_2-\vartheta_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\vartheta_1,\vartheta_2)d\vartheta_1d\vartheta_2d\tau_1d\tau_2.
\end{aligned}$$

Now we replace $\vartheta_1=\lambda_1$ and $\vartheta_2=\lambda_2$. Then we get

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}[P(x,y,D'_{x,y})\phi](t,u)\\
=\ &(\overline{C}_{\alpha_1})^2(\overline{C}_{\alpha_2})^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i[\tau_1 t-\lambda_1^2-(\tau_1-\lambda_1)^2]\cot\alpha_1+i[\tau_2 t-\lambda_2^2-(\tau_2-\lambda_2)^2]\cot\alpha_2}\\
&\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1d\lambda_2d\tau_1d\tau_2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}[(A'(x,y,D'_{x,y})B'(x,y,D'_{x,y}))\phi](t,u)\\
=\ &\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(t\tau_1-\tau_1^2)cot\alpha_1+i(u\tau_2-\tau_2^2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\\
&\times\mathscr{F}_{\alpha_1,\alpha_2}[B'(x,y,D'_{x,y})\phi](\tau_1,\tau_2)d\tau_1d\tau_2
\end{aligned}$$

and besides

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}[B'(x,y,D'_{x,y})\phi](\tau_1,\tau_2)\\
=\ &\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(\tau_1\lambda_1-\lambda_1^2)cot\alpha_1+i(\tau_2\lambda_2-\lambda_2^2)cot\alpha_2}\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,\tau_1,\tau_2)\\
&\times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1d\lambda_2
\end{aligned}$$

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and hence we shall obtain

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}[(A'(x,y,D'_{x,y})B'(x,y,D'_{x,y}))\phi](t,u)\\
=\ &\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\tau_1-\tau_1^2)cot\alpha_1+i(u\tau_2-\tau_2^2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\\
&\times\Bigg(\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\tau_1\lambda_1-\lambda_1^2)cot\alpha_1+i(\tau_2\lambda_2-\lambda_2^2)cot\alpha_2}\\
&\times\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,\tau_1,\tau_2)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\Bigg)d\tau_1 d\tau_2.
\end{aligned}$$

□

The absolute convergence of the fourth integrals here considered result from estimates

$$\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\leq \mathbb{M}_p(1+(t-\tau_1)^2csc^2\alpha+(u-\tau_2)^2csc^2\alpha_2)^{-p} \quad (24)$$

$$\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,\tau_1,\tau_2)\leq \mathbb{M}_p(1+(\tau_1-\lambda_1)^2csc^2\alpha+(\tau_2-\lambda_2)^2csc^2\alpha_2)^{-p}, \quad (25)$$

where $p=1,\ 2,\ 3,\ldots$to ∞. Therefore, we can express the difference

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}[(A'(x,y,D'_{x,y})B'(x,y,D'_{x,y})-P(x,y,D'_{x,y}))\phi](t,u)\\
=\ &(\overline{C}_{\alpha_1})^2(\overline{C}_{\alpha_2})^2\int_{\mathbb{R}^4}\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\Bigg[e^{i(\tau_1\lambda_1-\lambda_1^2)cot\alpha_1+i(\tau_2\lambda_2-\lambda_2^2)cot\alpha_2}\\
&\times\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,\tau_1,\tau_2)-e^{i[\tau_1 t-\lambda_1^2-(\tau_1-\lambda_1)^2]\cot\alpha_1+i[\tau_2 t-\lambda_2^2-(\tau_2-\lambda_2)^2]\cot\alpha_2}\\
&\times\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,t,u)\Bigg]\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\tau_1 d\tau_2.
\end{aligned}$$

Let us examine here the norm $L_2(\mathbb{R}\times\mathbb{R})$ of the expression

$$\begin{aligned}
&\mathscr{U}_s(t,u)\\
=\ &(\overline{C}_{\alpha_1})^2(\overline{C}_{\alpha_2})^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u)\\
&\times\Bigg[e^{i(\tau_1\lambda_1-\lambda_1^2)cot\alpha_1+i(\tau_2\lambda_2-\lambda_2^2)cot\alpha_2}\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,\tau_1,\tau_2)-e^{i[\tau_1 t-\lambda_1^2-(\tau_1-\lambda_1)^2]\cot\alpha_1}\\
&\times e^{i[\tau_2 t-\lambda_2^2-(\tau_2-\lambda_2)^2]\cot\alpha_2}\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,t,u)\Bigg]\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\tau_1 d\tau_2.
\end{aligned}$$

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$$
\begin{aligned}
& |\mathscr{U}_s(t,u)| \\
\leq\ & |\overline{C}_{\alpha_1}|^2|\overline{C}_{\alpha_2}|^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}|\widehat{a'}_{\alpha_1,\alpha_2}(t-\tau_1,u-\tau_2,t,u) \\
& \times\Big[|\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,\tau_1,\tau_2)|+|\widehat{b'}_{\alpha_1,\alpha_2}(\tau_1-\lambda_1,\tau_2-\lambda_2,t,u)|\Big] \\
& \times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\tau_1 d\tau_2 \\
\leq\ & |\overline{C}_{\alpha_1}|^2|\overline{C}_{\alpha_2}|^2 2\mathbb{M}_p^2\int_{\mathbb{R}^4}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}(1+(t-\tau_1)^2csc^2\alpha+(u-\tau_2)^2csc^2\alpha_2)^{-p} \\
& \times(1+(\tau_1-\lambda_1)^2csc^2\alpha+(\tau_2-\lambda_2)^2csc^2\alpha_2)^{-p}\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\tau_1 d\tau_2 \\
=\ & \mathscr{T}_s(t,u)\quad \text{(say)}.
\end{aligned}
$$

Firstly we want to show that

$$||\mathscr{U}_s||_{L_2(\mathbb{R}\times\mathbb{R})}\leq||\mathscr{T}_s||_{L_2(\mathbb{R}\times\mathbb{R})}.$$

For,

$$
\begin{aligned}
& |\mathscr{T}_s(t,u)| \\
=\ & |\overline{C}_{\alpha_1}|^2|\overline{C}_{\alpha_2}|^2 2\mathbb{M}_p^2\int_{\mathbb{R}^4}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}(1+(t-\tau_1)^2csc^2\alpha+(u-\tau_2)^2csc^2\alpha_2)^{-p} \\
& \times(1+(\tau_1-\lambda_1)^2csc^2\alpha+(\tau_2-\lambda_2)^2csc^2\alpha_2)^{-p}\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\tau_1 d\tau_2.
\end{aligned}
$$

Let us denote now:

$$
\begin{aligned}
& \mathscr{H}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2) \\
& =(1+(t-\tau_1)^2csc^2\alpha+(u-\tau_2)^2csc^2\alpha_2)^{-p}(1+(\tau_1-\lambda_1)^2csc^2\alpha+(\tau_2-\lambda_2)^2csc^2\alpha_2)^{-p},
\end{aligned}
$$

$$
\begin{aligned}
& \mathscr{L}_s(t,u,\tau_1,\tau_2) \\
=\ & \frac{(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}}{(1+|\tau_1|^2)^{\frac{s}{2}}(1+|\tau_2|^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathscr{H}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2.
\end{aligned}
$$

We remark that it follows, $\forall\ t,\ u\in\mathbb{R}$

$$
\begin{aligned}
|\mathscr{T}_s(t,u)| \leq\ & 2\mathbb{M}_p^2|\overline{C}_{\alpha_1}|^2|\overline{C}_{\alpha_2}|^2\int_{\mathbb{R}^4}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\mathscr{H}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2) \\
& \times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 d\tau_1 d\tau_2.
\end{aligned}
$$

Therefore, we have only to prove the inequality

$$
\begin{aligned}
& \left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^4}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\mathscr{H}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)|\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|d\lambda_1 d\lambda_2 d\tau_1 d\tau_2\right)^2 dudt\right)^{\frac{1}{2}} \\
\leq\ & C_{p,s,\alpha_1,\alpha_1}||\phi||_s,\quad \forall\ \phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}).
\end{aligned}
$$

In order to do that we shall prove here a more general result, which is given in the following Lemma 4.

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Lemma 4. *Let $\mathscr{R}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2) > 0$ be the function such that $\int_{\mathbb{R}}\int_{\mathbb{R}} \mathscr{R}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 < \infty$ for every t,u,τ_1,τ_2 fixed in $\mathbb{R}$.*
Firstly we denote

$$\rho_s(t,u,\tau_1,\tau_2) = \frac{(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}}{(1+|\tau_1|^2)^{\frac{s}{2}}(1+|\tau_2|^2)^{\frac{s}{2}}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathscr{R}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2$$

and we suppose

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)dtdu \leq \mathscr{L}, \quad \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)d\tau_1 d\tau_2 \leq \mathscr{L}, \quad \forall\, t,\ u,\ \tau_1,\ \tau_2 \in \mathbb{R}.$$

Then, there is a constant $C_{s,p,\alpha_1,\alpha_2}$ such that the inequality

$$\begin{aligned}&\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^4}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\mathscr{H}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)|\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|d\lambda_1 d\lambda_2 d\tau_1 d\tau_2\right)^2 dudt\right)^{\frac{1}{2}}\\ \leq\ & C_{p,s,\alpha_1,\alpha_1}||\phi||_s, \quad \forall\ \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}).\end{aligned}$$

Proof of Lemma 4 : We remark that in fact we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\mathscr{R}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)|\widehat{\phi}_{\alpha_1,\alpha_2}(\tau_1,\tau_2)|d\lambda_1 d\lambda_2 d\tau_1 d\tau_2\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)(1+|\tau_1|^2)^{\frac{s}{2}}(1+|\tau_2|^2)^{\frac{s}{2}}|\widehat{\phi}_{\alpha_1,\alpha_2}(\tau_1,\tau_2)|d\tau_1 d\tau_2.\end{aligned}$$

Let us put

$$\vartheta(\tau_1,\tau_2) = (1+|\tau_1|^2)^{\frac{s}{2}}(1+|\tau_2|^2)^{\frac{s}{2}}|\widehat{\phi}_{\alpha_1,\alpha_2}(\tau_1,\tau_2)|$$

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)\vartheta(\tau_1,\tau_2)d\tau_1 d\tau_2\\ =\ & \int_{\mathbb{R}}\int_{\mathbb{R}} \sqrt{\rho_s(t,u,\tau_1,\tau_2)}\sqrt{\rho_s(t,u,\tau_1,\tau_2)}\vartheta(\tau_1,\tau_2)d\tau_1 d\tau_2\\ \leq\ & \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)d\tau_1 d\tau_2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)\vartheta^2(\tau_1,\tau_2)d\tau_1 d\tau_2\right)^{\frac{1}{2}}\\ \leq\ & \sqrt{\mathscr{L}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)\vartheta^2(\tau_1,\tau_2)d\tau_1 d\tau_2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\int_{\mathbb{R}^4}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\mathscr{R}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)|\widehat{\phi}_{\alpha_1,\alpha_2}(\tau_1,\tau_2)|d\lambda_1 d\lambda_2 d\tau_1 d\tau_2\\ &\leq \sqrt{\mathscr{L}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_s(t,u,\tau_1,\tau_2)\vartheta^2(\tau_1,\tau_2)d\tau_1 d\tau_2\right)^{\frac{1}{2}}. \qquad (26)\end{aligned}$$

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$$
\begin{aligned}
&||\int_{\mathbb{R}^4}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\mathscr{R}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)|\widehat{\phi}_{\alpha_1,\alpha_2}(\tau_1,\tau_2)|d\lambda_1 d\lambda_2 d\tau_1 d\tau_2||_{L_2(\mathbb{R}^2)}\\
&\leq \sqrt{\mathscr{L}}\left(\int_{\mathbb{R}^4}\rho_s(t,u,\tau_1,\tau_2)\vartheta^2(\tau_1,\tau_2)d\tau_1 d\tau_2\right)dtdu\Bigg)^{\frac{1}{2}}\\
&\leq \mathscr{L}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\rho_s(t,u,\tau_1,\tau_2)\vartheta^2(\tau_1,\tau_2)d\tau_1 d\tau_2\right)^{\frac{1}{2}}\\
&=\mathscr{L}||\phi||_s.
\end{aligned}
$$

We shall apply Lemma 4, taking $\mathscr{R}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)=\mathscr{H}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)$ and $\rho_s(t,u,\lambda_1,\lambda_2)=\mathscr{L}_s(t,u,\lambda_1,\lambda_2)$. We see that $\int_{\mathbb{R}}\int_{\mathbb{R}}\mathscr{H}(t,u,\tau_1,\tau_2,\lambda_1,\lambda_2)d\tau_1 d\tau_2<\infty$, and it remains to prove the following Lemma 5.

Lemma 5. *We have* $\int_{\mathbb{R}}\int_{\mathbb{R}}\mathscr{L}_s(t,u,\lambda_1,\lambda_2)dtdu\leq L$, $\int_{\mathbb{R}}\int_{\mathbb{R}}\mathscr{L}_s(t,u,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\leq L$.

In fact

$$
\begin{aligned}
&\mathscr{L}_s(t,u,\lambda_1,\lambda_2)\\
=\;&\frac{(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}}{(1+|\lambda_1|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+(t-\tau_1)^2csc^2\alpha+(u-\tau_2)^2csc^2\alpha_2)^{-p}\\
&\times(1+(\tau_1-\lambda_1)^2csc^2\alpha+(\tau_2-\lambda_2)^2csc^2\alpha_2)^{-p}d\tau_1 d\tau_2\\
&\leq 2^{|s|}(1+|t-\lambda_1|^2)^{\frac{|s|}{2}}(1+|u-\lambda_2|^2)^{\frac{|s|}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+(t-\tau_1)^2csc^2\alpha+(u-\tau_2)^2csc^2\alpha_2)^{-p}\\
&\times(1+(\tau_1-\lambda_1)^2csc^2\alpha+(\tau_2-\lambda_2)^2csc^2\alpha_2)^{-p}d\tau_1 d\tau_2.
\end{aligned}
$$

Now from [29], we have

$$
\begin{aligned}
(1+|t-\lambda_1|^2)^{\frac{|s|}{2}} &\leq]C'(1+|t-\tau_1|^2)^{\frac{|s|}{2}}(1+|\tau_1-\lambda_1|^2)^{\frac{|s|}{2}}\\
(1+|u-\lambda_2|^2)^{\frac{|s|}{2}} &\leq C"(1+|u-\tau_2|^2)^{\frac{|s|}{2}}(1+|\tau_2-\lambda_2|^2)^{\frac{|s|}{2}}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\mathscr{L}_s(t,u,\lambda_1,\lambda_2)\\
\leq\;& 2^{|s|}C'C''\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t-\tau_1|^2)^{\frac{|s|}{2}}(1+|\tau_1-\lambda_1|^2)^{\frac{|s|}{2}}(1+|u-\tau_2|^2)^{\frac{|s|}{2}}(1+|\tau_2-\lambda_2|^2)^{\frac{|s|}{2}}\\
\times\;&(1+(t-\tau_1)^2csc^2\alpha+(u-\tau_2)^2csc^2\alpha_2)^{-p}(1+(\tau_1-\lambda_1)^2csc^2\alpha+(\tau_2-\lambda_2)^2csc^2\alpha_2)^{-p}d\tau_1 d\tau_2.
\end{aligned}
$$

We denote at this stage:

$$
\begin{aligned}
&\chi(\nu_1,\nu_2)\\
=\;&\int_{\mathbb{R}^2}(1+|\omega_1|^2)^{\frac{|s|}{2}}(1+|\omega_2|^2)^{\frac{|s|}{2}}(1+(\omega_1)^2csc^2\alpha+(\omega_2)^2csc^2\alpha_2)^{-p}(1+|\nu_1-\omega_1|^2)^{\frac{|s|}{2}}\\
&\times(1+|\nu_2-\omega_2|^2)^{\frac{|s|}{2}}(1+(\nu_1-\omega_1)^2csc^2\alpha+(\nu_2-\omega_2)^2csc^2\alpha_2)^{-p}d\omega_1 d\omega_2,\quad \nu_1,\ \nu_2\in\mathbb{R}
\end{aligned}
$$

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where p is large enough. We see that $\chi(v_1, v_2) \in L_2(\mathbb{R}\times\mathbb{R})$ as convolution of four integrable functions. Hence, we have

$$\begin{aligned}
& \chi(t-\lambda_1, u-\lambda_2) \\
= & \int_{\mathbb{R}^2} (1+|\omega_1|^2)^{\frac{|s|}{2}} (1+|\omega_2|^2)^{\frac{|s|}{2}} (1+(\omega_1)^2 csc^2\alpha + (\omega_2)^2 csc^2\alpha_2)^{-p} (1+|t-\lambda_1-\omega_1|^2)^{\frac{|s|}{2}} \\
& \times (1+|u-\lambda_2-\omega_2|^2)^{\frac{|s|}{2}} (1+(t-\lambda_1-\omega_1)^2 csc^2\alpha + (u-\lambda_2-\omega_2)^2 csc^2\alpha_2)^{-p} d\omega_1 d\omega_2 \\
= & \int_{\mathbb{R}^2} (1+|t-\tau_1|^2)^{\frac{|s|}{2}} (1+|\tau_1-\lambda_1|^2)^{\frac{|s|}{2}} (1+|u-\tau_2|^2)^{\frac{|s|}{2}} (1+|\tau_2-\lambda_2|^2)^{\frac{|s|}{2}} \\
\times & \quad (1+(t-\tau_1)^2 csc^2\alpha + (u-\tau_2)^2 csc^2\alpha_2)^{-p} (1+(\tau_1-\lambda_1)^2 csc^2\alpha + (\tau_2-\lambda_2)^2 csc^2\alpha_2)^{-p} d\tau_1 d\tau_2
\end{aligned}$$

by substituting $\omega_1 = t - \tau_1$ and $\omega_2 = t - \tau_2$.
Hence we get

$$\mathscr{L}_s(t,u,\lambda_1,\lambda_2) \le C_s''' \chi(t-\lambda_1, u-\lambda_2)$$

and obviously:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi(t-\lambda_1, u-\lambda_2) dt du < \infty, \quad \int_{\mathbb{R}}\int_{\mathbb{R}} \chi(t-\lambda_1, u-\lambda_2) d\lambda_1 d\lambda_2 < \infty$$

which proves the Lemma 5.
Hence, for Lemma 4 we have that

$$||\mathscr{U}_s||_{L_2(\mathbb{R}\times\mathbb{R})} \le C_{s,\alpha_1,\alpha_2} ||\phi||_s, \quad \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}) \text{ and } \forall s \in \mathbb{R}$$

and this proves Theorem 5.

Corollary 1. *If $A(x,y,D'_{x,y})$, $B(x,y,D'_{x,y})$ are pseudo-differential operators, the commutator $[A(x,y,D'_{x,y}), B(x,y,D'_{x,y})]$ is of order ≤ 0.*

Proof. We have

$$\begin{aligned}
& A(x,y,D'_{x,y})B(x,y,D'_{x,y}) - ab(x,y,D'_{x,y}) \\
= & [A(x,y,D'_{x,y}), B(x,y,D'_{x,y})] + A'(x,y,D'_{x,y})B'(x,y,D'_{x,y}) - P(x,y,D'_{x,y})
\end{aligned}$$

is of order ≤ 0 by Theorem 4 and Theorem 5. □

Remark 1. *Let $A(x,y,D'_{x,y})$ be a pseudo-differential operator associated with the symbol $a(x,y,\xi,\eta)$. We assume that $\lambda' \in \mathbb{C}$ ia an eigenvalue of $A(x,y,D'_{x,y})$ such that $|a(x,y,\xi,\eta) - \lambda'| > \rho > 0, \quad \forall x,y \in \mathbb{R},\ |\xi| = 1$ and $|\eta| = 1$. Then, any eigen-vector ϕ_0 corresponding to eigen-value λ' ia a $\mathbb{C}^\infty-$function. Therefore, $b(x,y,\xi,\eta) = (a(x,y,\xi,\eta) - \lambda')^{-1}$ is also a symbol. If $B(x,y,D'_{x,y})$ is associated to it. We that $B(x,y,D'_{x,y})(A(x,y,D'_{x,y}) - \lambda' E) = E + U$, where E is the identity operator and U has order ≤ 0. It follows that $\theta = B(A - \lambda' E)\phi_0 = \phi_0 + U\phi_0$. It implies that $\phi_0 = -U\phi_0$, $\phi_0 \in L_2(\mathbb{R}\times\mathbb{R})$, it follows that $U\phi_0 \in \mathscr{H}^1(\mathbb{R}\times\mathbb{R})$ and $\phi_0 \in \mathscr{H}^1(\mathbb{R}\times\mathbb{R})$.*
Let us consider now that the operator $\mathscr{I}_s = (1+|D'_{x,y}|^2)^{\frac{s}{2}}$, defined by $\mathscr{F}_{\alpha_1,\alpha_2}[\mathscr{I}_s\phi](\xi,\eta) = (1+|\xi|^2 \csc^2\alpha_1 + |\eta|^2 \csc^2\alpha_2)\widehat{\phi}_{\alpha_1,\alpha_2}$, $\forall \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$.

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Theorem 6. *Let $A(x,y,D'_{x,y})$ be a pseudo-differential operator associated with a symbol $a(x,y,\xi,\eta)$. Then we have*

$$||[A(x,y,D'_{x,y}),\mathscr{I}_s]|| \leq C_{\alpha_1,\alpha_2}||\phi||_{s-3}, \quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

Proof. We have

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}\left[A(x,y,D'_{x,y})\mathscr{I}_s\phi\right](\xi,\eta)\\
= \;& a(\infty,\infty,\xi,\eta)(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\\
&+\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{a'}_{\alpha_1,\alpha_2}(\xi-t,\eta-u,\xi,\eta)(1+|t|^2csc^2\alpha_1+|u|^2\csc^2\alpha_2)^{\frac{s}{2}}\\
&\times\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu,
\end{aligned}$$

and also

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}\left[\mathscr{I}_sA(x,y,D'_{x,y})\phi\right](\xi,\eta)\\
= \;& (1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}\mathscr{F}_{\alpha_1,\alpha_2}\left[A(x,y,D'_{x,y})\phi\right](\xi,\eta)\\
= \;& (1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}a(\infty,\infty,\xi,\eta)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\\
&+\overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{a'}_{\alpha_1,\alpha_2}(\xi-t,\eta-u,\xi,\eta)(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}\\
&\times\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu,
\end{aligned}$$

and hence it can be deduced that

$$\begin{aligned}
&\mathscr{F}_{\alpha_1,\alpha_2}\big([A(x,y,D'_{x,y}),\mathscr{I}_s]\phi\big)(\xi,\eta)\\
= \;& \overline{C}_{\alpha_1}\overline{C}_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{a'}_{\alpha_1,\alpha_2}(\xi-t,\eta-u,\xi,\eta)\\
&\times\big[(1+|t|^2csc^2\alpha_1+|u|^2\csc^2\alpha_2)^{\frac{s}{2}}-(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}\big]\\
&\times\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu.
\end{aligned}$$

Here

$$\begin{aligned}
&\mathscr{U}_s(\xi,\eta)\\
= \;& \int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{a'}_{\alpha_1,\alpha_2}(\xi-t,\eta-u,\xi,\eta)\\
&\times\big[(1+|t|^2csc^2\alpha_1+|u|^2\csc^2\alpha_2)^{\frac{s}{2}}-(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}\big]\\
&\times\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu.
\end{aligned}$$

$$\begin{aligned}
&|\mathscr{U}_s(\xi,\eta)|\\
\leq \;& \mathbb{M}_p\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi-t|^2csc^2\alpha_1+|\eta-u|^2\csc^2\alpha_2)^{-p}\\
&\times|(1+|t|^2csc^2\alpha_1+|u|^2\csc^2\alpha_2)^{\frac{s}{2}}-(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}|\\
&\times|\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|dtdu. \qquad (27)
\end{aligned}$$

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Let us remark here the elementary inequality, for $0<\theta_1<1$ and $0<\theta_2<1$.

$$\begin{aligned}
&(1+|t+u+\theta_1(\xi-t)csc\alpha_1+\theta_2(\eta-u)csc\alpha_2|^2)^{\frac{|s-1|}{2}}\\
\leq\ & 2^{\frac{|s-1|}{2}}(1+|t|^2)^{\frac{|s-1|}{2}}(1+|u|^2)^{\frac{|s-1|}{2}}(1+|\theta_1(\xi-t)csc\alpha_1|^2)^{\frac{|s-1|}{2}}\\
&\times(1+|\theta_2(\eta-u)csc\alpha_2|^2)^{\frac{|s-1|}{2}}
\end{aligned}$$

and therefore, as $\frac{|s-1|}{2}$, $0<\theta_1<1;\ 0<\theta_2<1:$

$$\begin{aligned}
(1+|\theta_1(\xi-t)csc\alpha_1|^2)^{\frac{|s-1|}{2}} &\leq (1+|\xi-t|^2csc^2\alpha_1)^{\frac{|s-1|}{2}}\\
(1+|\theta_2(\eta-u)csc\alpha_2|^2)^{\frac{|s-1|}{2}} &\leq (1+|\eta-u|^2csc^2\alpha_1)^{\frac{|s-1|}{2}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(1+|t+u+\theta_1(\xi-t)csc\alpha_1+\theta_2(\eta-u)csc\alpha_2|^2)^{\frac{|s-1|}{2}}\\
\leq\ & 2^{\frac{|s-1|}{2}}(1+|t|^2)^{\frac{|s-1|}{2}}(1+|u|^2)^{\frac{|s-1|}{2}}(1+|\xi-t|^2csc^2\alpha_1)^{\frac{|s-1|}{2}}\\
&\times(1+|\eta-u|^2csc^2\alpha_2)^{\frac{|s-1|}{2}}.
\end{aligned}$$

By Taylor's formula, we have

$$\begin{aligned}
&(1+|t|^2csc^2\alpha_1+|u|^2\csc^2\alpha_2)^{\frac{s}{2}}-(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}\\
&=\left((t-\xi)csc\alpha_1+(u-\eta)\csc\alpha_2,\ grad(1+t^2\csc^2\alpha_1+u^2\csc^2\alpha_2)^{\frac{s}{2}}\right)_{t=\zeta_1,\ u=\zeta_2}
\end{aligned}$$

where $\zeta_1=t+\theta_1(\xi-t)csc\alpha_1;\ \zeta_2=u+\theta_2(\eta-u)csc\alpha_2$.
It implies that

$$\begin{aligned}
&|(1+|t|^2csc^2\alpha_1+|u|^2\csc^2\alpha_2)^{\frac{s}{2}}-(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}|\\
&=|(t-\xi)csc\alpha_1+(u-\eta)\csc\alpha_2||grad(1+t^2\csc^2\alpha_1+u^2\csc^2\alpha_2)^{\frac{s}{2}}_{t=\zeta_1,\ u=\zeta_2}|.
\end{aligned}$$

Since we have

$$\frac{\partial^2}{\partial t\partial u}(1+t^2\csc^2\alpha_1+u^2\csc^2\alpha_2)^{\frac{s}{2}}=s(s-2)tu\csc^2\alpha_1\csc^2\alpha_2(1+t^2\csc^2\alpha_1+u^2\csc^2\alpha_2)^{\frac{s-4}{2}}.$$

It implies that

$$\begin{aligned}
&|grad(1+t^2\csc^2\alpha_1+u^2\csc^2\alpha_2)^{\frac{s}{2}}_{t=\zeta_1,\ u=\zeta_2}|\\
=\ & s(s-2)tu\csc^2\alpha_1\csc^2\alpha_2(1+t^2\csc^2\alpha_1+u^2\csc^2\alpha_2)^{\frac{s-4}{2}}\\
\leq\ & s(s-2)(1+t^2\csc^2\alpha_1+u^2\csc^2\alpha_2)^{\frac{s-3}{2}}
\end{aligned}$$

and hence

$$\begin{aligned}
&|(1+|t|^2csc^2\alpha_1+|u|^2\csc^2\alpha_2)^{\frac{s}{2}}-(1+|\xi|^2csc^2\alpha_1+|\eta|^2\csc^2\alpha_2)^{\frac{s}{2}}|\\
&\leq|(t-\xi)csc\alpha_1+(u-\eta)\csc\alpha_2|s(s-2)(1+|t+u+\theta_1(\xi-t)csc\alpha_1+\theta_2(\eta-u)csc\alpha_2|^2)^{\frac{|s-1|}{2}}\\
&\leq s(s-2)(1+|t-\xi|^2csc^2\alpha_1+|u-\eta|^2csc\alpha_2)^{\frac{1}{2}}(1+|t|^2)^{\frac{s-3}{2}}(1+|u|^2)^{\frac{s-3}{2}}\\
&\ \times(1+|t-\xi|^2csc^2\alpha_1+|u-\eta|^2csc\alpha_2)^{\frac{s-3}{2}}. \qquad (28)
\end{aligned}$$

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From (27) and (28), we get

$$
\begin{aligned}
& |\mathscr{U}_s(\xi,\eta)| \\
\leq\ & \mathbb{M}_p \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|\xi-t|^2 csc^2\alpha_1 + |\eta-u|^2\csc^2\alpha_2)^{-p} \\
& \times s(s-2)(1+|t-\xi|^2 csc^2\alpha_1 + |u-\eta|^2 csc\alpha_2)^{\frac{s-2}{2}}(1+|t|^2)^{\frac{s-3}{2}}(1+|u|^2)^{\frac{s-3}{2}} \\
& \times|\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|dtdu \\
=\ & \mathbb{M}_p s(s-2)\int_{\mathbb{R}}\int_{\mathbb{R}} (1+|\xi-t|^2 csc^2\alpha_1 + |\eta-u|^2\csc^2\alpha_2)^{-p+\frac{s-2}{2}}(1+|t|^2)^{\frac{s-3}{2}}(1+|u|^2)^{\frac{s-3}{2}} \\
& \times|\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|dtdu \\
=\ & \int_{\mathbb{R}}\int_{\mathbb{R}} h_{\alpha_1,\ \alpha_2}(\xi-t,\ \eta-u) f_{\alpha_1,\ \alpha_2}(t,u)dtdu \quad \text{(say)} \\
=\ & (h_{\alpha_1,\alpha_2} * f_{\alpha_1,\alpha_2})(\xi,\eta).
\end{aligned}
$$

If p is large, $h_{\alpha_1,\alpha_2} \in L_1(\mathbb{R}\times\mathbb{R})$. Also, then $(h_{\alpha_1,\alpha_2} * f_{\alpha_1,\alpha_2})(\xi,\eta)$ belongs to $L_2(\mathbb{R}\times\mathbb{R})$ and the inequality

$$||h_{\alpha_1,\alpha_2} * f_{\alpha_1,\alpha_2}||_{L_2(\mathbb{R}\times\mathbb{R})} \leq ||h_{\alpha_1,\alpha_2}||_{L_1(\mathbb{R}\times\mathbb{R})}||f_{\alpha_1,\alpha_2}||_{L_2(\mathbb{R}\times\mathbb{R})}.$$

It implies that

$$||\mathscr{U}_s||_{L_2(\mathbb{R}\times\mathbb{R})} \leq C_{\alpha_1,\alpha_2}||\phi||_{s-3}.$$

Hence

$$||[A(x,y,D'_{x,y}),\mathscr{I}_s]|| \leq C_{\alpha_1,\alpha_2}||\phi||_{s-3}, \quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}),\ s\in\mathbb{R}.$$

This completes the proof of Theorem 6. □

6 Declarations

Disclaimer (Artificial Intelligence). Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

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