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COMPARATIVE ANALYSIS OF CHEBYSHEV PSEUDOSPECTRAL METHODS FOR FREE VIBRATION OF UNIFORM, TAPERED, AND FUNCTIONALLY GRADED RODS

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Abstract

This paper presents a comprehensive comparative analysis of three Chebyshev pseudospectral methods—Gauss (CG), Radau (CGR), and Lobatto (CGL)—for analyzing free longitudinal vibrations of rods with varying geometric and material properties. The study employs a novel trigonometric transformation approach that enables analytic differentiation, eliminating the need for numerical differentiation matrices. The methodology encompasses uniform rods, linearly tapered rods, and functionally graded (FG) rods under various boundary conditions including fixed-fixed (F-F), fixed-free (F-Fr), and free-free (Fr-Fr) configurations. A systematic investigation of accuracy, convergence rates, and computational efficiency is conducted for each method across different rod types. The trigonometric formulation provides several advantages: derivatives are computed analytically via closed-form expressions, the method is naturally well-conditioned for smooth functions achieving spectral accuracy, and collocation points automatically concentrate near boundaries. Results demonstrate that all three methods achieve exponential convergence with relative errors below 10^{-12} for fundamental frequencies using $N = 20$-30 collocation points. The CGL method emerges as the most practical choice for engineering applications due to its natural boundary condition implementation and superior robustness, while CG and CGR methods offer marginally higher accuracy per degree of freedom. The study provides detailed recommendations for method selection based on problem complexity, accuracy requirements, and computational constraints.

Keywords: Pseudospectral methods; Chebyshev collocation; Trigonometric transformation; Rod vibration; Functionally graded materials; Tapered rods; Free vibration analysis; Analytic differentiation

1. Introduction

The accurate prediction of natural frequencies and mode shapes in one-dimensional structural elements is fundamental to the design and analysis of mechanical, civil, and aerospace structures [1,2]. Longitudinal vibrations of rods, in particular, represent a classical problem in structural dynamics with applications ranging from drill strings and cutting tools to ultrasonic transducers and nano-mechanical resonators [3,4].

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While uniform rods with constant cross-sectional area and material properties admit exact analytical solutions [5], practical engineering structures often feature variable geometry (tapered configurations) or spatially varying material properties (functionally graded materials, FGMs) that necessitate numerical solution techniques.

Traditional numerical methods for structural vibration problems include the finite element method (FEM) [6,7], differential quadrature method (DQM) [8], and various semi-analytical techniques [9]. While FEM offers versatility and can handle complex geometries, it typically requires a large number of elements to achieve high accuracy in frequency predictions, particularly for higher modes. In contrast, spectral and pseudospectral methods have emerged as highly efficient alternatives, offering exponential convergence rates and achieving machine precision with relatively few degrees of freedom [10,11].

Pseudospectral methods based on Chebyshev polynomials have gained significant attention in computational mechanics due to their superior approximation properties for smooth functions on bounded domains [12,13]. The method leverages Chebyshev interpolation at specially chosen collocation points to construct spectral approximations. Three primary variants exist based on the choice of collocation points: Chebyshev-Gauss (CG), Chebyshev-Radau (CGR), and Chebyshev-Lobatto (CGL) methods [14,15]. Each variant offers distinct advantages in terms of boundary condition implementation, matrix properties, and computational efficiency.

Conventional pseudospectral implementations rely on numerical differentiation matrices to approximate spatial derivatives [16,17]. While effective, this approach can introduce numerical conditioning issues and requires careful matrix construction. An alternative formulation employing trigonometric transformation enables analytic differentiation, offering superior numerical stability and computational efficiency [18,19]. This trigonometric approach exploits the fundamental identity $T_n(\cos\theta) = \cos(n\theta)$ to transform the problem into the frequency domain where derivatives are computed analytically.

Recent advances in functionally graded materials have stimulated research on vibration analysis of structures with continuously varying material properties [20,21]. FGMs, characterized by gradual spatial variation in composition and properties, offer superior performance in high-temperature applications and eliminate stress concentrations at material interfaces [22]. The longitudinal vibration of FG rods has been investigated using various numerical techniques [23,24], though systematic comparisons of different pseudospectral approaches with analytic differentiation for this class of problems remain limited. Similarly, tapered rods with variable cross-sectional area represent another important structural configuration that has received considerable attention [25,26,27].

Despite the growing body of literature on pseudospectral methods and their applications to structural dynamics, several gaps persist. First, comprehensive comparative studies examining multiple pseudospectral variants (CG, CGR, CGL) with trigonometric transformation for rod vibration problems are scarce. Second, the relative performance of these methods with analytic differentiation across different rod types—uniform, tapered, and functionally graded—has not been systematically documented. Third, practical guidance on method selection based on problem characteristics, accuracy requirements, and computational resources when using the trigonometric formulation is lacking.

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This paper addresses these gaps by presenting a systematic comparative analysis of CG, CGR, and CGL pseudospectral methods employing trigonometric transformation and analytic differentiation, applied to free vibration analysis of three rod types: (i) uniform rods with constant properties, (ii) linearly tapered rods with variable cross-section, and (iii) power-law functionally graded rods with variable material properties. The specific objectives are to:

1. Develop and implement all three pseudospectral formulations using trigonometric transformation for each rod type and boundary condition
2. Conduct rigorous convergence studies and accuracy assessments against analytical solutions where available
3. Compare computational efficiency and implementation complexity among the methods
4. Investigate the influence of taper parameters and material gradation on natural frequencies
5. Demonstrate the advantages of analytic differentiation over numerical differentiation matrices
6. Provide comprehensive recommendations for method selection in engineering practice

The remainder of this paper is organized as follows. Section 2 presents the mathematical formulation of the governing equations for each rod type. Section 3 describes the trigonometric transformation methodology and pseudospectral discretization procedures for CG, CGR, and CGL methods with analytic differentiation. Section 4 presents comprehensive numerical results including convergence studies, frequency analyses, and parametric investigations. Section 5 provides a detailed comparison of methods and practical recommendations. Section 6 concludes the paper with key findings and suggestions for future work.

2. Problem Formulation

2.1. Governing Equations

Consider a rod of length L with longitudinal coordinate $z \in [0, L]$, cross-sectional area $A(z)$, Young's modulus $E(z)$, and mass density $\rho(z)$. For free harmonic vibrations with circular frequency ω and displacement amplitude $\phi(z)$, the governing equation for longitudinal motion is given by [5]:

$$\frac{d}{dz}\left[E(z)A(z)\frac{d\phi}{dz}\right] + \omega^2 \rho(z)A(z)\phi = 0 \quad \text{for } z \in (0, L) \tag{1}$$

This represents a variable-coefficient ordinary differential equation that admits exact analytical solutions only for specific combinations of $A(z)$, $E(z)$, and $\rho(z)$ [28].

2.2. Uniform Rods

For a uniform rod with constant properties $A(z) = A_0$, $E(z) = E_0$, and $\rho(z) = \rho_0$, Eq. (1) simplifies to:

$$E_0 A_0 \frac{d^2\phi}{dz^2} + \omega^2 \rho_0 A_0 \phi = 0 \tag{2}$$

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or equivalently:

$$\frac{d^2\phi}{dz^2} + \beta^2\phi = 0 \tag{3}$$

where $\beta = \omega/c$ and $c = \sqrt{E_0/\rho_0}$ is the longitudinal wave speed. The general solution is:

$$\phi(z) = C_1\sin(\beta z) + C_2\cos(\beta z) \tag{4}$$

For fixed-fixed (F-F) boundary conditions with $\phi(0) = 0$ and $\phi(L) = 0$, the frequency equation is $\sin(\beta L) = 0$, yielding eigenvalues:

$$\omega_n = \frac{n\pi c}{L}, \quad n = 1,2,3, \ldots \tag{5}$$

2.3. Tapered Rods

For a linearly tapered rod with $A(z) = A_0[1 + \alpha(z/L)]$, where α is the taper parameter, and constant material properties $E(z) = E_0$ and $\rho(z) = \rho_0$, Eq. (1) becomes:

$$\frac{d}{dz}\left[A(z)\frac{d\phi}{dz}\right] + \frac{\omega^2\rho_0}{E_0}A(z)\phi = 0 \tag{6}$$

Expanding the derivative:

$$A(z)\frac{d^2\phi}{dz^2} + \frac{dA}{dz}\frac{d\phi}{dz} + \frac{\omega^2\rho_0}{E_0}A(z)\phi = 0 \tag{7}$$

For the linear taper, $dA/dz = A_0\alpha/L$. This equation generally lacks closed-form analytical solutions and requires numerical methods [29].

2.4. Functionally Graded Rods

For FG rods with constant cross-section $A(z) = A_0$ and power-law variation in material properties:

$$E(z) = E_1 + (E_2 - E_1)\left(\frac{z}{L}\right)^p \tag{8a}$$

$$\rho(z) = \rho_1 + (\rho_2 - \rho_1)\left(\frac{z}{L}\right)^p \tag{8b}$$

where $p > 0$ is the power-law exponent, subscript 1 denotes properties at $z = 0$, and subscript 2 denotes properties at $z = L$. The governing equation becomes:

$$\frac{d}{dz}\left[E(z)\frac{d\phi}{dz}\right] + \omega^2\rho(z)\phi = 0 \tag{9}$$

Expanding:

$$E(z)\frac{d^2\phi}{dz^2} + \frac{dE}{dz}\frac{d\phi}{dz} + \omega^2\rho(z)\phi = 0 \tag{10}$$

where:

$$\frac{dE}{dz} = \frac{p(E_2 - E_1)}{L}\left(\frac{z}{L}\right)^{p-1} \tag{11}$$

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2.5. Boundary Conditions

Three primary boundary condition types are considered:

Fixed-Fixed (F-F):

$$\phi(0) = 0, \quad \phi(L) = 0 \tag{12}$$

Fixed-Free (F-Fr):

$$\phi(0) = 0, \quad EA\frac{d\phi}{dz}|_{z=L} = 0 \tag{13}$$

Free-Free (Fr-Fr):

$$EA\frac{d\phi}{dz}|_{z=0} = 0, \quad EA\frac{d\phi}{dz}|_{z=L} = 0 \tag{14}$$

3. Trigonometric Transformation Methodology

3.1. Two-Stage Coordinate Transformation

The pseudospectral method proposed in this work employs a two-stage coordinate transformation that enables analytic differentiation. This approach eliminates the need for numerical differentiation matrices and provides superior numerical conditioning.

3.1.1. First Stage: Physical to Computational Domain

The physical domain $z \in [0, L]$ is mapped to the computational domain $x \in [-1,1]$ via the affine transformation:

$$x = 2\frac{z}{L} - 1 \tag{15}$$

or inversely:

$$z = \frac{L}{2}(1 + x) \tag{16}$$

Under this transformation, derivatives transform as:

$$\frac{d\phi}{dz} = \frac{2}{L}\frac{d\phi}{dx}, \quad \frac{d^2\phi}{dz^2} = \frac{4}{L^2}\frac{d^2\phi}{dx^2} \tag{17}$$

3.1.2. Second Stage: Computational to Trigonometric Domain

We now invoke the trigonometric representation of Chebyshev polynomials. Define the transformation:

$$x = \cos\theta, \quad \theta \in [0, \pi] \tag{18}$$

The Chebyshev polynomial of the first kind $T_k(x)$ satisfies the fundamental identity:

$$T_k(x) = T_k(\cos\theta) = \cos(k\theta) \tag{19}$$

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This allows the approximate solution to be written as:

$$\phi(x) = \sum_{k=0}^{N} a_k\, T_k(x) = \sum_{k=0}^{N} a_k \cos(k\theta) \tag{20}$$

where $\{a_k\}_{k=0}^{N}$ are the spectral coefficients to be determined.

3.2. Analytic Differentiation in Trigonometric Form

The power of this approach is enhanced by analytic differentiation. Computing derivatives via the chain rule:

$$\frac{d\phi}{dx} = \frac{d\phi}{d\theta}\frac{d\theta}{dx} \tag{21}$$

Since $x = \cos\theta$, we have:

$$\frac{dx}{d\theta} = -\sin\theta \tag{22}$$

Therefore:

$$\frac{d\theta}{dx} = -\frac{1}{\sin\theta} \tag{23}$$

3.2.1. First Derivative

Let $\psi(\theta) = \phi\big(x(\theta)\big)$. The first derivative in trigonometric coordinates becomes:

$$\frac{d\phi}{dx} = -\frac{1}{\sin\theta}\frac{d\psi}{d\theta} \tag{24}$$

Substituting the expansion (20):

$$\frac{d\psi}{d\theta} = \frac{d}{d\theta}\left[\sum_{k=0}^{N} a_k \cos(k\theta)\right] = -\sum_{k=1}^{N} k\, a_k \sin(k\theta) \tag{25}$$

Therefore:

$$\frac{d\phi}{dx} = \frac{1}{\sin\theta}\sum_{k=1}^{N} k\, a_k \sin(k\theta) \tag{26}$$

3.2.2. Second Derivative

For the second derivative:

$$\frac{d^2\phi}{dx^2} = \frac{d}{dx}\left(\frac{d\phi}{dx}\right) = -\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi}{d\theta}\right) \tag{27}$$

Applying the product rule:

$$\frac{d^2\phi}{dx^2} = -\frac{1}{\sin\theta}\left[-\frac{\cos\theta}{\sin^2\theta}\frac{d\psi}{d\theta} + \frac{1}{\sin\theta}\frac{d^2\psi}{d\theta^2}\right] \tag{28}$$

Simplifying:

$$\frac{d^2\phi}{dx^2} = \frac{\cos\theta}{\sin^3\theta}\frac{d\psi}{d\theta} - \frac{1}{\sin^2\theta}\frac{d^2\psi}{d\theta^2} \tag{29}$$

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where:

$$\frac{d^2\psi}{d\theta^2} = -\sum_{k=1}^{N} k^2\, a_k \cos(k\theta) \tag{30}$$

3.3. Collocation Points

3.3.1. Chebyshev-Gauss-Lobatto (CGL) Points

The Chebyshev-Gauss-Lobatto collocation points in the trigonometric domain are:

$$\theta_j = \frac{j\pi}{N}, \quad j = 0,1,\dots,N \tag{31}$$

Corresponding to computational domain points:

$$x_j = \cos\theta_j = \cos\left(\frac{j\pi}{N}\right) \tag{32}$$

These points include both boundaries: $x_0 = 1$ (corresponding to $\theta_0 = 0$) and $x_N = -1$ (corresponding to $\theta_N = \pi$).

3.3.2. Chebyshev-Gauss-Radau (CGR) Points

The Chebyshev-Gauss-Radau points include one boundary:

$$\theta_j = \frac{2j\pi}{2N+1}, \quad j = 0,1,\dots,N \tag{33}$$

corresponding to:

$$x_j = \cos\left(\frac{2j\pi}{2N+1}\right) \tag{34}$$

3.3.3. Chebyshev-Gauss (CG) Points

The Chebyshev-Gauss points lie strictly in the interior:

$$\theta_j = \frac{(2j+1)\pi}{2N}, \quad j = 0,1,\dots,N-1 \tag{35}$$

corresponding to:

$$x_j = \cos\left(\frac{(2j+1)\pi}{2N}\right) \tag{36}$$

3.4. Discretized System

At each interior collocation point θ_j, the derivatives evaluate to:

$$\left.\frac{d\phi}{dx}\right|_{x_j} = \frac{1}{\sin\theta_j}\sum_{k=1}^{N} k\, a_k \sin(k\theta_j) \tag{37}$$

$$\left.\frac{d^2\phi}{dx^2}\right|_{x_j} = \frac{\cos\theta_j}{\sin^3\theta_j}\sum_{k=1}^{N} k\, a_k \sin(k\theta_j) - \frac{1}{\sin^2\theta_j}\sum_{k=1}^{N} k^2\, a_k \cos(k\theta_j) \tag{38}$$

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3.4.1. Uniform Rods

For uniform rods, substituting into the governing equation yields:

$$\frac{4}{L^2}\left[\frac{\cos\theta_j}{\sin^3\theta_j}\sum_{k=1}^{N} k\, a_k \sin(k\theta_j) - \frac{1}{\sin^2\theta_j}\sum_{k=1}^{N} k^2\, a_k \cos(k\theta_j)\right] + \omega^2\frac{\rho_0}{E_0}\sum_{k=0}^{N} a_k \cos(k\theta_j) = 0 \quad (39)$$

This can be written in matrix form as:

$$\mathbf{Ka} = \Omega^2\mathbf{Ma} \quad (40)$$

where $\Omega = \omega L\sqrt{\rho_0/E_0}$ is the dimensionless frequency, $\mathbf{a} = [a_0, a_1, \ldots, a_N]^T$ is the vector of spectral coefficients, and the stiffness and mass matrices are constructed from the analytic expressions above.

3.4.2. Tapered Rods

For tapered rods with $A(x) = A_0[1 + \alpha(x + 1)/2]$, the combined stiffness function is:

$$S(x_j) = E_0 A(x_j) \quad (41)$$

and its logarithmic derivative:

$$\frac{d\ln S}{dx}|_{x_j} = \frac{1}{S(x_j)}\frac{dS}{dx}|_{x_j} = \frac{\alpha}{2[1+\alpha(x_j+1)/2]} \quad (42)$$

The discretized equation becomes:

$$\frac{4}{L^2}\left[S(x_j)\frac{d^2\phi}{dx^2}|_{x_j} + \frac{dS}{dx}|_{x_j}\frac{d\phi}{dx}|_{x_j}\right] + \Omega^2\rho_0 A(x_j)\phi(x_j) = 0 \quad (43)$$

3.4.3. Functionally Graded Rods

For FG rods with power-law variation, evaluated material properties at collocation points are:

$$E(x_j) = E_1 + (E_2 - E_1)\left(\frac{x_j+1}{2}\right)^p \quad (44)$$

$$\rho(x_j) = \rho_1 + (\rho_2 - \rho_1)\left(\frac{x_j+1}{2}\right)^p \quad (45)$$

The derivative of Young’s modulus:

$$\frac{dE}{dx}|_{x_j} = \frac{p(E_2-E_1)}{2}\left(\frac{x_j+1}{2}\right)^{p-1} \quad (46)$$

3.5. Matrix Assembly

The stiffness matrix **K** and mass matrix **M** are assembled as follows:

For each interior collocation point j (excluding boundaries for Dirichlet conditions):

$$K_{jk} = -\frac{4}{L^2}\left[S(x_j)\mathcal{D}_{jk}^{(2)} + \frac{dS}{dx}|_{x_j}\mathcal{D}_{jk}^{(1)}\right] \quad (47)$$

$$M_{jk} = \rho(x_j)A(x_j)\delta_{jk} \quad (48)$$

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where $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ are the analytic differentiation operators in trigonometric form:

$$\mathcal{D}_{jk}^{(1)} = \frac{k\sin(k\theta_j)}{\sin\theta_j} \tag{49}$$

$$\mathcal{D}_{jk}^{(2)} = \frac{\cos\theta_j}{\sin^3\theta_j} k\sin(k\theta_j) - \frac{k^2\cos(k\theta_j)}{\sin^2\theta_j} \tag{50}$$

3.6. Boundary Condition Enforcement

Boundary conditions are enforced through algebraic modification of boundary rows in the discretized system.

3.6.1. Fixed-Fixed (F-F)

At $\theta_0 = 0$ (corresponding to $z = 0$):

$$\phi(x_0) = \sum_{k=0}^{N} a_k \cos(0) = \sum_{k=0}^{N} a_k = 0 \tag{51}$$

At $\theta_N = \pi$ (corresponding to $z = L$):

$$\phi(x_N) = \sum_{k=0}^{N} a_k \cos(k\pi) = \sum_{k=0}^{N} (-1)^k a_k = 0 \tag{52}$$

These conditions replace the first and last rows of the matrix system.

3.6.2. Fixed-Free (F-Fr)

At $\theta_0 = 0$: Fixed condition as above.

At $\theta_N = \pi$: Free condition

$$\frac{d\phi}{dx}|_{x_N} = \frac{1}{\sin\pi} \sum_{k=1}^{N} k\, a_k \sin(k\pi) = 0 \tag{53}$$

Special treatment is required at $\theta = \pi$ where $\sin\pi = 0$. Using L'Hôpital's rule or limiting process:

$$\lim_{\theta\to\pi} \frac{\sum_{k=1}^{N} k a_k \sin(k\theta)}{\sin\theta} = \sum_{k=1}^{N} k\, a_k \cos(k\pi)(-1)^k = \sum_{k=1}^{N} (-1)^{k+1} k a_k = 0 \tag{54}$$

3.6.3. Free-Free (Fr-Fr)

Both boundary conditions involve derivatives, treated similarly to the F-Fr case at each boundary.

3.7. Computational Advantages

This trigonometric formulation offers several distinct advantages over conventional differentiation matrix approaches:

1. No numerical differentiation: Derivatives computed analytically via closed-form trigonometric expressions, eliminating numerical differentiation errors.

2. Well-conditioned for smooth functions: Spectral accuracy $O(\exp(-cN))$ where c depends on solution smoothness, not problem size.
3. Automatic boundary clustering: Collocation points naturally concentrate near endpoints where boundary layers form, without requiring manual mesh refinement
4. Simple matrix assembly: Each matrix element computed directly from analytic formulas without numerical interpolation
5. Eigenvalue problem structure: Reduces to standard form $\mathrm{Ka} = \Omega^2\mathrm{Ma}$ amenable to efficient eigensolvers
6. Superior conditioning: Avoids numerical cancellation errors inherent in finite-difference approximations of derivatives

The resulting generalized eigenvalue problem $\mathbf{Ka} = \Omega^2\mathbf{Ma}$ is solved using standard eigenvalue decomposition algorithms to obtain natural frequencies $\omega_n = \Omega_n\sqrt{E_{\text{ref}}/\rho_{\text{ref}}}/L$ and mode shapes.

4. Numerical Results and Discussion

4.1. Reference Parameters

All numerical computations employ the following reference parameters unless otherwise stated:

Parameter	Symbol	Value	Units
Rod Length	L	1.0	m
Reference Area	A_0	1.0×10^{-4}	m²
Steel Young's Modulus	E_{steel}	200	GPa
Steel Density	ρ_{steel}	7850	kg/m³
Aluminum Young's Modulus	E_{al}	70	GPa
Aluminum Density	ρ_{al}	2700	kg/m³
Wave Speed (Steel)	c	5048.4	m/s

Table 1: Reference material and geometric parameters

4.2. Uniform Rod Results

4.2.1. Analytical Benchmark

For uniform steel rods with the parameters in Table 1, the exact natural frequencies for the first five modes under different boundary conditions are:

BC Type	f_1 (Hz)	f_2 (Hz)	f_3 (Hz)	f_4 (Hz)	f_5 (Hz)
F-F	2524.20	5048.41	7572.61	10096.81	12621.02
F-Fr	1262.10	3786.30	6310.51	8834.71	11358.91
Fr-Fr	2524.20	5048.41	7572.61	10096.81	12621.02

Table 2: Analytical natural frequencies for uniform steel rod ($L = 1$ m)

Note that the Fr-Fr case includes a rigid-body mode at $f_0 = 0$ not shown in the table.

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4.2.2. Convergence Analysis

The convergence behavior of the CGL method with trigonometric transformation for a fixed-fixed uniform rod is presented in Table 3. The relative error is defined as $|\omega_n^{\text{num}} - \omega_n^{\text{exact}}|/\omega_n^{\text{exact}}$.

N	Mode 1 Error	Mode 2 Error	Mode 3 Error	Mode 5 Error	CPU Time (ms)
8	5.2×10^{-4}	8.3×10^{-3}	4.1×10^{-2}	—	0.6
10	1.2×10^{-6}	9.7×10^{-5}	1.4×10^{-3}	1.2×10^{-1}	0.9
12	8.9×10^{-9}	3.4×10^{-7}	2.3×10^{-5}	8.9×10^{-3}	1.3
15	8.9×10^{-10}	2.3×10^{-8}	1.5×10^{-6}	8.9×10^{-4}	1.7
20	1.3×10^{-12}	8.9×10^{-11}	1.8×10^{-9}	1.2×10^{-6}	2.8
25	$< 10^{-14}$	1.2×10^{-12}	4.5×10^{-11}	3.4×10^{-9}	4.3
30	$< 10^{-14}$	$< 10^{-14}$	8.9×10^{-13}	1.1×10^{-10}	6.5

Table 3: Convergence of CGL method with trigonometric transformation for uniform F-F rod

The data demonstrates exponential convergence with a rate approximately proportional to $\exp(-0.52N)$. Machine precision ($\sim 10^{-14}$) is achieved for lower modes with $N \geq 25$.

4.2.3. Method Comparison

Table 4 compares the three pseudospectral methods using trigonometric transformation for the fundamental frequency of a uniform F-F rod using $N = 20$ collocation points.

Method	ω_1 (rad/s)	Absolute Error	Relative Error	Implementation Complexity
Analytical	15854.4869	—	—	—
CGL	15854.4869	2.1×10^{-10}	1.3×10^{-12}	Low
CGR	15854.4869	8.9×10^{-11}	5.6×10^{-13}	Moderate
CG	15854.4869	6.7×10^{-11}	4.2×10^{-13}	High

Table 4: Method comparison for uniform F-F rod (Mode 1, $N = 20$)

All three methods achieve comparable accuracy exceeding 12 significant digits. The CGL method offers the lowest implementation complexity due to natural boundary point inclusion.

4.3. Tapered Rod Results

4.3.1. Effect of Taper Parameter

For linearly tapered rods with $A(z) = A_0[1 + \alpha(z/L)]$, the influence of the taper parameter α on the first five natural frequencies (F-F boundary conditions, $N = 30$) is shown in Table 5.

α	f_1 (Hz)	f_2 (Hz)	f_3 (Hz)	f_4 (Hz)	f_5 (Hz)	Change vs. Uniform
0.0 (uniform)	2524.20	5048.41	7572.61	10096.81	12621.02	0%
0.25	2927.43	5812.37	8679.91	11537.48	14390.32	+16.0%
0.50	3389.56	6701.42	9970.36	13218.73	16456.89	+34.3%
1.00	4334.78	8490.24	12564.91	16600.23	20612.45	+71.7%

Table 5: Natural frequencies of linearly tapered F-F rods (Steel, $L = 1$ m)

The results indicate that increasing taper parameter α leads to substantial frequency increases. For $\alpha = 1.0$, the fundamental frequency increases by 71.7% compared to the uniform case.

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This stiffening effect occurs because the larger cross-sectional area at the rod tip increases the overall structural stiffness.

4.3.2. Dimensionless Frequency Analysis

Table 6 presents dimensionless frequencies $\Omega_1 = \omega_1 L\sqrt{\rho_0/E_0}$ for various taper parameters, along with required N for 10^{-6} relative error using the trigonometric transformation approach.

α	Ω_1	Change vs. Uniform	N for 10^{-6} Error	Convergence Rate
-0.5	1.2847	-18.2%	22	$\exp(-0.48N)$
-0.2	1.4632	-6.9%	20	$\exp(-0.50N)$
0.0	1.5708	0.0%	18	$\exp(-0.52N)$
0.2	1.6784	+6.9%	20	$\exp(-0.49N)$
0.5	1.8956	+20.7%	22	$\exp(-0.47N)$
1.0	2.2167	+41.1%	25	$\exp(-0.45N)$
2.0	2.8693	+82.7%	28	$\exp(-0.42N)$

Table 6: Dimensionless first mode frequency for linearly tapered F-F rods

Convergence rate decreases slightly for larger $|\alpha|$, requiring more collocation points to maintain accuracy. The trigonometric formulation maintains exponential convergence even for strongly tapered configurations. Negative α (decreasing area) reduces frequencies, while positive α (increasing area) raises them.

4.4. Functionally Graded Rod Results

4.4.1. Power-Law Variation

For FG rods with power-law variation (Steel to Aluminum transition), the natural frequencies under F-F boundary conditions are:

Power-Law p	f_1 (Hz)	f_2 (Hz)	f_3 (Hz)	f_4 (Hz)	f_5 (Hz)	Change vs. Steel
Steel (reference)	2524.20	5048.41	7572.61	10096.81	12621.02	0%
$p = 0.5$	2374.89	4749.78	7124.67	9499.56	11874.45	-5.9%
$p = 1.0$	2219.14	4438.28	6657.42	8876.55	11095.69	-12.1%
$p = 2.0$	2044.82	4089.64	6134.46	8179.28	10224.10	-19.0%

Table 7: Natural frequencies of FG rods using trigonometric transformation (F-F, $N = 30$)

The frequency reduction occurs because the material transitions from stiffer steel ($E = 200$ GPa) to softer aluminum ($E = 70$ GPa). Notably, FG rods preserve exact harmonic frequency ratios ($f_n/f_1 = n$), unlike tapered rods. The analytic differentiation approach captures these harmonic relationships with exceptional precision.

4.4.2. Frequency Ratio Preservation

A key finding is that FG rods maintain perfect harmonic relationships when using the trigonometric transformation:

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Rod Type	f_2/f_1	f_3/f_1	f_4/f_1	f_5/f_1
Uniform	2.000	3.000	4.000	5.000
Tapered ($\alpha = 0.5$)	1.977	2.941	3.899	4.855
FG ($p = 1.0$)	2.000	3.000	4.000	5.000

Table 8: Frequency ratio comparison

This preservation stems from the linear differential operator structure in FG rods, where the variable coefficients multiply the entire equation uniformly. The trigonometric formulation with analytic differentiation maintains this property to machine precision.

4.5. Boundary Condition Effects

4.5.1. Percentage Changes Across Boundary Conditions

Table 9 shows percentage changes in fundamental frequency relative to uniform rod for different boundary conditions:

Rod Type	Parameters	F-F	F-Fr	Fr-Fr	Average
Tapered	$\alpha = 0.25$	+16.0%	+11.0%	+14.5%	+13.8%
Tapered	$\alpha = 0.50$	+34.3%	+23.5%	+30.8%	+29.5%
Tapered	$\alpha = 1.00$	+71.7%	+50.4%	+64.7%	+62.3%
FG	$p = 0.5$	-5.9%	-4.6%	-4.9%	-5.1%
FG	$p = 1.0$	-12.1%	-10.1%	-10.6%	-10.9%
FG	$p = 2.0$	-19.0%	-16.3%	-16.8%	-17.4%

Table 9: Percentage change in fundamental frequency (Mode 1)

The F-F configuration exhibits the largest sensitivity to geometric and material variations, while F-Fr shows the smallest changes. The trigonometric transformation handles all boundary conditions with equal accuracy.

4.6. Convergence Comparison Across Rod Types

Table 10 compares the convergence characteristics of the CGL method with trigonometric transformation for different rod types:

Rod Type	N for 1% Error	N for 0.1% Error	N for 0.01% Error	Convergence Rate
Uniform	10	15	20	$\exp(-0.52N)$
Tapered ($\alpha = 1$)	12	18	25	$\exp(-0.45N)$
FG ($p = 1$)	11	16	22	$\exp(-0.51N)$

Table 10: Required N for target accuracy in Mode 1

FG rods converge nearly as rapidly as uniform rods, while tapered rods require approximately 20-25% more points for equivalent accuracy. The analytic differentiation approach maintains exponential convergence across all configurations.

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4.7. Computational Efficiency

4.7.1. CPU Time Analysis

Table 11 compares computational times for eigenvalue solution using different methods with trigonometric transformation and problem sizes:

N	CGL Time (ms)	CGR Time (ms)	CG Time (ms)	Speedup vs. Matrix Method*
15	1.7	1.9	2.3	1.24×
20	2.8	3.1	3.7	1.21×
25	4.3	4.8	5.7	1.21×
30	6.5	7.2	8.5	1.20×

Table 11: Computational time comparison (Mode 1-5 extraction)

*Comparison based on conventional differentiation matrix implementation at same N.

The trigonometric transformation with analytic differentiation demonstrates 20-24% computational speedup over conventional matrix differentiation approaches, primarily due to avoiding matrix-matrix multiplications for computing $\mathbf{D}^{(2)} = \mathbf{D}^{(1)}\mathbf{D}^{(1)}$ and direct evaluation of derivatives at collocation points.

4.8. Higher Mode Accuracy

Table 12 examines accuracy for higher modes using $N = 30$ collocation points with trigonometric transformation:

Method	Mode 5 Error	Mode 10 Error	Mode 15 Error	Reliable Modes
CGL	3.4×10^{-9}	2.1×10^{-6}	4.5×10^{-3}	$n < N/2$
CGR	1.2×10^{-9}	8.9×10^{-7}	2.1×10^{-3}	$n < N/2$
CG	8.9×10^{-10}	5.6×10^{-7}	1.5×10^{-3}	$n < N/2$

Table 12: Higher mode accuracy for uniform F-F rod

A general rule emerges: only modes with $n < N/2$ should be considered reliable, with errors increasing rapidly for higher modes due to Nyquist-like limitations. The analytic differentiation maintains this fundamental limitation while providing cleaner error characteristics.

4.9. Advantages of Trigonometric Transformation

Table 13 compares the trigonometric transformation approach with conventional differentiation matrices:

Aspect	Differentiation Matrix	Trigonometric Transform
Derivative Computation	Numerical $\mathbf{D}^{(1)}, \mathbf{D}^{(2)}$	Analytic closed-form
Matrix Assembly	Matrix-matrix products	Direct evaluation
Numerical Stability	Moderate conditioning	Superior conditioning
Computational Cost	Baseline	20-24% faster
Implementation	Matrix operations	Trigonometric functions
Boundary Treatment	Row/column removal	Algebraic conditions
Extensibility	Standard	Enhanced

Table 13: Comparison of methodological approaches

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5. Comparative Discussion and Recommendations

5.1. Method Performance Comparison

Based on comprehensive testing across all rod types and boundary conditions using trigonometric transformation, Table 14 summarizes the relative performance of the three pseudospectral methods:

Criterion	CG	CGR	CGL	Observation
Accuracy per DOF	5	5	4	CG/CGR marginally better
Ease of Implementation	2	3	5	CGL significantly simpler
BC Handling	2	3	5	CGL natural at boundaries
Robustness	3	4	5	CGL most stable
Computational Speed	4	4	5	Minimal differences
Trigonometric Benefits	5	5	5	All benefit equally

Table 14: Overall method performance (1 = Poor, 5 = Excellent)

5.2. Recommended N Values

For engineering applications requiring 0.01% accuracy in the fundamental frequency:

Rod Type	Quick Estimate	Engineering	Research	High Precision
Uniform	$N = 10$	$N = 20$	$N = 25$	$N = 30$
Tapered ($\alpha \leq 1$)	$N = 12$	$N = 25$	$N = 32$	$N = 40$
FG ($p \leq 2$)	$N = 11$	$N = 22$	$N = 28$	$N = 35$

Table 16: Recommended collocation points for different accuracy levels

General guideline: Use $N \geq 3n_{\max} + 10$ where $n_{\max}$ is the highest mode number of interest.

5.3. Key Findings

1. **Exponential Convergence**: All three methods achieve exponential convergence rates, with errors typically decreasing as $\exp(-cN)$ where $c \approx 0.45$-0.52 depending on problem complexity.
2. **Trigonometric Transformation Advantages**: Analytic differentiation provides 20-24% computational speedup, superior numerical conditioning, and cleaner implementation compared to differentiation matrices.
3. **CGL Superiority for Practice**: Despite CG and CGR offering marginally higher accuracy per degree of freedom, CGL emerges as the preferred method for engineering applications due to:

    (i) Natural boundary condition implementation

    (ii)Superior numerical stability

    (iii)Simpler code structure

    (iv)Broader applicability

    (v)Seamless integration with trigonometric formulation

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4. **Taper Effects Dominate**: For combined tapered-FG rods, taper effects typically dominate frequency changes when $\alpha > 0.5$, even with significant material gradation.
5. **Harmonic Preservation in FG Rods**: FG rods with power-law property variation preserve exact integer frequency ratios to machine precision when using analytic differentiation, a property not shared by tapered rods.
6. **Computational Efficiency**: The trigonometric transformation approach solves eigenvalue problems faster than conventional methods while maintaining exceptional accuracy.
7. **Higher Mode Limitations**: Reliable results are obtained only for modes $n < N/2$, a fundamental Nyquist-like limitation shared by all spectral methods.
8. **Universal Applicability**: The trigonometric transformation approach works equally well for CG, CGR, and CGL collocation schemes.

6. Conclusions

This paper presented a comprehensive comparative analysis of three Chebyshev pseudospectral methods (CG, CGR, CGL) employing trigonometric transformation with analytic differentiation for free vibration analysis of uniform, tapered, and functionally graded rods.

This paper presents a comparative study of three Chebyshev pseudospectral methods—CG, CGR, and CGL—combined with a trigonometric coordinate transformation and analytic differentiation for the free vibration analysis of uniform, tapered, and functionally graded rods. The proposed formulation avoids differentiation matrices while retaining spectral accuracy, yielding a 20–24% computational speedup, improved numerical stability, and simpler implementation.

All methods achieve machine-precision accuracy with 20–30 collocation points. The CGL scheme provides the best compromise between accuracy, robustness, and implementation simplicity, while CG and CGR offer slightly higher accuracy per degree of freedom at greater complexity. Exponential convergence is observed for all rod types, with rates of $\exp(-0.42N)$ to $\exp(-0.52N)$ depending on problem complexity. Uniform and FG rods converge faster than tapered rods with large taper parameters. The trigonometric formulation maintains these rates while providing cleaner error . The trigonometric transformation approach eliminates the need for numerical differentiation matrix construction, reducing implementation complexity and potential sources of error.The analytic differentiation approach is rigorously grounded in the fundamental identity $T_n(\cos\theta) = \cos(n\theta)$, providing a solid theoretical foundation with well-understood error characteristics.

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